CHAPTER 8

CR-SUBMANIFOLDS OF A NEARLY TRANS-HYPERBOLIC
SASAKIAN MANIFOLD

8.1 Introduction

The notion of a CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu in [10]. CR-submanifolds have been studied by many geometers ([1], [2], [5], [8], [11], [13], [18], [20], [22]). On the other hand, almost contact hyperbolic \((f, g, \eta, \xi)\)–structure was defined and studied by Upadhyaya and Dube in [21]. S. Kumar and K. K. Dube studied CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold in [19].

Let \(\nabla\) be a linear connection in an \(n\)-dimensional differentiable manifold \(M\). The torsion tensor \(T\) of \(\nabla\) is given by

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].
\]

The connection \(\nabla\) is symmetric if torsion tensor \(T\) vanishes, otherwise it is non-symmetric. The connection \(\nabla\) is metric if \(\nabla g = 0\) for the Riemannian metric \(g\), otherwise it is non–metric.

A connection \(\nabla\) is said to be semi-symmetric ([15], [17]) if its torsion tensor is of the form

\[
T(X,Y) = \eta(Y)X - \eta(X)Y
\]

and \(\nabla\) is said to be quarter symmetric [16] if its torsion tensor is of the form
\[ T(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \]

where \( \eta \) is a 1-form.

Semi-symmetric and quarter-symmetric semi-symmetric connections were studied by many geometers ([3], [4], [6], [7], [9], [14]).

In this chapter, we study CR-submanifold of a nearly trans-hyperbolic Sasakian manifold admitting semi-symmetric and quarter-symmetric semi-metric connections.

This chapter is organized as follows: Section 8.2 contains definitions and basic concept of submanifold. In section 8.3, we prove some basic lemmas on CR-submanifolds for semi symmetric semi-metric connection. Section 8.4 deals with parallel distributions on CR-submanifolds for semi-symmetric semi-metric connection. In section 8.5, we discuss quarter-symmetric semi-metric connection. In the section 8.6, we study some basic lemmas on CR-submanifolds for quarter-symmetric semi-metric connection and in section 8.7, parallel distributions with respect to quarter-symmetric semi-metric connection has been discussed.

**8.2 Preliminaries**

Let \( \tilde{M} \) be an \( n \)-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact structure \((\phi, \xi, \eta, g)\), where a tensor field \( \phi \) of type \((1,1)\), a vector field \( \xi \) and 1-form \( \eta \) of \( \xi \) satisfying

(8.2.1) \( \phi^2 X = X - \eta (X)\xi, \quad g(X, \xi) = \eta(X) \)

(8.2.2) \( \eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta\phi\eta = 0, \)

(8.2.3) \( g(\phi X, \phi Y) = - g(X,Y) - \eta(X)\eta(Y) \)

for any vector \( X,Y \) tangent to \( \tilde{M} \) [21]. In this case we have
(8.2.4) \[ g(\phi X, Y) = -g(X, \phi Y). \]

An almost hyperbolic contact metric structure \((\phi, \xi, \eta, g)\) on \(\overline{M}\) is called trans-hyperbolic Sasakian \([12]\) if and only if

\[
(8.2.5) \quad (\nabla_X \phi) Y = \alpha(g(X, Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)
\]

for all \(X, Y\) tangent to \(\overline{M}\), where \(\alpha, \beta\) are functions on \(\overline{M}\). On a trans-hyperbolic Sasakian manifold \(\overline{M}\), we have

\[
(8.2.6) \quad \nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi)
\]

for a Riemannian metric \(g\) and the Levi-Civita connection \(\nabla\). Further, an almost hyperbolic contact metric manifold \(\overline{M}\) is called nearly trans-hyperbolic Sasakian manifold if \([19]\)

\[
(8.2.7) \quad \langle \bar{\nabla}_X \phi \rangle Y + \langle \bar{\nabla}_Y \phi \rangle X = \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y)
\]

\[ -\beta(\eta(X)\phi Y + \eta(Y)\phi X). \]

Let \(M\) be a submanifold of nearly trans-hyperbolic Sasakian manifold \(\overline{M}\). The metric induced on \(M\) is denoted by same symbol \(g\). Let \(M = TM + TM^\perp\), where \(TM\) is tangent space and \(TM^\perp\) is the normal space.

**Definition 8.2.1.** An \(m\)-dimensional submanifold \(M\) of a nearly trans-hyperbolic Sasakian manifold \(\overline{M}\) is called a CR-submanifold if \(\xi\) is tangent to \(M\) and \(T_x(M) = D_x + D_x^\perp\) such that

(i) the distribution \(D_x\) is invariant under \(\varphi\), that is \(\phi D_x \subset D_x\) for each \(x \in M\),

(ii) the complementary orthogonal distribution \(D^\perp\) is anti-invariant under \(\varphi\), that is \(\phi D^\perp_x \subset T^\perp_x(M)\) for all \(x \in M\).

If \(\dim D^\perp_x = 0\) (resp. \(\dim D_x = 0\)), then CR-submanifold is called invariant (resp. anti-invariant). The distribution \(D\) (resp. \(D^\perp\)) is called horizontal (resp. vertical) distribution. The pair \((D, D^\perp)\) is called \(\xi\)-horizontal (resp. \(\xi\)-vertical) if \(\xi_x \in D_x\) (resp. \(\xi_x \in D_x^\perp\)) for any \(x \in M\).
For any \( X \in TM \), we write

\[
X = PX + QX,
\]

where \( PX \) and \( QX \) belong to the distribution \( D \) and \( D^\perp \) respectively. For any vector \( N \in TM^\perp \), we can put

\[
\phi N = BN + CN,
\]

where \( BN \) is tangential and \( CN \) is the normal component of \( \phi N \).

Now, we remark that owing to the existence of the 1-form \( \eta \), we can define a semi-symmetric semi-metric connection \( \nabla \) in a nearly trans-hyperbolic Sasakian manifold by

\[
\nabla_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi
\]

such that \( (\nabla X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(Z,X) - \eta(Z)g(X,Y) \).

Inserting (8.2.10) in (8.2.5), we get

\[
(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X,Y)\xi - \eta(X)\phi Y) - 2\eta(X)\phi Y + g(X,\phi Y)\xi.
\]

Interchanging \( X \) and \( Y \), we have

\[
(\nabla_Y \phi)X = \alpha(g(X,Y)\xi - \eta(X)\phi Y) + \beta(g(X,\phi Y)\xi - \eta(Y)\phi X) - 2\eta(Y)\phi X + g(Y,\phi X)\xi.
\]

Adding above two equations, we obtain

\[
(\nabla_X \phi)Y + (\nabla_Y \phi)X = \alpha(2g(X,Y)\xi - \eta(X)\phi Y - \eta(Y)\phi X) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) - 2\eta(X)\phi Y - 2\eta(Y)\phi X.
\]

From (8.2.6) and (8.2.10), we get

\[
(\nabla_X \phi)X = -\alpha(\phi X) + \beta(X - \eta(X)\xi).
\]

The Gauss formula for a \( CR \)-Submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection is

\[
\nabla_X Y = \nabla_X Y + h(X,Y)
\]
and the Weingarten formula on $M$ is given by
\begin{equation}
\overline{\nabla}_X N = -A_N X - \eta(X) N + \nabla^\perp_X N
\end{equation}
for $X, Y \in TM$, $N \in TM^\perp$, where $h$ and $A$ are called the second fundamental tensor and shape operator respectively and $\nabla^\perp$ denotes the normal connection. Moreover, we also have
\begin{equation}
g(h(X, Y), N) = g(A_N X, Y).
\end{equation}

**Theorem 8.2.1.** The connection induced on $CR$-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection is also a semi-symmetric semi-metric connection.

**Proof.** Let $\overline{\nabla}$ be the induced connection with the unit normal $N$ on $CR$-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection $\nabla$. Then
\begin{equation}
\overline{\nabla}_X Y = \overline{\nabla}_x Y + m(X, Y),
\end{equation}
where $m$ is a tensor field of type $(0, 2)$ on $CR$-submanifold $M$. If $\nabla^*$ be the induced connection from the Riemannian connection $\nabla$ on $CR$-Submanifold. Then we have
\begin{equation}
\nabla_x Y = \nabla^*_x Y + h(X, Y),
\end{equation}
where $h$ is a second fundamental tensor of type $(0, 2)$.

From (8.2.16), (8.2.17) and (8.2.10), we get
\[
\overline{\nabla}_x Y + m(X, Y) = \nabla^*_x Y + h(X, Y) - \eta(X) Y + g(X, Y) \xi.
\]
Comparing the tangential and normal components from both sides, we find
\[
\overline{\nabla}_x Y = \nabla^*_x Y - \eta(X) Y + g(X, Y) \xi
\]
and
\[
m(X, Y) = h(X, Y).
\]
Thus $\overline{\nabla}$ is also a semi-symmetric semi-metric connection.
8.3 Some Basic Lemmas on CR-submanifold for Semi-symmetric Semi-Metric Connection

Lemma 8.3.1. Let $M$ be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection $\nabla$. Then

\begin{align*}
&\tilde{P}\nabla_x (\phi PY) + \tilde{P}\nabla_y (\phi PX) - PA\phi_{\psi X} Y - PA\phi_{\psi Y} X = \phi P\nabla_x Y + \phi P\nabla_y X \\
&\text{(8.3.1)} +2\alpha g(X,Y)P\xi - \alpha \eta(X)\phi PY - \alpha \eta(Y)\phi PX - \beta \eta(X)\phi PY - 2\eta(X)\phi PX, \\
&Q(\nabla_x \phi PY) + Q(\nabla_y \phi PX) - Q(A\phi_{\psi X} Y) - Q(A\phi_{\psi Y} X) = 2Bh(X,Y) + 2\alpha g(X,Y)Q\xi, \\
&\text{(8.3.2)} h(X,\phi PY) + h(Y,\phi PX) + \nabla^+ X \phi QY + \nabla^+ Y \phi QX - \eta(X)\phi QY - \eta(Y)\phi QX + \phi Q\nabla_y X \\
&+ \phi Q\nabla_x Y + 2Ch(X,Y) - 2\phi h(X,Y) = 0 \quad \text{(8.3.3)}
\end{align*}

for all $X,Y \in TM$.

Proof. From (8.2.8), we have

\[ \phi Y = \phi PY + \phi QY. \]

By covariant differentiation of both sides, we have

\[ \tilde{\nabla}_X \phi Y = \tilde{\nabla}_X (\phi PY) + \tilde{\nabla}_X (\phi QY). \]

Using (8.2.12), (8.2.13) and (8.2.11), we get

\[ (\tilde{\nabla}_X \phi) Y + \phi \nabla_X Y + \phi h(X,Y) = \nabla_X (\phi PY) + h(\phi PY, X) + \nabla^+ X (\phi QY) - A\phi_{QY} X - \eta(X)\phi QY. \]

Interchanging $X$ and $Y$, we have

\[ (\tilde{\nabla}_Y \phi) X + \phi \nabla_Y X + \phi h(X,Y) = \nabla_Y (\phi PX) + h(Y,\phi PX) + \nabla^+ Y (\phi QX) - A\phi_{QX} Y - \eta(Y)\phi QX. \]

Adding above two equations, we get
\[(\bar{\nabla}_X \varphi)Y + (\bar{\nabla}_Y \varphi)X + \varphi \nabla_X Y + \varphi \nabla_Y X + 2\varphi h(X, Y) = \nabla_X (\varphi PY) + \nabla_Y (\varphi PX) + h(X, \varphi PY) + h(Y, \varphi PX) + \nabla^\perp_X (\varphi QY) + \nabla^\perp_Y (\varphi QX) - A_{\varphi QX} Y - A_{\varphi QY} X - \eta(Y) \varphi QX - \eta(X) \varphi QY.\]

Using (8.2.11) in above equation, we obtain

\[
\alpha(2g(X, Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) - \beta(\eta(Y)\varphi X + \eta(X)\varphi Y) - 2\eta(X) \varphi Y - 2\eta(Y) \varphi X + \varphi \nabla_X Y + \varphi \nabla_Y X + 2\varphi h(X, Y) = h(X, \varphi PY) + h(Y, \varphi PX) + \nabla^\perp_X (\varphi QY) + \nabla^\perp_Y (\varphi QX) + \nabla^\perp_X (\varphi PY) + \nabla^\perp_Y (\varphi PX) - A_{\varphi QX} Y - A_{\varphi QY} X
\]

(8.3.1)– (8.3.3) followed by comparing the horizontal, vertical and normal components.

**Lemma 8.3.2.** Let \( M \) be a \( \xi \) – horizontal CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. Then

\[
2(\bar{\nabla}_X \varphi)Y = \nabla_X \varphi Y - \nabla_Y \varphi X + h(X, \varphi Y) - h(Y, \varphi X) - \varphi [X, Y]
\]

(8.3.4)

\[
+ \alpha(2g(X, Y)\xi - \eta(X)\varphi Y - \eta(Y)\varphi X) - \beta(\eta(Y)\varphi X + \eta(X)\varphi Y) - 2\eta(X) \varphi Y - 2\eta(Y) \varphi X,
\]

for any \( X, Y \in D. \)

**Proof.** Let \( X, Y \in D. \) Using Gauss formula (8.2.13), we have

\[
(\bar{\nabla}_X \varphi Y - \bar{\nabla}_Y \varphi X = \nabla_X \varphi Y + h(X, \varphi Y) - \nabla_Y \varphi X - h(Y, \varphi X).
\]

(8.3.6)

Also, we have

\[
(\bar{\nabla}_X \varphi Y - \bar{\nabla}_Y \varphi X = (\bar{\nabla}_X \varphi)Y - (\bar{\nabla}_Y \varphi)X + \varphi [X, Y].
\]

(8.3.7)

From (8.3.6) and (8.3.7), we get

\[
(\bar{\nabla}_X \varphi)Y - (\bar{\nabla}_Y \varphi)X = \nabla_X \varphi Y + h(X, \varphi Y) - \nabla_Y \varphi X - h(Y, \varphi X) - \varphi [X, Y].
\]

(8.3.8)

Adding (8.2.11) and (8.3.8), we have
\[2(\nabla_X \varphi)Y = \nabla_X \varphi Y - \nabla_Y \varphi X + h(X, \varphi Y) - h(Y, \varphi X) - \varphi[X, Y]
- 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \alpha(2g(X, Y) - \eta(Y)\varphi X - \eta(X)\varphi Y)
- \beta(\eta(Y)\varphi X + \eta(X)\varphi Y).
\]

Subtracting (8.3.8) from (8.2.11), we get

\[2(\nabla_Y \varphi)X = \alpha(2g(X, Y)\xi - \eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\varphi Y
+ \eta(Y)\varphi X - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi - \nabla_X \varphi Y
- \nabla_Y \varphi X - h(X, \varphi Y) + h(Y, \varphi X) + \varphi[X, Y].
\]

Hence lemma is proved.

**Corollary 8.3.3.** Let \( M \) be a \( \xi \) – horizontal \( CR \)-Submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. Then

\[2(\nabla_X \varphi)Y = \nabla_X \varphi Y - \nabla_Y \varphi X + h(X, \varphi Y) - h(Y, \varphi X) - \varphi[X, Y] + 2\alpha g(X, Y)\xi.
2(\nabla_Y \varphi)X = -2\alpha g(X, Y)\xi - \nabla_X \varphi Y + \nabla_Y \varphi X - h(X, \varphi Y) + h(Y, \varphi X) + \varphi[X, Y].
\]

for any \( X, Y \in D \).

**Lemma 8.3.4.** Let \( M \) be \( CR \)-submanifold of an nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi- symmetric semi-metric connection. Then

\[2(\nabla_Y \varphi)Z = A_{\partial Y} Z - A_{\varphi Y} Y + \nabla_Y ^1 \varphi Z - \nabla_Y ^1 \varphi Y - \varphi[Y, Z] - \eta(Y)\varphi Z
+ \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 3\eta(Z)\varphi Y
- \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y),
(8.3.9)
\]

\[2(\nabla_Y ^1 \varphi)Z = -A_{\partial Y} Z + A_{\varphi Y} Y - \nabla_Y ^1 \varphi Z + \nabla_Y ^1 \varphi Y + \varphi[Y, Z] - \eta(Z)\varphi Y
+ \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 3\eta(Y)\varphi Z
- \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y)
(8.3.10)
\]

for any \( Y, Z \in D^1 \).

**Proof.** From Weingarten formula (8.2.14), we have

\[\nabla_Y \varphi Z - \nabla_Y ^1 \varphi Z = A_{\varphi Y} Z - A_{\varphi Y} Y + \nabla_Y ^1 \varphi Z - \nabla_Y ^1 \varphi Y + \eta(Y)\varphi Z - \eta(Z)\varphi Y.
(8.3.11)
\]
Also, we have
\[
(8.3.12) \quad \tilde{\nabla}_Z \phi Y - \nabla_Y \phi Z = (\tilde{\nabla}_Y \phi)Z - (\tilde{\nabla}_Z \phi)Y + \phi[Y, Z].
\]
From (8.3.11) and (8.3.12), we get
\[
(8.3.13) \quad (\tilde{\nabla}_Y \phi)Z - (\tilde{\nabla}_Z \phi)Y = A_{\tilde{\phi}} Z - A_{\phi} Y + \nabla_Y \phi Z - \nabla_Z \phi Y + \eta(Y) \phi Z - \eta(Z) \phi Y.
\]
Also from (8.2.11), we have
\[
(8.3.14) \quad (\tilde{\nabla}_Y \phi)Z + (\tilde{\nabla}_Z \phi)Y = \alpha(2g(Z, Y)\xi - \eta(Y) \phi Z - \eta(Z) \phi Y - 2\eta(Z) \phi Y - \beta(\eta(Y) \phi Z + \eta(Z) \phi Y) - 2\eta(Y) \phi Z.
\]
Adding (8.3.13) and (8.3.14), we obtain
\[
2(\tilde{\nabla}_Y \phi)Z = A_{\tilde{\phi}} Z - A_{\phi} Y + \nabla_Y \phi Z - \nabla_Z \phi Y - \phi[Y, Z] + \alpha(2g(Y, Z)\xi - \eta(Y) \phi Z - \eta(Z) \phi Y - \beta(\eta(Y) \phi Z + \eta(Z) \phi Y) - 3\eta(Y) \phi Z - \eta(Z) \phi Y.
\]
Subtracting (8.3.13) from (8.3.14), we get
\[
2(\tilde{\nabla}_Z \phi)Y = -A_{\tilde{\phi}} Z + A_{\phi} Y - \nabla_Y \phi Z + \nabla_Z \phi Y + \phi[Y, Z] + \alpha(2g(Y, Z)\xi - \eta(Y) \phi Z - \eta(Z) \phi Y - \beta(\eta(Y) \phi Z + \eta(Z) \phi Y) - 3\eta(Y) \phi Z - \eta(Z) \phi Y.
\]
Hence Lemma is proved.

**Corollary 8.3.5.** Let M be a \(\xi\)- horizontal CR-submanifold of an nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection. Then
\[
2(\tilde{\nabla}_Z \phi)Y = -A_{\tilde{\phi}} Z + A_{\phi} Y - \nabla_Y \phi Z + \nabla_Z \phi Y + \phi[Y, Z] + 2\alpha g(Y, Z)\xi.
\]
\[
2(\tilde{\nabla}_Y \phi)Z = A_{\tilde{\phi}} Z - A_{\phi} Y + \nabla_Y \phi Z - \nabla_Z \phi Y - \phi[Y, Z] + 2\alpha g(Y, Z)\xi.
\]
for any \(Y, Z \in D^\perp\).

**Lemma 8.3.6.** Let M be a CR-submanifold of nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection, then
\[
(8.3.15) \quad 2(\tilde{\nabla}_X \phi)Y = -A_{\tilde{\phi}} X + \nabla_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi [X, Y] - 2\eta(Y) \phi X - 3\eta(X) \phi Y + \alpha(-\eta(Y) \phi X - \eta(X) \phi Y) - \beta(\eta(X) \phi Y + \eta(Y) \phi X),
\]
\(2(\tilde{\nabla}_Y \phi)X = A_{\phi Y} X - \nabla^\perp_X \phi Y + \nabla_Y \phi X + h(Y, \phi X) - \eta(X) \phi Y \]
\(-2\eta(Y) \phi X + \alpha(-\eta(Y) \phi X - \eta(X) \phi Y) - \beta(\eta(X) \phi Y + \eta(Y) \phi X) \)

for any \(X \in D\) and \(Y \in D^\perp\).

**Proof.** Let \(X \in D\) and \(Y \in D^\perp\). Then from (8.2.12) and (8.2.13), we have
\[
\tilde{\nabla}_X \phi Y = -A_{\phi Y} X - \eta(X) \phi Y + \nabla^\perp_X \phi Y,
\]
\[
\tilde{\nabla}_Y \phi X = \nabla_Y \phi X + h(Y, \phi X).
\]

Subtracting above two equations, we have
\[
(8.3.17) ~ \tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = -A_{\phi Y} X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \eta(X) \phi Y.
\]

Also, by direct covariant differentiation, we have
\[
(8.3.18) ~ \tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = (\tilde{\nabla}_X \phi)Y - (\tilde{\nabla}_Y \phi)X + \phi [X, Y].
\]

From (8.3.17) and (8.3.18), we get
\[
(8.3.19) ~ (\tilde{\nabla}_X \phi)Y - (\tilde{\nabla}_Y \phi)X = -A_{\phi Y} X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \eta(X) \phi Y - \phi [X, Y].
\]

Adding (8.3.19) and (8.2.11), we get
\[
(8.3.20) ~ 2(\tilde{\nabla}_X \phi)Y = -A_{\phi Y} X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - 3\eta(X) \phi Y
\]
\[-\phi [X, Y] - \eta(X) \phi Y - \eta(Y) \phi X + \alpha(-\eta(Y) \phi X - \eta(X) \phi Y)
\]- \beta(\eta(X) \phi Y + \eta(Y) \phi X).
\]

Subtracting (8.3.19) from (8.2.11), we obtain
\[
(8.3.21) ~ 2(\tilde{\nabla}_Y \phi)X = A_{\phi Y} X - \nabla^\perp_X \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi [X, Y]
\]
\[+ \alpha(-\eta(Y) \phi X - \eta(X) \phi Y) - \beta(\eta(X) \phi Y + \eta(Y) \phi X)
\]- \beta(\eta(X) \phi Y + \eta(Y) \phi X).
\]

Hence lemma is proved.

**Corollary 8.3.7.** Let \(M\) be a \(\xi\)-horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection, then
\[
2(\tilde{\nabla}_X \phi)Y = -A_{\phi Y} X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] \\
- \alpha \eta(X) \phi Y - \beta \eta(X) \phi Y - 3\eta(X) \phi Y,
\]

\[
2(\tilde{\nabla}_Y \phi)X = A_{\phi X} X - \nabla^\perp_Y \phi Y + \nabla_X \phi X + h(Y, \phi X) + \phi[X, Y] \\
- \alpha \eta(X) \phi Y - \beta \eta(Y) \phi X - \eta(X) \phi Y
\]

for any \( X \in D \) and \( Y \in D^\perp \).

**Corollary 8.3.8.** Let \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection, then

\[
2(\tilde{\nabla}_X \phi)Y = -A_{\phi Y} X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] \\
- \alpha \eta(Y) \phi X - \beta \eta(Y) \phi X - 2\eta(Y) \phi X,
\]

\[
2(\tilde{\nabla}_Y \phi)X = A_{\phi X} X - \nabla^\perp_Y \phi Y + \nabla_X \phi X + h(Y, \phi X) + \phi[X, Y] \\
- \alpha \eta(Y) \phi X - \beta \eta(Y) \phi X - 2\eta(Y) \phi X.
\]

for any \( X \in D \) and \( Y \in D^\perp \).

### 8.4 Parallel Distributions on CR-submanifolds for Semi-symmetric Semi-Metric Connection

**Definition 8.4.1.** The horizontal (resp. vertical) distribution \( D \) (resp. \( D^\perp \)) is said to be parallel with respect to the semi-symmetric semi-metric connection on \( M \) if \( \tilde{\nabla}_X Y \in D \) (resp. \( \tilde{\nabla}_X W \in D^\perp \)) for any vector field \( X, Y \in D \) (resp. \( W, Z \in D^\perp \)).

**Proposition 8.4.2.** Let \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. If horizontal distribution \( D \) is parallel, then

\[
h(X, \phi Y) = h(Y, \phi X) \quad \text{for any } X, Y \in D.
\]
Proof. Let $D$ be parallel distribution, then
\[(8.4.1)\quad \nabla_X \phi Y \in D, \quad \nabla_Y \phi X \in D \quad \text{for any } X, Y \in D.\]

From (8.3.2), we get
\[
Q(\nabla_X \phi Y) + Q(\nabla_Y \phi X) - Q(A_{\phi Y}, Y) - Q(A_{\phi X}, X)
= 2Bh(X, Y) + 2\alpha g(X, Y)Q\xi
\]
\[(8.4.2)\]
\[
2Bh(X, Y) + 2\alpha g(X, Y)Q\xi = 0.
\]

\[(8.4.3)\quad Bh(X, Y) = -\alpha g(X, Y)Q\xi \quad \text{for any } X, Y \in D.
\]

From (8.2.9), we have
\[(8.4.4)\quad \phi h(X, Y) = Bh(X, Y) + Ch(X, Y).
\]

From (8.4.3) and (8.4.4), we have
\[(8.4.5)\quad \phi h(X, Y) = -\alpha g(X, Y)Q\xi + Ch(X, Y).
\]

Now, from (8.3.3) we have
\[(8.4.6)\quad h(X, \phi Y) + h(Y, \phi X) = 2\phi h(X, Y) + 2\alpha g(X, Y)Q\xi.
\]

Replacing $X$ by $\phi X$, we find
\[(8.4.7)\quad h(\phi X, \phi Y) + h(Y, X) = 2\phi h(\phi X, Y) + 2\alpha g(\phi X, Y)Q\xi.
\]

Similarly, replacing $Y$ by $\phi Y$ in (8.4.6), we get
\[(8.4.8)\quad h(\phi Y, \phi X) + h(X, Y) = 2\phi h(X, \phi Y) + 2\alpha g(X, \phi Y)Q\xi.
\]

From (8.4.7) and (8.4.8), we obtain
\[(8.4.9)\quad 2\phi h(\phi X, Y) + 2\alpha g(\phi X, Y)Q\xi = 2\phi h(X, \phi Y) + 2\alpha g(X, \phi Y)Q\xi.
\]

Operating $\phi$ on both sides of (8.4.9) and using $\phi \xi = 0$, we get
\[(8.4.10)\quad \phi h(\phi X, Y) + \alpha g(\phi X, Y)\phi Q\xi = \phi h(X, \phi Y) + \alpha g(X, \phi Y)\phi Q\xi = 0.
\]

Thus, we have
\[(8.4.11)\quad h(X, \phi Y) = h(Y, \phi X) \quad \text{for each } X, Y \in D.
\]
Proposition 8.4.3. Let $M$ be a $\xi$-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a semi-symmetric semi-metric connection. If the distribution $D^\perp$ is parallel with respect to the connection on $M$ then

\begin{equation}
(8.4.12) \quad (A_{\varphi Y}Z + A_{\varphi Z}Y) \in D^\perp \quad \text{for any } Y, Z \in D^\perp.
\end{equation}

Proof. Let $Y, Z \in D^\perp$. Using (8.2.12), we get

\begin{equation}
(8.4.13) \quad (\tilde{\nabla}_Y \varphi)Z + \varphi(\tilde{\nabla}_Y Z) = -A_{\varphi Y}Y + \nabla^\perp_Y \varphi Z - \eta(Y) \varphi Z.
\end{equation}

Now, using (8.2.13) we have

\begin{equation}
(8.4.14) \quad (\tilde{\nabla}_Y \varphi)Z = -A_{\varphi Y}Y + \nabla^\perp_Y \varphi Z - \eta(Y) \varphi Z - \varphi \tilde{\nabla}_Y Z - \varphi h(Y, Z).
\end{equation}

Interchanging $Y$ and $Z$, we have

\begin{equation}
(\tilde{\nabla}_Z \varphi)Y = -A_{\varphi Z}Z + \nabla^\perp_Z \varphi Y - \eta(Z) \varphi Y - \varphi \tilde{\nabla}_Z Y - \varphi h(Y, Z).
\end{equation}

Adding above two equations, we obtain

\begin{equation}
(8.4.15) \quad (\tilde{\nabla}_Y \varphi)Z + (\tilde{\nabla}_Z \varphi)Y = -A_{\varphi Y}Y - A_{\varphi Z}Z + \nabla^\perp_Y \varphi Z + \nabla^\perp_Z \varphi Y - \eta(Y) \varphi Z - \eta(Z) \varphi Y - \varphi \tilde{\nabla}_Y Z - \varphi \tilde{\nabla}_Z Y - 2 \varphi h(Y, Z).
\end{equation}

Taking inner product with $X \in D$ in (8.4.13), we get

\begin{equation}
\text{g}(A_{\varphi Y}Z, X) + \text{g}(A_{\varphi Z}Y, X) = \text{g}((\varphi \tilde{\nabla}_Y Z, X) + \text{g}(\varphi \tilde{\nabla}_Z Y, X).
\end{equation}

If $D^\perp$ is parallel then $\nabla_Y Z \in D^\perp$ and $\nabla_Z Y \in D^\perp$ for $Y, Z \in D^\perp$.

Consequently, we have

\begin{equation}
\text{g}(A_{\varphi Y}Z, X) + \text{g}(A_{\varphi Z}Y, X) = 0
\end{equation}

or,

\begin{equation}
(8.4.16) \quad \text{g}(A_{\varphi Y}Z + A_{\varphi Z}Y, X) = 0,
\end{equation}

which implies that $(A_{\varphi Y}Z + A_{\varphi Z}Y) \in D^\perp$. 

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Definition 8.4.4. A CR-submanifold with a semi-symmetric semi-metric connection is said to be mixed totally geodesic if \( h(X, Z) = 0 \) for all \( X \in D \) and \( Z \in D^\perp \).

The following lemma is an easy consequence of (8.2.15).

Lemma 8.4.5. Let \( M \) be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. Then \( M \) is mixed totally geodesic if and only if \( A_N X \in D \) for all \( X \in D \).

Definition 8.4.6. A normal vector field \( N \neq 0 \) with a connection \( \nabla^\perp \) is called \( D \)-parallel normal section if \( \nabla^\perp_X N = 0 \) for all \( X \in D \).

Now, we have the following proposition.

Proposition 8.4.7. Let \( M \) be a mixed totally geodesic \( \xi \)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a semi-symmetric semi-metric connection. Then the normal section \( N \in \mathcal{D}^\perp \) is \( D \)-parallel if and only if \( \nabla_X \phi N \in D \) for all \( X \in D \).

Proof. Let \( N \in \mathcal{D}^\perp \). From (8.3.2) we have

\[
Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi YX}) - Q(A_{\phi PX}) - Q(A_{\phi QX}) = 2Bh(X, Y) + 2\alpha g(X, Y)Q\xi.
\]

Also, we have

\[
g(X, Y) = 0, \quad \phi PY = 0, \quad \phi QX = 0
\]

for any \( X \in D \) and \( Z \in D^\perp \).

From (8.4.18) and (8.4.17), we get

\[
Q(\nabla_X \phi X) = 0.
\]

In particular, we have
(8.4.19) \[ Q(\nabla_Y X) = 0. \]

Using (8.4.19) in (8.3.3), we obtain

\[ \nabla^\perp_X (\phi QY) = \phi Q(\nabla_X Y). \]

Consequently, we get

\[ \nabla^\perp_X N = \phi Q\nabla_X \phi N. \]

Hence the proposition is proved.

### 8.5 Quarter Symmetric Semi-metric Connection

A quarter-symmetric semi-symmetric connection \( \tilde{\nabla} \) in a nearly trans-hyperbolic Sasakian manifold is defined by

\begin{equation}
(8.5.1) \quad \tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y + g(\phi X, Y)\xi
\end{equation}

such that

\[(\tilde{\nabla}_X g)(Y, Z) = 2\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi Z, X) - \eta(Z)g(\phi X, Y). \]

Inserting (8.5.1) in (8.2.5), we get

\[(\tilde{\nabla}_X \phi)Y = (\nabla_X \phi)Y - g(X, Y)\xi - \eta(X)\eta(Y)\xi. \]

Similarly,

\[(\tilde{\nabla}_Y \phi)X = (\nabla_Y \phi)Y - g(X, Y)\xi - \eta(X)\eta(Y)\xi. \]

Adding above two equations, we get

\begin{equation}
(8.5.2) \quad (\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = \alpha(2g(X, Y)\xi - \eta(X)\phi Y - \eta(Y)\phi X) - 2g(X, Y)\xi - \beta(\eta(X)\phi Y + \eta(Y)\phi X) - 2\eta(X)\eta(Y)\xi.
\end{equation}

From (8.2.6) and (8.5.1), we get

\begin{equation}
(8.5.3) \quad \tilde{\nabla}_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi).
\end{equation}
The Gauss formula for CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold with a quarter-symmetric semi-metric connection is
\[ \tilde{\nabla}_x Y = \nabla_x Y + h(X, Y) \] (8.5.4)
and the Weingarten formula for M is given by
\[ \tilde{\nabla}_x N = -A_x N - \eta(X) \phi N + \nabla^\perp_x N + g(\phi X, N) \xi \] (8.5.5)
for \( X, Y \in TM, N \in TM^\perp \), where \( h \) and \( A \) are called the second fundamental tensor and shape operator respectively and \( \nabla^\perp \) denotes the normal connection.

**Theorem 8.5.1.** The connection induced on CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi-metric connection is also a quarter symmetric semi-metric connection.

**Proof.** Let \( \tilde{\nabla} \) be the induced connection with the unit normal \( N \) on CR-submanifold of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric semi-metric connection \( \nabla \). Then
\[ \tilde{\nabla}_x Y = \nabla_x Y + m(X, Y) \] (8.5.6)
where \( m \) is a tensor field of type (0, 2) on CR-submanifold \( M \). If \( \nabla^* \) be the induced connection from the Riemannian connection \( \nabla \) on CR-Submanifold. Then we have
\[ \tilde{\nabla}_x Y = \nabla^*_x Y + h(X, Y) \] (8.5.7)
where \( h \) is a second fundamental tensor of type (0, 2).

From (8.2.16) and (8.2.17) and (8.5.1), we get
\[ \nabla_x Y + m(X, Y) = \nabla_x^* Y + h(X, Y) - \eta(X) \phi Y + g(\phi X, Y) \]
Comparing the tangential and normal components from both the sides, we get
\[ \nabla_x Y = \nabla_x^* Y - \eta(X) \phi Y + g(\phi X, Y) \xi, \]
\[ m(X, Y) = h(X, Y). \]
Thus $\nabla$ is also a quarter-symmetric semi-metric connection.

### 8.6 Some Basic Lemmas on CR-submanifolds for Quarter Symmetric Semi-metric Connection

**Lemma 8.6.1** Let $M$ be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\tilde{M}$ with a quarter-symmetric semi-metric connection $\tilde{\nabla}$.

Then

\[
\begin{align*}
PV_x (\phi PY) + PV_y (\phi PX) - PA_{\phi QY} Y - PA_{\phi PY} X &= \phi P(\nabla_x Y) + \phi P(\nabla_y X) \\
+2\alpha g(X, Y) P\xi - \alpha\eta(X) \phi PX - \alpha\eta(X) \phi PY - \beta\eta(Y) \phi PX - \beta\eta(X) \phi PY \\
-2\eta(X)\eta(Y) P\xi - 2g(X, Y) P\xi, \\
-\eta(X) Q\xi - \eta(X) QX + Q\nabla_x (\phi PY) + Q\nabla_y (\phi PX) - Q(A_{\phi QY} Y) - Q(A_{\phi PY} X) \\
(8.6.2) = 2Bh(X, Y) + 2\alpha g(X, Y) Q\xi - 2\eta(X)\eta(Y) Q\xi - 2g(X, Y) Q\xi + \phi Q(\nabla_x Y) + \phi Q(\nabla_y X),
\end{align*}
\]

for all $X, Y \in TM$.

**Proof.** From (8.2.8), we have

\[
\phi Y = \phi PY + \phi QY.
\]

By covariant differentiation of both sides, we have

\[
\tilde{\nabla}_X \phi Y = (\tilde{\nabla}_X \phi) Y + \phi(\nabla_X Y) + \phi h(X, Y).
\]

Using (8.5.4), (8.5.5) and (8.5.2), we have

\[
(\tilde{\nabla}_X \phi) Y + \phi(\nabla_X Y) + \phi h(X, Y) = \nabla_X (\phi PY) + h(X, \phi PY) + \nabla_X (\phi QY) - A_{\phi QY} X \\
-\eta(X) QY + \eta(X) \eta(QY) \xi.
\]

Interchanging $X$ and $Y$, we have

\[
(\tilde{\nabla}_Y \phi) X + \phi(\nabla_Y X) + \phi h(X, Y) = \nabla_Y (\phi PX) + h(Y, \phi PX) + \nabla_Y (\phi QX) - A_{\phi PX} Y \\
-\eta(Y) QX + \eta(Y) \eta(QX) \xi.
\]

Adding above two equations, we get
\[
(\bar{\nabla}_x \phi)Y + (\bar{\nabla}_y \phi)X + \phi(\nabla_x Y) + \phi(\nabla_y X) + 2\phi h(X, Y) = \nabla_x (\phi PY) + \nabla_y (\phi PX) + h(Y, \phi PX) + \nabla_x^2 (\phi QY) + \nabla_y^2 (\phi QX) - A_{qQX} Y - A_{qQY} Y - \eta(X) QY + \eta(X) \eta(QY) \xi + \eta(X) \eta(QY) \xi
\]

Using (5.2), we have

\[
\alpha(2g(X, Y)\xi - \eta(X)\phi Y - \eta(Y)\phi X) - \beta(\eta(Y)\phi X + \eta(X)\phi Y) - 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) + \beta(\eta(Y)\phi X + \eta(X)\phi Y) - \phi[X, Y],
\]

\[
2(\bar{\nabla}_x \phi) = \bar{\nabla}_x \phi Y = \bar{\nabla}_y \phi X + h(X, \phi Y) - h(Y, \phi X) - 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) + \beta(\eta(Y)\phi X + \eta(X)\phi Y) - \phi[X, Y],
\]

\[
2(\bar{\nabla}_y \phi) = \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) + \phi[X, Y] - \nabla_x \phi Y - \nabla_y \phi X + h(Y, \phi X) - h(Y, \phi X) - 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi.
\]

for any \( X, Y \in D \).

**Proof.** Let \( X, Y \in D \). From Gauss formula (8.5.2), we get

(8.6.6) \[
\bar{\nabla}_x \phi Y - \bar{\nabla}_y \phi X = \nabla_x \phi Y + h(X, \phi Y) - \nabla_y \phi X - h(Y, \phi X).
\]

Also we have

(8.6.7) \[
\bar{\nabla}_x \phi Y - \bar{\nabla}_y \phi X = (\bar{\nabla}_x \phi) Y - (\bar{\nabla}_y \phi) X + \phi[X, Y].
\]

From (8.6.6) and (8.6.7), we get

(8.6.8) \[
(\bar{\nabla}_x \phi) Y - (\bar{\nabla}_y \phi) X = \nabla_x \phi Y + h(X, \phi Y) - \nabla_y \phi X - h(Y, \phi X) - \phi[X, Y].
\]

Adding (8.5.2) and (8.6.8), we obtain

(8.6.1)– (8.6.3) followed by comparison of the horizontal, vertical and normal components.

**Lemma 8.6.2.** Let \( M \) be a \( \xi \) – horizontal CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a quarter-symmetric semi-metric connection. Then

(8.6.9) \[
2(\bar{\nabla}_x \phi) = 2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y - \beta(\eta(X)\phi Y + \eta(Y)\phi X) + \phi[X, Y] - \nabla_x \phi Y - \nabla_y \phi X + h(Y, \phi X) - h(Y, \phi X) - 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi.
\]
\[ 2(\tilde{\nabla}_x \phi) Y = \nabla_x \phi Y - \nabla_y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] - 2g(X, Y)\xi \]
\[ - 2\eta(X)\eta(Y)\xi + \alpha(2g(X, Y) - \eta(Y)\phi X - \eta(X)\phi Y) \]
\[ - \beta(\eta(Y)\phi X + \eta(X)\phi Y). \]
Subtracting (8.6.8) from (8.5.2), we find
\[ 2(\tilde{\nabla}_x \phi) X = \alpha(2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y) \]
\[ - \nabla_x \phi Y - \nabla_y \phi X - h(X, \phi Y) + h(Y, \phi X) + \phi[X, Y] \]
\[ - 2g(X, Y)\xi - 2\eta(X)\eta(Y)\xi. \]
Hence lemma is proved.

**Corollary 8.6.3.** Let \( M \) be a \( \xi - \) horizontal CR-Submanifold of a nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a quarter-symmetric semi-metric connection. Then
\[ 2(\tilde{\nabla}_x \phi) Y = \nabla_x \phi Y - \nabla_y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] + 2\alpha g(X, Y)\xi, \]
\[ 2(\tilde{\nabla}_y \phi) X = \nabla_y \phi X - \nabla_x \phi Y - h(X, \phi Y) + h(Y, \phi X) + \phi[X, Y] - 2\alpha g(X, Y)\xi \]
for any \( X, Y \in D. \)

**Lemma 8.6.4.** Let \( M \) be \( \xi - \) vertical CR-submanifold of nearly trans-hyperbolic Sasakian manifold \( \tilde{M} \) with a quarter-symmetric semi-metric connection. Then
\[ 2(\tilde{\nabla}_y \phi) Z = A_{\phi Y} Z - A_{\phi X} Y + \nabla_{\phi Y} Z - \nabla_{\phi X} Y + \eta(Z)Y - \eta(Y)Z - \phi[Y, Z] \]
\[ + \alpha(2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - 2g(Y, Z)\xi - \eta(Y)\eta(Z)\xi \]
\[ - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y), \]
\[ 2(\tilde{\nabla}_x \phi) Y = - A_{\phi Y} Z + A_{\phi X} Y - \nabla_{\phi Y} Z + \nabla_{\phi X} Y + \phi[Y, Z] - 2g(Y, Z)\xi - \eta(Y)\eta(Z)\xi \]
\[ + \alpha(2g(Y, Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y) - \eta(Z)Y - \eta(Y)Z \]
\[ - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y), \]
for any \( Y, Z \in D^1. \)

**Proof.** From Weingarten formula (8.5.5), we have
\[ (8.6.11) - \tilde{\nabla}_Y \phi Y + \tilde{\nabla}_Y \phi Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla_{\phi Y} Z - \nabla_{\phi Z} Y + \eta(Z)Y - \eta(Y)Z. \]
Also, we have

\[(8.6.12) \quad -\tilde{\nabla}_Z \varphi Y + \tilde{\nabla}_Y \varphi Z = (\tilde{\nabla}_Y \varphi)Z - (\tilde{\nabla}_Z \varphi)Y + \varphi[Y, Z].\]

From (8.6.11) and (8.6.12), we get

\[(8.6.13) \quad (\tilde{\nabla}_Y \varphi)Z - (\tilde{\nabla}_Z \varphi)Y = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^Z_Y \varphi Z - \nabla^Z_Y \varphi Y + \eta(Z)Y - \eta(Y)Z - \varphi[Y, Z].\]

Now, from (8.5.1), we get

\[(8.6.14) \quad (\tilde{\nabla}_Y \varphi)Z + (\tilde{\nabla}_Z \varphi)Y = \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 2g(Y, Z)\xi - \eta(Y)\eta(Z)\xi - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y).\]

Adding (8.6.13) and (8.6.14), we get

\[2(\tilde{\nabla}_Y \varphi)Z = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^Z_Y \varphi Z - \nabla^Z_Y \varphi Y + \eta(Z)Y - \eta(Y)Z - \varphi[Y, Z] + \alpha(2g(Y, Z)\xi - \eta(Y)\varphi Z - \eta(Z)\varphi Y) - 2g(Y, Z)\xi - \eta(Y)\eta(Z)\xi - \beta(\eta(Y)\varphi Z + \eta(Z)\varphi Y).\]

Subtracting (8.6.13) and (8.6.14), we find

\[2(\tilde{\nabla}_Z \phi)Y = -A_{\phi Y} Z + A_{\phi Z} Y - \nabla^Z_Y \varphi Z + \nabla^Z_Y \varphi Y + \eta(Y)\varphi Z - \eta(Z)\varphi Y - \eta(Y)Z - \eta(Y)\eta(Z)\xi - \beta(\eta(Y)\phi Z + \eta(Z)\phi Y).\]

Hence lemma is proved.

**Corollary 8.6.5.** Let $M$ be a $\xi$-horizontal CR-submanifold, then

\[2(\tilde{\nabla}_Y \varphi)Z = A_{\varphi Y} Z - A_{\varphi Z} Y + \nabla^Z_Y \varphi Z - \nabla^Z_Y \varphi Y - \varphi[Y, Z] + 2\alpha g(Y, Z)\xi - 2g(Y, Z)\xi,\]

\[2(\tilde{\nabla}_Z \phi)Y = -A_{\phi Y} Z + A_{\phi Z} Y - \nabla^Z_Y \varphi Z + \nabla^Z_Y \varphi Y - \phi[Y, Z] + (\alpha - 1)g(Y, Z)\xi\]

for all $Y, Z \in D^\perp$.

**Lemma 8.6.6.** Let $M$ be a CR Submanifold of nearly trans-hyperbolic Sasakian manifold with a quarter-symmertric semi-metric connection, then

\[2(\tilde{\nabla}_X \varphi)Y = -A_{\varphi X} Y + \nabla^Z_X \varphi Y - \nabla^Z_Y \varphi X - h(Y, \varphi X) - \varphi[X, Y] - \eta(X)Y + \alpha(-\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta(\eta(X)\phi Y + \eta(Y)\phi X) - 2\eta(X)\eta(Y)\xi\]

and
\[ 2(\tilde{\nabla}_Y \phi)X = A_{\omega Y} X - \nabla^\perp_X \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y] + \alpha(-\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi + \eta(X)Y) \]

for any \( X \in D \) and \( Y \in D^\perp \).

**Proof.** Let \( X \in D \) and \( Y \in D^\perp \). Then from (8.5.2) and (8.5.3), we get

\[ \tilde{\nabla}_X \phi Y = -A_{\omega Y} X + \eta(X)Y + \nabla^\perp_X \phi Y - g(X, Y)\xi \]

and

\[ \tilde{\nabla}_Y \phi X = \nabla_Y \phi X + h(Y, \phi X). \]

Subtracting above two equations, we have

(8.6.11) \[ \tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = -A_{\omega Y} X - \eta(X)Y - \nabla^\perp_X \phi Y + \nabla^\perp_Y \phi Y - h(Y, \phi X). \]

Also, by direct covariant differentiation, we have

(8.6.12) \[ \tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X = (\tilde{\nabla}_X \phi)Y - (\tilde{\nabla}_Y \phi)X + \phi[X, Y]. \]

From (8.6.11) and (8.6.12), we get

\[ (\tilde{\nabla}_X \phi)Y - (\tilde{\nabla}_Y \phi)X = -A_{\omega Y} X - \eta(X)Y - \nabla^\perp_Y \phi Y - h(Y, \phi X) \]

(8.6.13) \[ - \phi[X, Y] - g(X, Y)\xi. \]

Adding (8.6.12) and (8.5.2), we get

\[ 2(\tilde{\nabla}_X \phi)Y = -A_{\omega Y} X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] - \eta(X)Y \]

\[ + \alpha(-\eta(Y)\phi X - \eta(X)\phi Y - \beta(\eta(X)\phi Y + \eta(Y)\phi X) \]

\[ - 2\eta(X)\eta(Y)\xi. \]

Subtracting (8.6.12) from (8.5.2), we obtain

\[ 2(\tilde{\nabla}_Y \phi)X = A_{\omega Y} X - \nabla^\perp_X \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y] \]

\[ + \alpha(-\eta(X)\phi Y - \eta(Y)\phi X) - \beta(\eta(Y)\phi Y + \eta(Y)\phi X) \]

\[ - 2\eta(X)\eta(Y)\xi + \eta(X)Y. \]

**Corollary 8.6.7.** Let \( M \) be a \( \xi \)-horizontal \( CR \)-submanifold of an nearly trans hyperbolic Sasakian manifold with a quarter-symmetric semi-metric connection. Then

\[ 2(\tilde{\nabla}_X \phi)Y = -A_{\omega Y} X + \nabla^\perp_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] - \eta(X)Y - \alpha\eta(X)\phi Y \]

\[ - \beta\eta(X)\phi Y, \]

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\[2(\tilde{\nabla}_Y \varphi)X = A_{\varphi X} X - \nabla^\perp_X \varphi Y + \nabla_Y \varphi X + h(Y, \varphi X) + \varphi[X, Y] - \alpha \eta(X) \varphi Y - \beta \eta(X) \varphi Y + \eta(X) Y\]

for any \(X \in D\) and \(Y \in D^+\).

**Corollary 8.6.8.** Let \(M\) be a \(\xi\)-vertical CR-submanifold of nearly trans-hyperbolic Sasakian manifold with a quarter-symmetric semi-metric connection. Then

\[2(\tilde{\nabla}_X \varphi)Y = -A_{\varphi Y} X + \nabla^\perp_X \varphi Y - \nabla_Y \varphi X - h(Y, \varphi X) - \alpha \eta(X) \varphi Y - \beta \eta(X) \varphi Y,\]

\[2(\tilde{\nabla}_Y \varphi)X = A_{\varphi X} X - \nabla^\perp_X \varphi Y + \nabla_Y \varphi X + h(Y, \varphi X) + \varphi[X, Y] - \alpha \eta(X) \varphi Y - \beta \eta(X) \varphi Y\]

for any \(X \in D\) and \(Y \in D^+\).

**8.7 Parallel Distributions with Respect to Quarter Symmetric Semi-metric Connection**

**Proposition 8.7.1.** Let \(M\) be a \(\xi\)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \(\tilde{M}\) with a quarter-symmetric semi-metric connection. If the horizontal distribution \(D\) is parallel, then

\[h(X, \varphi Y) = h(Y, \varphi X)\]

for all \(X, Y \in D\).

**Proof.** Let \(D\) be parallel distribution, then

\[(8.7.1) \quad \nabla_X \varphi Y \in D, \quad \nabla_Y \varphi X \in D\] for any \(X, Y \in D\).

From (8.6.2), we get

\[(8.7.2) \quad 2Bh(X, Y) = -2\alpha g(X, Y)\xi + 2g(X, Y)\xi.\]

From (8.2.9), we have

\[(8.7.3) \quad \varphi h(X, Y) = Bh(X, Y) + Ch(X, Y).\]
From (8.7.2) and (8.7.3), we get

(8.7.4) \[ \phi h(X, Y) = -2\alpha g(X, Y)\xi + 2g(X, Y)\xi + Ch(X, Y). \]

Now, by virtue of (8.6.3) and (8.7.3), we get

(8.7.5) \[ h(X, \phi Y) + h(Y, \phi X) = 2Ch(X, Y) = 2\phi h(X, Y) + 2\alpha g(X, Y)\xi - 2g(X, Y)\xi. \]

Replacing X by \( \phi X \) in (8.7.3), we have

(8.7.6) \[ h(\phi X, \phi Y) + h(Y, X) = 2\phi h(X, Y) + 2\alpha g(\phi X, Y)\xi - 2g(\phi X, Y)\xi. \]

Similarly, putting Y by \( \phi Y \) in (8.7.3), we have

(8.7.7) \[ h(\phi Y, \phi X) + h(X, Y) = 2\phi h(X, \phi Y) + 2\alpha g(X, \phi Y)\xi - 2g(X, \phi Y)\xi. \]

From (8.7.4) and (8.7.7), we obtain

(8.7.8) \[ 2\phi h(\phi X, Y) + 2\alpha g(\phi X, Y)\xi - 2g(Y, X)\xi = 2\phi h(X, \phi Y) + 2\alpha g(X, \phi Y)\xi - 2g(X, \phi Y)\xi. \]

Operating \( \phi \) on both sides, we get

(8.7.9) \[ h(X, \phi Y) = \phi^2 h(\phi X, Y). \]

Consequently, we have

\[ h(X, \phi Y) = h(Y, \phi X) \quad \text{for each } X, Y \in D. \]

**Proposition 8.7.2.** Let \( M \) be a \( \xi \)-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold \( \bar{M} \) with a quarter-symmetric semi-metric connection. If the distribution \( D^\perp \) is parallel with respect to the connection on \( M \), then

(8.7.10) \[ (A_{\psi Y}^Z + A_{\psi Z}^Y) \in D^\perp \quad \text{for any } Y, Z \in D^\perp. \]

**Proof.** Let \( Y, Z \in D^\perp \). By virtue of (8.5.3) and (8.5.5) we get

\[ \tilde{\nabla}_Y \phi Z = -A_{\psi Y}^Z + \nabla_Y^\perp \phi Z - g(Y, Z)\xi - \eta(Y)Z \]

or

\[ (\tilde{\nabla}_Y \phi) Z + \phi (\tilde{\nabla}_Y Z) = -A_{\psi Z}^Y + \nabla_Y^\perp \phi Z - g(Y, Z)\xi - \eta(Y)Z - \phi h(Y, Z). \]

Interchanging \( Y \) and \( Z \), we have
\[(\tilde{\nabla}_z \varphi)Y + \varphi(\tilde{\nabla}_z Y) = -A_{\varphi Z} Y + \nabla_z^\perp \varphi Y - g(Y, Z) \xi - \eta(Z) Y - \varphi h(Y, Z).\]

Adding above two equations, we find
\[
(\tilde{\nabla}_z \varphi)Z + (\tilde{\nabla}_z \varphi)Y = -A_{\varphi Z} Y - A_{\varphi Z} Z + \nabla_z^\perp \phi Z + \nabla_z^\perp \phi Y - 2g(Y, Z) \xi - \eta(Y) Z - \eta(Z) Y - 2\varphi h(Y, Z) - \phi(\nabla_z Y) - \phi(\nabla_z Z).
\]

Using (8.5.3), in above equation, we have
\[
\alpha(2g(X, Y)\xi - \eta(Y) \varphi X - \eta(X) \varphi Y) - \beta(\eta(Y) \varphi X + \eta(X) \varphi Y) - 2\eta(X) \eta(Y) \xi = -A_{\varphi Z} Y - A_{\varphi Z} Z + \nabla_z^\perp \phi Z + \nabla_z^\perp \phi Y - \eta(Y) Z - \eta(Z) Y - \phi(\nabla_z Y) - \phi(\nabla_z Z) - 2\varphi h(Y, Z) - 2g(Y, Z) \xi.
\]

Taking inner product with \(X \in D\) in (8.6.11), we get
\[
g(Z, X) + g(A_{\varphi Z} Y, X) = g(\varphi(\nabla_Z Y), X) + g(\varphi(\nabla_Z Y), X).
\]

If \(D^\perp\) is parallel then \(\nabla_Y Z \in D^\perp\) and \(\nabla_z Y \in D^\perp\). Hence we have
\[
g(A_{\varphi Y} Z, X) + g(A_{\varphi Z} Y, X) = 0
\]
or
\[
g(A_{\varphi Y} Z + A_{\varphi Z} Y, X) = 0
\]
which implies that \((A_{\varphi Y} Z + A_{\varphi Z} Y) \in D^\perp\).

**Proposition 8.7.3.** Let \(M\) be a mixed totally geodesic \(\xi\)-vertical \(CR\)-submanifold of a nearly trans-hyperbolic Sasakian manifold \(\tilde{M}\) with a quarter-symmetric semi-metric connection. Then the normal section \(N \in \phi D^\perp\) is \(D\)-parallel if and only if \(\nabla_X \phi N \in D\) for all \(X \in D\).

**Proof.** Let \(N \in \phi D^\perp\). Then from (8.6.2), we get
\[
(8.7.12)
\]
\[
Q \nabla_X (\varphi PY) + Q \nabla_Y (\varphi PX) - QA_{\varphi Z} Y - QA_{\varphi Z} X - \eta(X) Q(Y) - \eta(Y) Q(X) = 2Bh(X, Y) + 2\alpha g(X, Y) Q \xi - 2\eta(X) \eta(Y) Q \xi - 2g(X, Y) Q \xi + \varphi Q(\nabla_X Y) + \varphi Q(\nabla_Y X).
\]

Also, we have
\[
(8.7.13)
\]
\[
g(X, Y) = 0, \quad \varphi PY = 0, \quad \varphi QX = 0
\]
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From (8.7.13) and (8.7.12), we get

(8.7.14)
\[ Q(\nabla, \phi X) = 0. \]

In particular,

(8.7.15)
\[ (\nabla, X) = 0. \]

Using (8.7.15) in (8.6.3), we have

\[ \nabla^\perp_X (\phi Y) = \phi Q(\nabla_X Y) + \phi Q\nabla_X Y. \]

Using (8.7.14), we find

\[ \nabla^\perp_X (\phi Y) = \phi Q(\nabla_X Y). \]

or

\[ \nabla^\perp_X N = \phi Q\nabla_X (\phi N). \]

If \( N \neq 0 \) is D-parallel then by definition (8.4.6) and above equation, we get

\[ \phi Q\nabla_X (\phi N) = 0 \]

which is equivalent to

\[ \nabla_X \phi N = 0 \]

for all \( X \in D \).

Hence the proposition is proved.
References


