In this chapter some standard definitions relating to Boolean like gamma near-rings and some important results like; If $\Gamma_N$ is a Boolean like $\Gamma$-near-ring then $a\gamma b = a\gamma b\gamma a$ for all $a, b \in \Gamma_N$, $\gamma \in \Gamma$; If $\Gamma_N$ is a Boolean like $\Gamma$-near-ring then $a\gamma b\gamma c = a\gamma c\gamma b$ for $a, b, c \in \Gamma_N$, $\gamma \in \Gamma$ are proved. Further an ordering is defined on $\Gamma_N$ by $a \leq b$ if $a = b\gamma a$ for all $\gamma \in \Gamma$ and for $\Gamma_N$ a Boolean like $\Gamma$-near-ring, some interesting results like; Let $a, b \in \Gamma_N$, If $a \leq b$ then $a = b\gamma a = a\gamma b$; Zero is the least element of $\Gamma_N$ and the Boolean like $\Gamma$-near-ring $\Gamma_N$ is zero-symmetric; Suppose $\Gamma_N$ has the greatest element then the Boolean like $\Gamma$-near-ring $\Gamma_N$ is a Boolean like $\Gamma$-near-ring with identity. If $\Gamma_N$ is a $u$-directed then $\Gamma_N$ is a commutative Boolean like $\Gamma$-near-ring. If $\Gamma_N$ is a $\Gamma$-meet semi lattice then $\Gamma_N$ is zero-symmetric, Let $a, b \in \Gamma_N$, $a\gamma b = a \land b \forall \gamma \in \Gamma$ iff $a\gamma b = b\gamma a \forall \gamma \in \Gamma$; $a \land b = a\gamma b \land b\gamma a$ for all $a, b \in \Gamma_N$ and $\forall \gamma \in \Gamma$. $(a \land b) \land (a\gamma b - b\gamma a) = 0$ for all $a, b \in \Gamma_N$ and $\forall \gamma \in \Gamma$; are proved.

**Definition 4.1** A $\Gamma$-near-ring $\Gamma_N$ is a system consisting of

(i) a group $(\Gamma_N, +)$ (not necessarily Abelian)

(ii) a non-empty set $\Gamma$
(iii) a mapping \((a, \alpha, b) \rightarrow a\alpha b\) of \(\Gamma_N \times \Gamma \times \Gamma_N \rightarrow \Gamma_N\)

satisfying the following conditions:

(a) \((a + b) \alpha c = a\alpha c + b\alpha c\) \(\forall a, b, c \in \Gamma_N\) and \(\alpha \in \Gamma\).

(b) \((a\alpha b) \beta c = a\alpha (b\beta c)\) \(\forall a, b, c \in \Gamma_N\) and \(\alpha, \beta \in \Gamma\).

**Note 4.2**

(i) The identity 0 in \((\Gamma_N, +)\) is called the zero element of \(\Gamma_N\).

(ii) Clearly \(0\gamma b = 0\) \(\forall \gamma \in \Gamma, b \in \Gamma_N\).

(iii) The inverse of \(a \in \Gamma_N\) is denoted by \(- a\).

(iv) \((- a) \gamma b = - a\gamma b\).

**Example 4.3** Let \((G, +)\) be a group. Let \(X\) be a nonempty set. Let \(\Gamma_N = \{f / f: X \rightarrow G\}\) and let \(\Gamma = \{g / g: G \rightarrow X\}\). Then \(\Gamma_N\) is a \(\Gamma\)-near-ring under the mapping \((f, g, h) \rightarrow fgh\) of \(\Gamma_N \times \Gamma \times \Gamma_N \rightarrow \Gamma_N\) where \(fgh\) is the composite of \(f, g, h\).

**Definition 4.4** An element \(d \in \Gamma_N\) is called a **distributive element** if for all \(n, n^1 \in \Gamma_N, \gamma \in \Gamma\), \(d\gamma (n + n^1) = d\gamma n + d\gamma n^1\).

**Definition 4.5** A \(\Gamma\)-Near ring \(\Gamma_N\) is said to be a **Zero-Symmetric** if \(n\gamma 0 = 0\) for all \(n \in \Gamma_N, \gamma \in \Gamma\).
**Definition 4.6** A $\Gamma$-Near ring $\Gamma_N$ is said to be **Weak Commutative** if for all $x, y, z \in \Gamma_N$, $\gamma \in \Gamma$, $x\gamma yz = x\gamma z\gamma y$.

**Definition 4.7** A $\Gamma$ Near-ring $\Gamma_N$ with unit element is called a **Boolean like $\Gamma$-near-ring** provided $a\gamma a^*b\gamma b^* = 0$, $a\gamma a = a$ and $a \Delta a = a$ for every $a, b \in \Gamma_N$, $\gamma \in \Gamma$.

**Lemma 4.8** If $\Gamma_N$ is a Boolean like $\Gamma$-near-ring then $a\gamma b = a\gamma b\gamma a$ for all $a, b \in \Gamma_N$, $\gamma \in \Gamma$.

**Proof**: Let $\Gamma_N$ is a Boolean like $\Gamma$-near-ring and $a, b \in \Gamma_N$, $\gamma \in \Gamma$.

Now

\[
(a\gamma b - a\gamma b\gamma a) \gamma = a\gamma b\gamma a - a\gamma b\gamma a
\]

\[
= a\gamma b\gamma a - a\gamma b\gamma a
\]

\[
= 0.
\]

Therefore

\[
(a\gamma b - a\gamma b\gamma a) \gamma = 0. \quad (1)
\]

and $a\gamma (a\gamma b - a\gamma b\gamma a)$

\[
= (a\gamma (a\gamma b - a\gamma b\gamma a)) \gamma (a\gamma (a\gamma b - a\gamma b\gamma a))
\]

\[
= a\gamma (((a\gamma b - a\gamma b\gamma a) \gamma) (a\gamma b - a\gamma b\gamma a))
\]

\[
= a\gamma 0\gamma (a\gamma b - a\gamma b\gamma a)
\]

\[\text{by (1)}\]

\[
= a\gamma 0.
\]
Also $(\alpha \gamma \beta - \alpha \gamma \beta \gamma) \gamma \alpha \beta \gamma$

$$= \alpha \gamma \beta \alpha \beta \gamma - \alpha \gamma \beta \alpha \beta \gamma \gamma \alpha \beta \gamma$$

$$= \alpha \gamma \beta \gamma - \alpha \gamma \beta \gamma$$

$$= 0.$$

As above it follows that

$$\alpha \gamma \beta \gamma (\alpha \gamma \beta - \alpha \gamma \beta \gamma) = \alpha \gamma \beta \gamma 0$$

$\text{Illy}$

$$\alpha \gamma \beta (\gamma \alpha \beta - \alpha \gamma \beta \gamma) = \alpha \gamma \beta \gamma$$

Now $(\alpha \gamma \beta - \alpha \gamma \beta \gamma) = (\alpha \gamma \beta - \alpha \gamma \beta \gamma) \gamma (\alpha \gamma \beta - \alpha \gamma \beta \gamma)$

$$= \alpha \gamma \beta (\alpha \gamma \beta - \alpha \gamma \beta \gamma) - \alpha \gamma \beta \gamma (\alpha \gamma \beta - \alpha \gamma \beta \gamma)$$

$$= \alpha \gamma \beta 0 - \alpha \gamma \beta \gamma 0$$

$$= (\alpha \gamma \beta - \alpha \gamma \beta \gamma) \gamma 0$$

$$= (\alpha \gamma \beta - \alpha \gamma \beta \gamma) \gamma (\alpha \gamma \beta - \alpha \gamma \beta \gamma) \gamma a$$

$$= (\alpha \gamma \beta - \alpha \gamma \beta \gamma) \gamma a$$

$$= 0.$$

Therefore $(\alpha \gamma \beta - \alpha \gamma \beta \gamma) = 0.$

Hence $\alpha \gamma \beta = \alpha \gamma \beta \gamma$ for all $a, b \in \Gamma _N, \gamma \in \Gamma$. 

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Theorem 4.9 If $\Gamma_N$ is a Boolean like $\Gamma$-near-ring then

$$a\gamma b\gamma c = a\gamma c\gamma b$$ for $a, b, c \in \Gamma_N, \gamma \in \Gamma$ i.e. $\Gamma_N$ is weak commutative.

Proof: Let $\Gamma_N$ be a Boolean like $\Gamma$-near-ring and $a, b, c \in \Gamma_N, \gamma \in \Gamma$

Consider $a\gamma b\gamma c - a\gamma c\gamma b = a\gamma b\gamma c - a\gamma c\gamma b$ \hspace{1cm} by 4.8

$$= (a - a\gamma c) \gamma b\gamma c$$

$$= (a - a\gamma c) \gamma b\gamma (a - a\gamma c) \gamma c$$ \hspace{1cm} by 4.8

$$= (a - a\gamma c) \gamma b\gamma 0$$ \hspace{1cm} by 4.7

$$= a\gamma b\gamma 0 - a\gamma c\gamma b\gamma 0.$$ \hspace{1cm} (1)

Replacing $b$ by $b\gamma c$ in (1) and by 4.8

$$a\gamma b\gamma c - a\gamma c\gamma b = a\gamma b\gamma c\gamma 0 - a\gamma c\gamma b\gamma 0.$$ \hspace{1cm} (2)

From (1) & (2),

$$a\gamma b\gamma 0 = a\gamma b\gamma c\gamma 0$$ for all $a, b, c \in \Gamma_N, \gamma \in \Gamma.$ \hspace{1cm} (3)

Substituting $b = a$ in (3) we have

$$a\gamma a\gamma 0 = a\gamma a\gamma c\gamma 0$$

$$\Rightarrow a\gamma 0 = a\gamma c\gamma 0$$ for all $a, c \in \Gamma_N, \gamma \in \Gamma$ \hspace{1cm} (4)
By (1) & (4) we get

\[ a\gamma b c - a\gamma c b = a\gamma b 0 - a\gamma c b 0 \]

\[ = a\gamma 0 - a\gamma 0 \]

\[ = 0. \]

Therefore \( a\gamma b c = a\gamma c b. \)

Hence \( \Gamma_N \) is Weak Commutative.

**Lemma 4.10** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring. If \( d \) is a distributive element in \( \Gamma_N \), then \( d + d = 0 \) and hence \( d = -d. \)

**Proof:** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring and \( d \) is a distributive element.

Consider \( d + d \)

\[ d + d = (d + d) \gamma (d + d) \]

\[ = d\gamma (d + d) + d\gamma (d + d) \]

\[ = d\gamma d + d\gamma d + d\gamma d + d\gamma d \]

\[ = d + d + d + d. \quad \text{by } 4.7 \]

Therefore \( d + d = 0 \Rightarrow d = -d. \)

**Definition 4.11** A \( \Gamma \)-near-ring \( \Gamma_N \) is called a **distributive \( \Gamma \)-near-ring** if every element of \( \Gamma_N \) is a distributive element.
**Definition 4.12** An element \( e \in \Gamma_N \) is called a **left identity** if 
\[ e \gamma a = a, \ \forall a \in \Gamma_N. \]

**ORDERING ON BOOLEAN LIKE \( \Gamma \)-NEAR-RINGS**

**Definition 4.13** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring. Define a **relation** “\( \leq \)’’ on \( \Gamma_N \) by \( a \leq b \) if \( a = b \gamma a \) for all \( \gamma \in \Gamma \).

**Proposition 4.14** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring and \( a, b \in \Gamma_N \). If \( a \leq b \) then \( a = b \gamma a = a \gamma b \).

**Proof:** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring and \( a, b \in \Gamma_N \). We know by 4.13 \( a \leq b \Rightarrow a = b \gamma a \)

Now \[ a \gamma b = b \gamma a \gamma b \quad \text{(since } a = b \gamma a \text{)} \]

\[ = b \gamma b \gamma a \quad \text{by } 4.9 \]

\[ = b \gamma a. \quad \text{by } 4.7 \]

Therefore \( a \gamma b = b \gamma a \).

**Corollary 4.15** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring with 0 as the least element of \( \Gamma_N \). Then the Boolean like \( \Gamma \)-near-ring \( \Gamma_N \) is zero-symmetric.

**Proof:** As \( \Gamma_N \) a Boolean like \( \Gamma \)-near-ring and since 0 is the least element of \( \Gamma_N \).
We have \( 0 \leq a \) for all \( a \in \Gamma_N \)

by 4.14 \( a\gamma 0 = 0\gamma a = 0 \)

Therefore \( \Gamma_N \) is a zero-symmetric Boolean like \( \Gamma \)-near-ring.

**Theorem 4.16** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring. Suppose \( \Gamma_N \) has the greatest element then the Boolean like \( \Gamma \)-near-ring \( \Gamma_N \) is a Boolean like \( \Gamma \)-near-ring with identity.

**Proof:** Suppose \( \Gamma_N \) has the greatest element say 1

then \( a \leq 1 \) for all \( a \in \Gamma_N \)

\[ \Rightarrow a = l\gamma a \]

and therefore \( a\gamma 1 = 1\gamma a = a \) for all \( a \in \Gamma_N \).

Thus 1 is the identity element of \( \Gamma_N \).

Thus \( \Gamma_N \) is a Boolean like \( \Gamma \)-near-ring with identity.

**Lemma 4.17** Let \( \Gamma_N \) be a Boolean like \( \Gamma \)-near-ring. If \( \Gamma_N \) is u-directed then \( \Gamma_N \) is a commutative Boolean like \( \Gamma \)-near-ring.

**Proof:** Let \( \Gamma_N \) be u-directed.

Let \( a, b \in \Gamma_N \)

Since \( \Gamma_N \) is u-directed, \( \exists c \in \Gamma_N \) such that \( c = a \lor b \)
Now $a \leq c$ and $b \leq c$

$\Rightarrow a = c\gamma$ and $b = c\gamma$ \quad \forall \gamma \in \Gamma.

Now $a\gamma b = c\gamma a\gamma b$

$= c\gamma b\gamma a$ \qquad \text{by 4.9}$

$= b\gamma a$

Therefore $a\gamma b = b\gamma a$ for all $a, b \in \Gamma_N$.

Thus $\Gamma_N$ is a commutative Boolean like $\Gamma$-near-ring.

**Theorem 4.18** Let $\Gamma_N$ be a Boolean like $\Gamma$-near-ring. If $\Gamma_N$ is a $\Gamma$-meet semi lattice then we have the following.

1. $\Gamma_N$ is zero-symmetric.

2. Let $a, b \in \Gamma_N$, $a\gamma b = a \land b \quad \forall \gamma \in \Gamma$ iff $a\gamma b = b\gamma a \quad \forall \gamma \in \Gamma$.

3. $a \land b = a\gamma b \land b\gamma a$ for all $a, b \in \Gamma_N$ and $\forall \gamma \in \Gamma$.

4. $(a \land b) \land (a\gamma b - b\gamma a) = 0$ for all $a, b \in \Gamma_N$ and $\forall \gamma \in \Gamma$.

**Proof:** Suppose $\Gamma_N$ is a $\Gamma$-meet semi lattice

1. For any $a \in \Gamma_N$, g.l.b $\{0, a\}$ exists and let it be ‘$e$’

   therefore $e \leq 0$, $e \leq a$

   $\Rightarrow e = 0\gamma e$ and $e = a\gamma e$ \quad $\forall \gamma \in \Gamma$

   $\Rightarrow e = 0$ and $0 = a\gamma 0$
\[ a \gamma 0 = 0. \]

Therefore \( \Gamma_N \) is zero-symmetric.

2. \( a \gamma b = a \land b \iff b \gamma a = a \land b \iff a \gamma b = b \gamma a \) for all \( a, b \in \Gamma_N \).

Suppose \( a \gamma b = a \land b \)

then \( a \gamma b \leq a \) and \( a \gamma b \leq b \)

\[ \Rightarrow \quad a \gamma b = a \gamma a \land b \text{ and } a \gamma b = b \gamma a \gamma b \]

\[ \Rightarrow \quad a \gamma b = b \gamma a \land b = b \gamma b \gamma a = b \gamma a \]

Conversely suppose that \( a \gamma b = b \gamma a \).

Now \( a \gamma a \gamma b = a \gamma b \), hence \( a \gamma b \leq a \)

and \( b \gamma a \gamma b = b \gamma a = a \gamma b \), hence \( a \gamma b \leq b \).

Therefore \( a \gamma b \) is a lower bound of \( \{ a, b \} \).

Suppose \( c \leq a \) and \( c \leq b \) then \( c = a \gamma c, c = b \gamma c \).

Now \( c = a \gamma c = a \gamma b \gamma c \)

\[ \Rightarrow \quad c \leq a \gamma b \]

Therefore \( \gamma.l.b \{ a, b \} = a \land b = a \gamma b \).

Similarly we can prove that \( b \gamma a = a \land b \iff a \gamma b = b \gamma a \).

Therefore \( \gamma.l.b \{ a, b \} = a \land b = b \gamma a \).
3. For all $\gamma \in \Gamma$

\[ a \land b = a\gamma b \land b\gamma a \]

Let \( c = a \land b \) & \( d = a\gamma b \land b\gamma a \)

\[ \Rightarrow c \leq a \quad \text{and} \quad c \leq b \]

\[ \Rightarrow c = a\gamma c \quad \text{and} \quad c = b\gamma c \]

\[ \Rightarrow c = a\gamma c = a\gamma b\gamma c \quad \text{and} \quad c = b\gamma c = b\gamma a\gamma c \]

\[ \Rightarrow c \leq a\gamma b \quad \text{and} \quad c \leq b\gamma a. \]

Since \( d = a\gamma b \land b\gamma a \) we have \( c \leq d \).

Also \( d \leq a\gamma b \) and \( d \leq b\gamma a \).

Now \( a\gamma b \leq a \) and \( b\gamma a \leq b \)

\[ \Rightarrow d \leq a\gamma b \leq a \quad \text{and} \quad d \leq b\gamma a \leq b \]

\[ \Rightarrow d \leq a \quad \text{and} \quad d \leq b. \]

Since \( c = a \land b \), we have \( d \leq c \).

Therefore \( c = d \).

4. \( (a \land b) \land (a\gamma b - b\gamma a) = 0 \)

Let \( x = (a \land b) \land (a\gamma b - b\gamma a) \)

then \( x \leq (a \land b) \) and \( x \leq (a\gamma b - b\gamma a) \)
\[ \Rightarrow \quad x \leq a, \quad x \leq b, \quad x \leq (a \gamma b - b \gamma a) \]

\[ \Rightarrow \quad x = a \gamma x, \quad x = b \gamma x, \quad x = (a \gamma b - b \gamma a) \gamma x \]

\[ \Rightarrow \quad a \gamma x = b \gamma x, \quad x = a \gamma b \gamma x - b \gamma a \gamma x \]

\[ = a \gamma a \gamma x - b \gamma b \gamma x \]

\[ = a \gamma x - b \gamma x \]

\[ = x - x \]

\[ = 0. \]

Therefore \( (a \land b) \land (a \gamma b - b \gamma a) = 0 \) for all \( \gamma \in \Gamma \).

**Definition 4.19** Suppose \((P, \leq)\) is a partially ordered set. For any subset \(A\) of \(P\),

Let \(L(A) = \{x \in P / x \leq a, \forall a \in A\}\) and \(U(A) = \{x \in P / a \leq x, \forall a \in A\}\).

For convenience we write \(L(x)\) for \(L(\{x\})\) and \(U(x)\) for \(U(\{x\})\).

Write \(L(P) = \{L(A) / A \text{ is a non-empty finite subset of } P\}\) and

\(U(P) = \{U(A) / A \text{ is a non-empty finite subset of } P\}\).

**Result 4.20** Let \(\Gamma_N\) be a Boolean like \(\Gamma\)-near-ring. If \(\Gamma_N\) is a generalized meet semi-lattice then \(\Gamma_N\) is a zero-symmetric.

**Proof:** Let \(\Gamma_N\) be a generalized meet semi-lattice.

Claim: \(\Gamma_N\) is zero-symmetric.
Let \( a \in \Gamma_N \). Consider the finite subset \( A = \{0, a\} \) of \( \Gamma_N \). There exists a non-empty finite subset \( B \) of \( \Gamma_N \) such that \( x \in L(A) \) iff \( x \leq b \) for some \( b \in B \) where \( L(A) \) is the set of all lower bounds of \( A \).

Let \( b \in B \), clearly \( b \leq b \)

\[
\begin{align*}
b \in L(A) \\
\text{Therefore } b \text{ is a lower bound of } A.
\end{align*}
\]

i.e \( b \leq 0 \) and \( b \leq a \)

\[
\Rightarrow \quad b = 0 \gamma b \quad \text{and} \quad b = a \gamma b
\]

\[
\Rightarrow \quad b = 0 \quad \text{and} \quad b = a \gamma b
\]

\[
\Rightarrow \quad a \gamma 0 = 0.
\]

Therefore \( \Gamma_N \) is zero-symmetric.

**Definition 4.21** A poset \( P \) is called **distributive** if both the meet semi-lattices \( L(P) \) and \( U(P) \) are distributive.

**Theorem 4.22** Let \( \Gamma_N \) be a Boolean like near-ring. If \( \Gamma_N \) is a distributive poset then the Boolean like \( \Gamma \)-near-ring \( \Gamma_N \) is commutative.

**Proof:** Let \( \Gamma_N \) be a distributive poset.

Since \( \Gamma_N \) is a distributive poset, \( L(\Gamma_N) \) and \( U(\Gamma_N) \) are meet distributive semi-lattices.
For any $L(A), L(B), L(C) \in L(\Gamma_N)$ such that

$L(A) \land L(B) \subseteq L(C)$, there exists $L(X), L(Y)$ in $L(\Gamma_N)$ such that

$L(A) \subseteq L(X), L(B) \subseteq L(Y)$ and $L(X) \land L(Y) = L(C)$.

Also for any $U(A), U(B), U(C) \in U(\Gamma_N)$ such that

$U(A) \land U(B) \subseteq U(C)$, there exists $U(X), U(Y)$ in $U(\Gamma_N)$ such that

$U(A) \subseteq U(X), U(B) \subseteq U(Y)$ and $U(X) \land U(Y) = U(C)$.

Let $a, b \in \Gamma_N$, Clearly $L(a) \land L(b) \subseteq L(b)$

Therefore there exists $L(a) \subseteq L(X), L(b) \subseteq L(Y)$ and $L(X) \land L(Y) = L(b)$.

Hence $b \in L(X)$ and $b \in L(Y)$.

Choose $x \in X$, then $b \leq x$ and $a \leq x$

\[ \Rightarrow b = x \gamma b \text{ and } a = x \gamma a \]

Now $a \gamma b = x \gamma a \gamma b$

\[ = x \gamma b \gamma a \]

\[ = b \gamma a. \quad \text{by 4.9} \]

\[ \Rightarrow a \gamma b = b \gamma a. \]

Therefore Boolean like $\Gamma$-near-ring $\Gamma_N$ is commutative.