CHAPTER-3: ONE-DIMENSIONAL GROUND WATER RECHARGE THROUGH POROUS MEDIA

3.1 : Introduction

The unsteady and unsaturated flow of water through soils is due to content changes as a function of time and the entire pore spaces are not completely filled with flowing liquid respectively. Knowledge concerning such flow helps some workers like hydrologist, agriculturalists, many fields of science and engineering. The water infiltration system and the underground disposal of seepage and waste water are encountered by these flows, which are described by nonlinear partial differential equation.

The mathematical model conforms to the hydrological situation of one dimensional vertical ground water recharge by spreading [1]. Such flow is of great importance in water resource science, soil engineering and agricultural sciences.

In the present chapter, we have obtained a numerical solution of the problem by S.O.R. method.

This chapter has been divided in two parts:

(A) One-dimensional flow through unsaturated porous media

(B) Miscible fluid flow through porous media

Part (A) One-dimensional flow through unsaturated porous media

3.2: Statement of The Problem (A)

In the investigated mathematical model, we consider that the groundwater recharge takes place over a large basin of such geological configuration that the sides are limited by rigid boundaries and the bottom by a thick layer of water table. In this case, the flow is assumed vertically downwards through unsaturated porous media.
It is assumed that the diffusivity co-efficient is equivalent to its average value over the whole range of moisture content, and the permeability of the media is continuous linear function of the moisture content. The theoretical formulation of the problem yields a nonlinear partial differential equation.

3.3 : Mathematical Formulation of The Problem (A)

From Klute [89], the equation of continuity for an unsaturated medium is given by

\[
\frac{\partial}{\partial t} (\rho_s \theta) = -\nabla \cdot \mathbf{M} \quad \text{------------------------ (3.3.1)}
\]

Where \( \rho_s \) the bulk density of the medium is, \( \theta \) is its moisture content on a dry weight basis, and \( \mathbf{M} \) is the mass flux of moisture.

From Darcy’s law [1, 2, 3] for the motion of water in a porous medium, we get,

\[
\mathbf{V} = -k \nabla \varphi \quad \text{------------------------ (3.3.2)}
\]

Where \( \nabla \varphi \) represents the gradient of the moisture potential, the volume flux of moisture, and \( k \) the co-efficient of aqueous conductivity.

Combining equations (3.3.1) and (3.3.2) we obtain,

\[
\frac{\partial}{\partial t} (\rho_s \theta) = \nabla \cdot (\rho k \nabla \varphi) \quad \text{------------------------ (3.3.3)}
\]

Where \( \mathbf{M} = \rho \mathbf{V} \). \( \rho \) is the flux density.

Since in the present case, we consider that the flow takes place only in the vertical direction, equation (3.3.3) reduces to,

\[
\rho_s \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( \rho k \frac{\partial \psi}{\partial z} \right) - \frac{\partial}{\partial z} (\rho g) \quad \text{------------------------ (3.3.4)}
\]

Where \( \psi \) the capillary pressure potential, \( g \) is the gravitational constant and \( \varphi = \psi - gz \) [5-9] The positive direction of the \( z \)-axis is the same as that of the gravity.
Considering $\theta$ and $\psi$ to be connected by a single valued function, we may write (3.3.4) as,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D \frac{\partial \theta}{\partial z} \right) - \frac{\rho}{\rho_s} k \frac{\partial k}{\partial z} \quad \text{----------------------- (3.3.5)}$$

Where $D = \frac{\rho}{\rho_s} k \frac{\partial \psi}{\partial \theta}$ and is called diffusivity co-efficient.

Replacing $D$ by average value $D_a$ and assuming $k = k_0 \theta$, we have

$$\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \frac{\rho}{\rho_s} k_0 \frac{\partial \theta}{\partial z} \quad \text{----------------------- (3.3.6)}$$

Considering the water table to be situated at a depth $L$, and putting:

$$\frac{z}{L} = \xi, \quad \frac{tD_a}{L^2} = T, \quad \beta = \frac{\rho}{\rho_s} \frac{k_0}{D_a}$$

We may write the boundary value problem as:

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial \xi^2} - \beta \frac{\partial \theta}{\partial \xi} \quad \text{----------------------- (3.3.7)}$$

$\xi$ = Penetration depth (dimensionless)

$T$ = Time (dimensionless)

$\beta$ = Flow parameter (cm$^2$)

$\rho$ = Mass density of water (gm)

$\rho_s$ = Bulk density of the medium on dry weight basis (g/cm$^3$)

$k_0$ = Slope of the permeability vs moisture content plot (cm/sec$^{-1}$)

$D_a$ = Average value of the diffusivity co-efficient over the whole range of moisture content (cm$^2$ sec$^{-1}$)
It may be mentioned for definiteness that a set of appropriate boundary conditions are
\[ \theta(0, T) = \theta_0, \quad \frac{\partial \theta}{\partial \xi}(1, T) = 0 \quad (3.3.8) \]
\[ \theta(\xi, 0) = 0 \quad (3.3.9) \]

Where the moisture content throughout the region is zero initially, at the layer \( z = 0 \) it is \( \theta_0 \), and at the water table (\( z = L \)) it is assumed to remain 100% throughout the process of investigation. It may be remarked that the effect of capillary action at the stationary groundwater level, being small, is neglected.

The following values of the various parameters have been considered in the analysis:

Let \( \beta = 3.5, \quad \theta_0 = 0.5, \quad h = \frac{1}{10} \quad \text{and} \quad k = 0.1 \)

The numerical values are shown by the table. Curves indicating the behavior of the moisture content corresponding to various time period.

### 3.4 : Mathematical Solution of The Problem (A)

Using Crank- Nikolson method [64], we have,

\[ s_{w_{i+1,j}} = s_{w_{i,j}} \]

\[ + \frac{k}{2h^2} \left( s_{w_{i+1,j} - 2s_{w_{i,j}} + s_{w_{i-1,j}} + s_{w_{i+1,j+1}}} \right) \]

\[ - 2s_{w_{i+1,j}} + s_{w_{i+1,j+1}} \]

\[ - \frac{\beta k}{2h} \left( s_{w_{i+1,j}} - s_{w_{i,j}} + s_{w_{i+1,j+1}} - s_{w_{i,j+1}} \right) \]

Let \( r = \frac{k}{h^2} \) and

\[ c_i = s_{w_{i,j}} + \frac{r}{2} \left( s_{w_{i+1,j}} - 2s_{w_{i,j}} + s_{w_{i-1,j}} \right) - \frac{\beta rh}{2} \left( s_{w_{i+1,j}} - s_{w_{i,j}} \right) \]
Using Successive Over-Relaxation Method, we have

\[ s_{w_{i+1}} = (1 - \omega)s_{w_i} + \omega \left[ \frac{r}{2\left(1 + r - \frac{\beta rh}{2}\right)} \left( s_{w_{i+1},j} - s_{w_{i-1},j+1} \right) + \frac{c_i}{1 + r - \frac{\beta rh}{2}} \right] \]

Choose \( k = 0.1, \ h = 0.1, \ \beta = 3.5, \ \omega = 1.4, \)

\[ s_{w_{i+1}} = -0.4s_{w_i} + 1.4 \left[ 0.35135s_{w_{i+1},j} + 0.54054s_{w_{i-1},j+1} + \frac{c_i}{9.25} \right] \]

Where \( c_i = -7.25s_{w_i} + 3.25s_{w_{i+1},j} + 5s_{w_{i-1},j} \)

Numerical calculation at different values of \( T \) and \( \xi \) are shown in the following table.
<table>
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<th>ξ</th>
<th>T=0.1</th>
<th>T=0.2</th>
<th>T=0.3</th>
<th>T=0.4</th>
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<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
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<tr>
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<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

**Graphical Representation:**

![Graph](image_url)

**Figure 3 (a) : Penetration depth (ξ) → moisture (θ)**
Figure 3 (b) : Time (T) → moisture(θ)

3.5 : Interpretation

The solution is obtained with $\beta = 3.5$, $\theta_0 = 0.5$, $h= 0.1$, $k=0.1$. We consider that the sides of basin are limited by rigid boundaries and bottom by a thick layer of water table so that water flows only in positive direction.

From figure 3 (a), it is clear that $\theta = \theta_0 = 0.5$ at layer $x=0$ and at water table ($x=1$), it is assumed to remain 100% throughout the process of investigation. Initially it is 0.5 and then increases to 1. Keeping $T$ constant, for different values of $X$, moisture content $\theta$ is increasing.

From figure 3 (b), it is clear that initially $\theta = 0$ throughout the region. As time ($T$) increases, the moisture content $\theta$ also increases at each point of the basin and after some time, it become constant.

3.6: Scope of The Problem

Engineers in several fields have to learn the mechanism of drainage and to apply to problems of water supply, land reclamation and stabilization of foundations and sub grade, and also to the fields of petroleum production and agriculture.
Drainage in general is any provision for the removal of excess water. The common objective of land projects to prevent or eliminate either water logging or inundation or otherwise productive land. Drainage of projected land refers principally to the disposal of surplus natural water adversely affecting irrigation. Practically every area where irrigation has been carried on for time has been affected by high water table. Therefore provision for adequate drainage is an essential part of planning, construction and operation of an irrigation project.

For agriculture purpose, the continued presence of water in excess of that needed for vegetation is harmful. Prolonged saturation of soil excludes air essentially for healthy plant growth and the soil becomes cold, sour and unproductive. Consequently unsaturated or irrigated soils is a necessary evil, so to this type of drainage where originally saturation conditions are existing up to the top.

**Part (B) : Multifluid Miscible Fluid Flow Through Porous Media**

**3.7 : Introduction**

In this part, the phenomenon of longitudinal dispersion in the flow of two miscible fluids through porous media is discussed. The phenomenon of longitudinal dispersion is the process by which miscible fluids in the laminar flow mix in the direction of the flow has been discussed. The hydrodynamic dispersion is the macroscopic outcome of the actual movements of individual tracer particles throughout the pores and various physical and chemical phenomenon that take place within the pores. This phenomenon simultaneously occurs due to molecular diffusion and convection.

This phenomenon plays an important role in the sea water intrusion into reservoir at river mouths and in the underground recharge of waste water. Immiscible flooding, that is, the oil is displaced by one of the LPG (liquid petroleum gas) products, Ethane, Propen or Butane. If the reservoir conditions are such that the LPG is in the liquid phase then it is miscible with the oil and theoretically all residual oil can be recovered.

This problem has been discussed by several authors from different viewpoints, namely Scheidegger [74], Greenkorn [75], Schwartz [76] etc.
The governing equation of this case is partial differential equation. It is converted in Ordinary differential equation and then its series solution is obtained by Power series method.

### 3.8 : Statement of the Problem (B)

Many Important problems in water resources engineering involve the mass-transport of a miscible fluid in a flow.

A fluid is considered to be a continuous material and hence in addition to the velocity of a fluid element, the molecules in this element have random motion. As a result of the random motion, molecules of a certain material in high concentration at one point will spread with time. So the velocity considered here is time dependent. The net molecular motion from a point of higher concentration to one of lower concentration is called molecular diffusion.

Fluid flows in nature are usually turbulent, but we have considered the porous medium through which the fluid flows, to be homogeneous and for this reason, in the direction of flow, we assume laminar flow in which miscible fluids mix.

In moving through the random passages of the medium, two fluid elements adjacent to each other at one time will separate, as they may take different routes. The geometrical dispersion is coupled with molecular diffusion and dispersion due to no uniformity of the velocity across the cross-section of the passages. By considering the passage as randomly connected tubes, de Jong (1) and safman (2) have shown that the dispersion in an isotropic medium can be described with a coefficient $D$ for longitudinal dispersion in the direction of seepage velocity.

### 3.9: Mathematical Formulation of The Problem (B)

The equation of continuity for the mixture is given by

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
$$

(3.9.1)

Where $\rho$ is the density for mixture and $\vec{v}$ is the pore (seepage) velocity (vector).
The equation of diffusion for a fluid flow through homogeneous porous medium with no addition or subtraction of the dispersing material, is given by

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\vec{v}) = \nabla \cdot \left[ \rho \vec{D} \nabla \left( \frac{c}{\rho} \right) \right]$$

(3.9.2)

Where \( c \) is the concentration of the fluid A in the other host fluid B (i.e. \( c \) is the mass of A per unit volume of the mixture), \( \vec{D} \) is the tensor co-efficient of dispersion with nine components \( D_{ij} \).

In a laminar flow through homogeneous porous medium at constant temperature, \( \rho \) may be considered to be constant. The equation (3.9.1) gives

$$\nabla \cdot \vec{v} = 0$$

And equation (3.9.2) becomes

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\vec{v}) = \nabla \cdot \left[ \vec{D} \nabla c \right]$$

(3.9.3)

When the seepage velocity \( \vec{v} \) is along the X-axis, the non-zero components are

\( D_{11} = D_L \) and \( D_{22} = D_T \) (coefficient of transverse dispersion), and other \( D_{ij} \) are zero. Thus equation (3.9.3), in this case, becomes,

$$\frac{\partial c}{\partial t} + \vec{u} \frac{\partial c}{\partial x} = D_L \frac{\partial^2 c}{\partial x^2}$$

(3.9.4)

Where \( \vec{u} \) is the component of velocity along X-axis which is time dependent and \( D_L \) is the longitudinal dispersion co-efficient.

An appropriate initial and boundary conditions are:

\[ c(x, 0) = \gamma_0 , 0 \leq x \leq L , c(0, t) = \gamma_1 \text{ and } c(L, t) = \gamma_2 \text{ t > 0} \]

(3.9.5)

Where \( \gamma_0 \) is the initial concentration of the tracer

\( \gamma_1 \) is the concentration at \( x = 0 \)
\( \gamma_2 \) is the concentration at \( x = L \)

Since \( u \) is the cross-sectional timedependent flow velocity through porous medium, it is regarded as \(- \frac{1}{\sqrt{t}}\) (where negative sign shows the decreasing of velocity in the direction of flow) for definiteness. Again by setting dimensionless variables \( X = \frac{x}{L} \), the equation (3.9.4) together with (3.9.5) becomes

\[
\frac{\partial c}{\partial t} - \frac{1}{L\sqrt{t}} \frac{\partial c}{\partial X} = \frac{D_L}{L^2} \frac{\partial^2 c}{\partial X^2} \quad \text{------------------- (3.9.6)}
\]

And \( c(X, 0) = \gamma_0, 0 \leq X \leq \gamma_1 \) and \( c(0, t) = \gamma_2, t > 0 \)

\text{------------------- (3.9.7)}

Now we use \( \alpha \) similarity (Boltzmann) transformation \[13\] to convert equation (3.9.6) into an ordinary different equation. For that we consider

\[
\eta = X^\alpha t^\beta \quad \text{------------------- (3.9.8)}
\]

Where \( \alpha \) and \( \beta \) are to be determined so as the resulting equation in \( \eta \) will be free from \( X \) and \( t \). Substituting (3.9.8) in (3.9.6), we get

\[
\beta X^2 \frac{\eta}{t} c' - \frac{\alpha}{L\sqrt{t}} \eta X c' = \frac{D_L}{L^2} \left[ \alpha (\alpha - 1) \eta c' + \alpha^2 \eta^2 c'' \right] \quad \text{------------------- (3.9.9)}
\]

Now the right hand side of equation (3.9.9) is the free of \( X \) and \( t \), thus for the full equation to be a function only of \( \eta \). That is \( \frac{x^2}{t} = f(\eta) \) Which determines the values of \( \alpha \) and \( \beta \).

Here, we choose \( f(\eta) = \eta^2 \); so that \( \alpha = 1 \) and \( \beta = -\frac{1}{2} \). The transformation (3.3.8) is called the Boltzmann transformation, and under it equation (3.9.6) becomes

\[
D_L \frac{d^2 c}{d\eta^2} + \left[ \frac{L^2}{2} \eta + L \right] \frac{dc}{d\eta} = 0, \quad D_L \neq 0 \quad \text{------------------- (3.9.10)}
\]

Under the transformation (3.9.8), the first boundary condition becomes (since \( t > 0 \))

\[
c(0) = \gamma_1 \quad \text{------------------- (3.9.11)}
\]
And the initial and second auxiliary conditions are consolidate only if

\[ Y_0 = Y_2 \]

and hence we have

\[ c(1) = \gamma_2, \ c(L) = \gamma_2 \quad \text{--------- (3.9.12)} \]

### 3.10 : Mathematical Solution of Problem (B)

Using series solution for (3.9.10)

Let \[ c = \sum_{0}^{\infty} a_k \eta^k \], \( a_0 \neq 0 \quad \text{------------------ (3.10.1)} \)

Substituting (3.10.1) in (3.9.10), we get

\[ \sum_{2}^{\infty} D_L k(k - 1)a_k \eta^{k-2} + \sum_{1}^{\infty} \left( \frac{L^2 \eta}{2} + L \right) k a_k \eta^{k-1} = 0 \]

\[ \sum_{2}^{\infty} D_L k(k - 1)a_k \eta^{k-2} + \sum_{1}^{\infty} \frac{L^2}{2} k a_k \eta^{k} + \sum_{1}^{\infty} L k a_k \eta^{k-1} = 0 \]

\[ \sum_{2}^{\infty} D_L (k - 1)a_k \eta^{k-2} + \sum_{3}^{\infty} \frac{L^2}{2} (k - 2)a_{k-2} \eta^{k-2} + \sum_{2}^{\infty} L (k - 1)a_{k-1} \eta^{k-2} = 0 \]

\[ 2a_2 D_L + L a_1 + \sum_{3}^{\infty} \left[ D_L k(k - 1)a_k + \frac{L^2}{2} (k - 2)a_{k-2} + L(k - 1)a_{k-1} \right] \eta^{k-2} = 0 \]

Comparing co-efficients of both sides, we get,

\[ 2a_2 D_L + L a_1 = 0 \Rightarrow a_2 = -\frac{L a_1}{2 D_L} \quad \text{------------------ (3.10.2)} \]

\[ a_k = -\frac{L}{D_L k} a_{k-1} - \frac{L^2}{2 D_L k(k-1)} a_{k-2}, \ k \geq 3 \quad \text{--------- (3.10.3)} \]
\[ k = 3, a_3 = \frac{L^2}{6D_L} \left[ \frac{1}{D_L} - \frac{1}{2} \right] a_1 \] .................................................. (3.10.4)

\[ k = 4, a_4 = -\frac{L^3}{24D_L} \left[ \frac{1}{D_L} - \frac{3}{2} \right] a_1 \] .................................................. (3.10.5)

Substituting (3.10.2), (3.10.4) and (3.10.5) in (3.10.1) we get

\[
c = a_0 + a_1 \left[ \eta - \frac{L}{2D_L} \eta^2 + \frac{L^2}{6D_L} \left[ \frac{1}{D_L} - \frac{1}{2} \right] \eta^3 - \frac{L^3}{24D_L} \left[ \frac{1}{D_L} - \frac{3}{2} \right] \eta^4 + \ldots \ldots \right].
\]

.................................................. (3.10.6)

\[ c(0) = \gamma_1 \Rightarrow a_0 = \gamma_1, \quad c(1) = \gamma_2 \Rightarrow a_1 = \frac{\gamma_2 - \gamma_1}{\eta} \]

Where \[ \eta = \left[ 1 - \frac{L}{2D_L} + \frac{L^2}{6D_L} \left[ \frac{1}{D_L} - \frac{1}{2} \right] - \frac{L^3}{24D_L} \left[ \frac{1}{D_L} - \frac{3}{2} \right] + \ldots \ldots \right] \neq 0 \]

From (3.10.6), we get

\[
c = \gamma_1 + \gamma_2 - \gamma_1 \left[ \frac{x}{L\sqrt{t}} - \frac{L}{2D_L} \frac{x^2}{L^2t} + \frac{L^2}{6D_L} \left[ \frac{1}{D_L} - \frac{1}{2} \right] \frac{x^3}{L^3t^{3/2}} - \frac{L^3}{24D_L} \left[ \frac{1}{D_L} - \frac{3}{2} \right] \frac{x^4}{L^4t^2} \ldots \right] \]

\[ \therefore \eta = \frac{x}{L\sqrt{t}} \]

\[
c = \gamma_1 + \frac{\gamma_2 - \gamma_1}{y} \left[ \frac{x}{\sqrt{t}} - \frac{x^2}{2D_Lt} + \left[ \frac{1}{D_L} - \frac{1}{2} \left[ \frac{x^3}{6D_Lt^{3/2}} - \frac{1}{D_L} - \frac{3}{2} \right] \frac{x^4}{24D_Lt^2} + \ldots \ldots \right]. \]

.................................................. (3.10.7)

### 3.11: Concluding Remark

Equation (3.10.7) is a series solution of the problem (3.9.4) together with the conditions (3.9.5).

As time (t) tends to \( \infty \), concentration (c) tends to \( \gamma_1 \)
i.e. it reach to concentration at the boundary $x = 0$.

As $x$ tends to 0, $c$ tends to $\gamma_1$

As $x$ tends to $L$, $c$ tends to $\gamma_2 (= \gamma_0)$ . (choose $t=1$)

3.12 : Scope of The Problem

When sewage is discharged into a body of water, it is important to certain how the sewage is dispersed in the receiving water. To determine the rate of evaporation from the surface of a reservoir, it is necessary to know the rate at which water vapor near this surface is carried into the air above.

This type of problem also has an importance in spreading of contaminant in a canal, evaporation of water vapor from water surface, contaminant of an aquifer.