CHAPTER III

One Dimensional Drift Phase Space Holes

3.1 Introduction:

In this chapter we shall study the maximum entropy solutions of the drift kinetic equation in one dimension. In the last chapter, coherent stationary states of the magnetised, inhomogeneous system were studied in the fluid limit. Thus the phase space effects were excluded and important fluid nonlinear terms identified and examined in various parameter regimes. New effects, leading to the formation of vortices in physical space, were studied. We now wish to take into account wave-particle interaction to study the formation of phase space structures.

As a first step towards extending Dupree's formalism (1982) of phase space holes as maximum entropy states, we would like to consider the simplest system. The one dimensional DKE is an ideal point to begin the formulation of this concept for a magnetised, inhomogeneous plasma. In order to study some possible nonlinear steady state properties of this wave, it is important to note that a kinetic drift wave is unstable in the linear limit, as against the fluid case, which is always stable, since it does not take into account
resonant wave-particle interactions in phase space. In the kinetic limit, the resonant electrons near the parallel phase velocity undergo large ExB displacements in the direction of the density gradient. As a result of this gradient there are more number of particles giving energy to the wave than taking energy from it, leading to wave growth. In order to achieve a steady state, there must exist some mechanism to counter this process. In general the nonlinear terms in the DKE arise due to the nonlinear ExB advection of particles and their motion in the parallel direction. However, in the limit when \( k_x \ll k_y \), the ExB nonlinearity is ignored, effectively reducing the system to one dimension. The dominant nonlinear mechanism here is then due to parallel motion. This gives rise to phase space trapping of particles. In this chapter we shall set up one dimensional phase space holes due to this effect. There is a qualitative difference between electron and ion trapping in the case of a drift wave. Since electrons are responsible for the instability mechanism, their trapping should give a direct modification of the growth rate. The ion trapping effects are small even in the kinetic limit since their thermal velocities are generally smaller than the wave phase velocities.

Oraevskii, Sagdeev, Galeev and Rudakov (1969a) considered the quasilinear theory of stability and saturation of drift waves due to parallel trapping. It was shown that coherent parallel trapping in a single mode as well as quasilinear diffusion due to many waves can cause, a)
steepening of the velocity distribution of electrons and b) a local flattening of the spatial gradient, near the parallel resonance. While the former is a stabilising effect, the latter leads to an effective reduction in the growth rate. This can be seen from the expression for the linear electron response, calculated from the drift kinetic equation.

\[ \delta f_e = (\omega_r + i \gamma \omega_c - k_{||} V_{||}) \frac{M}{m} k_y \tilde{\phi} \left( \frac{k_{||}}{k_L} \partial_{V_{||}} + \frac{m}{M} \partial_x \right) f_{0e} \]

The first derivative describes standard Landau damping due to parallel acceleration and causes the net shift in the perturbed electron density to run behind the potential. The second derivative term is due to the ExB drift of electrons near resonance. This makes the net electron density run ahead of the potential. When \( \omega_r > \omega_c \), as is usually the case, the latter effect dominates, giving a net growth. Therefore, a) enhances damping and b) makes the growth mechanism vanish.

The problem of maximum entropy steady states of the DKE with electron parallel trapping effects was examined by Terry et al. (1987) in the full three dimensional geometry. They made the assumption that, the electron distribution function could be taken to be a function of \( \phi \) alone. So, \( f(x,y,z,V_{||}) = f(\phi) \). In effect they ignored any independent \( x \)-dependence of \( f \) in the form of a density gradient. However, this cannot be assumed to be the only configuration of the saturated state. In more realistic situations some modified form of the density gradient is maintained in the steady state phase. Also, in more than one dimensions the ExB nonlinearity.
leading to perpendicular physical space trapping of resonant particles becomes dominant at amplitudes lower than those required for parallel trapping to be effective. The assumption of \( f(\phi) \) ignored this effect completely since \( \nabla \mathcal{E} \cdot \nabla f(\phi) = 0 \), when, \( f = f(\phi) \). This reduced their problem to an effective one dimensional case and may therefore be treated as a special case of the present formulation.

In the following sections we shall systematically reduce the system to a single dimension and formulate the most general maximum entropy state. Both electron and ion trapping effects are considered in various limiting cases of the hole velocity. Finally we shall apply this formalism to the case of a plasma with an equilibrium temperature gradient. It was pointed out in section (2.2) that the presence of a \( \nabla \mathcal{E} \) scalar nonlinearity gives interesting solutions in the one dimensional fluid limit. This case will be modified with the most probable phase space response with trapping effects. Numerical solutions of the resulting equations for the steady state potential will be presented.

3.2 Distribution function and entropy:

We begin by writing the drift kinetic equation (DKE) for a magnetised, inhomogeneous plasma with a density gradient. The equilibrium consists of a plasma in a slab geometry with a uniform B field in the z-direction. There is an equilibrium density gradient in the x-direction, \( n_0(x) \).
The DKE is,

\[ \partial_t f + \nabla \cdot (\nu f) + \nu \nabla \cdot f + \frac{q}{m} E \nabla \nu f = 0, \]

\[ \nu \nu = \nu \nu + \nu \nu, \quad \nabla \nu = -\nabla \nu. \]

Consider the electron DKE in this geometry.

\[ \partial_t f + \frac{e}{B} \partial_y \nu \nu \nu \nu f - \frac{e}{B} \partial_x \nu \nu \nu \nu f + \nu \nu \nabla \nu f + \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu f = 0. \tag{3.1} \]

This equation was studied by Terry et al. (1987) for the three-dimensional parallel trapping problem. However they assumed that the second and third terms vanished identically when \( f = f(\nu) \). With this ansatz the only constant of motion in the steady state is the parallel kinetic energy,

\[ \mathcal{K} = \frac{1}{2} m \nu \nu - e \nu. \]

Therefore the general solution of equation (3.1) in the steady state is,

\[ f = f(\mathcal{K}). \]

This reduces the problem effectively to the one-dimensional case. Let us consider the more general problem and not say anything about the dependence of \( f \) on \( x, y, z, \nu \nu \). We want to look for stationary nonlinear solutions of the plane wave type in a moving frame. Then we can transform to a moving frame with velocity \( u \),

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\[ \eta = y + \Theta z - u t, \]

so that, \( f(x,y,z,v_{\|},t) = f(x,\eta,v_{\|}). \) \( u \) is the hole velocity in the perpendicular direction and, typically, is close to the drift velocity, \( V_d. \) The equation (3.1) has in general the following two conserved quantities.

\[ E = \frac{1}{2} m v^2 - e \Theta^2 \Phi^\sim, \]

\[ K = V + \omega c e \Theta^2 x, \]

where \( V = \Theta v_{\|} - u. \) Therefore now the general solutions of equation (3.1) is any arbitrary function of \( K \) and \( E, \)

\[ f(x,\eta,v_{\|}) = f(K,E). \] (3.2)

\( E \) is the parallel kinetic energy and \( K \) the canonical momentum. This gives the complete description of the steady state. The parallel trapping of electrons is present through the dependence on \( E. \) In simple terms, the resonant particles at \( u/ \) will get trapped in a region of velocity space around it, \( |V| \lesssim (2 e \Theta^2 \Phi^\sim / m)^{1/2}. \) In the treatment of Terry et al. (1987) the dependence of \( K \) has been ignored, removing all gradients from the system. The equation (3.2) includes both the parallel as well as perpendicular trapping information. In order to study the one dimensional system we make the following assumptions.

It is specified that, in the unperturbed state, the density gradient has an exponential form,

\[ \frac{\partial \rho f}{f} = K \eta. \]
Further, we make the approximation, \( k_x \ll k_y \). Then, in one dimension, the equation (3.1) reduces to,

\[
\nabla \partial_\eta f + \frac{e B^2}{m} \partial_\eta \tilde{\phi} \partial_\eta f = \frac{e}{B} \partial_\eta \tilde{\phi} k_n f.
\]

(3.3)

The RHS gives the correction due to the presence of the density gradient to the one dimensional unmagnetised case studied by Dupree (1982). The solutions of this equation are given by,

\[
\int (V, \eta) = g(\xi) \exp \left( \frac{k_n V}{\lambda_n} \right),
\]

(3.4)

\( \lambda_n = \omega_{ce} \theta_c^2 \) and \( g(E) \) is any arbitrary function of \( E \). This is the correct distribution function to be used to study the parallel trapping problem. \( k_n = 0 \) will reduce this to the case of Terry et al. (1987).

Particle trapping effects are now incorporated through the choice of \( g(E) \). In the original BGK approach (1957) the potential structure \( \tilde{\phi}(\eta) \) and the untrapped particle distribution were specified. Then, in the simple case of a electron plasma wave, the Poisson equation was solved for the trapped distribution. However, this solution is artificial. Depending on the choice of \( \tilde{\phi}(\eta) \) there are an infinity of such solutions. One could equally well solve the inverse problem. Specify all the distributions and solve for \( \tilde{\phi}(\eta) \).

This was attempted by BGK (1957) and later applied to the study of nonlinear ion acoustic solitary waves by Schamel.
(1972). But the arbitrariness in the choice of these functions would persist.

In order to remove this ambiguity, the following approach, as suggested by Dupree (1982), is taken. We are not looking for any ad-hoc steady state. Only the most probable one. So the restriction on the choice of \( g(E) \) is that it should be commensurate with the maximum entropy state of the system. The boundaries of this structure will also be determined self-consistently. The resulting potential \( \tilde{\phi}(\gamma) \) would then describe a phase space hole in the sense of Dupree.

It is to be remembered, however, that it is only the form of \( g(E) \) in equation (3.4) that is being determined, not of \( f(V, \gamma) \) as a whole.

The form of the entropy to be used to describe a collisionless Boltzmann system is an area of much investigation. Lynden-Bell (1967) has proposed an alternative form of the entropy. However, the subject has not produced final answers yet. For lack of a better alternative and from the point of view of familiarity, we continue to use the Maxwell-Boltzmann form of entropy,

\[
S = - \int f \ln f \, d\xi \, d\gamma .
\]

It is obvious from results of the statistical mechanics of the Boltzmann equation that the maximum entropy form of \( f(\gamma, \xi) \) would be similar to the Maxwellian. However we shall carry out the process to see what the specifications are.
3.3 **Entropy Maximisation**

We shall continue to study the system where the kinetic species is the electrons. Ions will be described by fluid equations. This is a good approximation when the parallel phase velocity \( u/\theta \) is close to or larger than the electron thermal velocity. Then the ions will not see the parallel resonance and can be treated adequately as a fluid. Let \( F_0(V) \) be the equilibrium distribution of electrons. Let \( u/\theta = u' \) be the parallel hole velocity. Then the amount of phase space density that is required to fill the area of the trapping region in order to create a phase space hole is, \( F_0(u') \). This phase space density is lost by the untrapped region in a reversible fashion. So, the creation of the hole rearranges the phase space in the hole region irreversibly and contributes to the change in entropy. But the untrapped region remains unchanged except for losing some density and does not contribute to the change of entropy. Then the entropy of the entire system may be written as,

\[
\mathcal{S} = n \int \int \left[ f_h \ln f_h - F_0(\nu) \ln F_0(\nu) \right] d\eta d\nu + \mathcal{S}_i, \tag{3.5}
\]

where \( \mathcal{S}_i \) is the initial entropy.

\[
f_h(\nu, \eta) = g(\nu) \exp \left( \frac{\nu \eta}{\kappa e} \right), \tag{3.6}
\]

as given by equation (3.4). We now maximise the entropy subject to certain constraints. We want to keep the total
mass, momentum and energy of the system constant during the maximisation process. Assuming that the region outside the hole may be treated as a linear dielectric, we can write the constraint equations to be:

\[
\begin{bmatrix}
M_0 \\
\mathbf{p}_0 \\
T_0
\end{bmatrix} = \eta \int \int d\nu d\eta \left( f_h - f_0(u) \right) \begin{bmatrix}
m \\
mg \nu \\
E
\end{bmatrix}
\]

(3.7)

where \(M_0, \mathbf{p}_0, T_0\) are the mass, momentum and energy of the hole. The definition for the case of general oscillatory solutions will be modified and \(M_0, \mathbf{p}_0, T_0\) will be defined as quantities per unit wavelength. These quantities, in physical terms, refer to the hole density, hole velocity and hole temperature, respectively.

The potential \(\tilde{\phi}(\eta)\) is given self consistently by the quasineutrality condition,

\[
\eta e \approx \eta i
\]

(3.8)

We now wish to find \(f_h(\nu, \eta)\) and the hole boundary in velocity space, \(V(\eta)\) that will make maximum. Using Lagrange multipliers, \(a, b\) and \(-\zeta^{-1}\), for \(M_0, \mathbf{p}_0\) and \(T_0\) respectively, we get two equations corresponding to independent variations in \(f_h(\nu, \eta)\) and \(V(\eta)\) on the boundary,

\[
\delta \sigma = 0 = \int \int d\eta d\nu \left( 1 + \ln g(E) + \frac{k n V}{\epsilon e} + a + b \nu - E/\zeta \right) \delta f_h
\]

(3.9)

\[
\delta \sigma = 0 = \int d\eta \left[ V(\eta) \left( f_h ln f_h - f_0 ln f_0 \right) + (a + b \nu - \frac{E}{\epsilon}) (f_h - f_0) \right]
\]

(3.10)
Since both $\int f_h$ and $\int v(\eta)$ are arbitrary, from (3.9) we obtain,

$$g(E) = \exp \left[ \frac{E}{\varepsilon} - (b + \frac{K\eta}{\varepsilon}) v - a - 1 \right].$$

However, we know from equation (3.4) that $g(E)$ is a function of $E$ alone, fixing the value of 'b' to be,

$$b = - \frac{K\eta}{\varepsilon}.$$

Therefore,

$$g(E) = \exp \left( \frac{E}{\varepsilon} - a - 1 \right).$$

Equation (3.10) gives the condition that, on the hole boundary,

$$f_0(\nu) = f_h(\nu, \eta),$$

giving

$$f_h(\nu, \eta) = f_0(\nu) \exp \left( \frac{E}{\varepsilon} + \frac{K\eta}{\varepsilon} \nu \right), \quad (3.10)$$

when,

$$\frac{E}{\varepsilon} + \frac{K\eta}{\varepsilon} \nu = 0; \quad \Rightarrow \quad \nu = -\frac{K\eta}{m\varepsilon} \pm \left[ \frac{2\varepsilon E \phi^2}{m} + \frac{2K\eta}{m^2\varepsilon^2} \right].$$

Thus the maximum entropy form of equation (3.5) is determined together with the relevant limits in velocity space where it is valid. Making the untrapped distribution continuous at the boundaries, we can write,

$$f_\nu(\nu, \eta) = \exp \left\{ -\frac{1}{\varepsilon} \left[ \pm \left( \frac{E + \frac{K\eta}{\varepsilon} \nu}{\nu} - \nu \right)^2 \right] \right\},$$

when, \( \nu < \nu_- \), \( \nu > \nu_+ \)

\( (3.11) \)

This represents a shifted Maxwellian. As $K\eta \to 0$ this
entire set reduces to the one studied by Schamel et al. (1972) for nonlinear ion acoustic waves. However, they had also taken account of ion trapping. When $K_n = 0$, the drift wave branch will be absent and these equations will describe the parallel propagating ion acoustic wave. In the limit of $\frac{2}{\beta^2} < 1$, the trapping will be a very small effect and the Boltzmann response for electrons recovered. $\mathcal{U}$ is the temperature of the trapped region and for a hole in phase density, must be positive, $-\mathcal{U} < 0$.

The electron density is given by

$$n_e (\eta) = \int \nu_u f_u (\nu, \eta) d\nu + \int \nu_h f_h (\nu, \eta) d\nu.$$  \hfill (3.12)

The fluid limit is obtained when $\mathcal{U} \to 0$, giving,

$$n_e (\eta) = e \chi \hat{p} \left( \frac{e \hat{\phi}}{te} \right).$$

Thus (3.12) together with (3.10) and (3.11) gives the most probable electron response. Note that by fixing the value of 'b', we are being restricted to a single value of hole momentum $P_0$. The choice of $\mathcal{U}$ and $u'$ will give the corresponding values of $T_0$ and $M_0$. Thus all parameters have been determined without ambiguity.

As said earlier the ions will be treated as a cold fluid, in the limit, $\omega \gg k_{\parallel} V_i \nu_i$.

We use the equations of continuity and parallel momentum for ions in the low frequency approximation, in the limit $k_x$.
\[ \vec{V}_p = \frac{-c \beta \omega_c i (V_i \partial_\eta \tilde{\phi})}{1 + \frac{c \beta \omega_c i}{\omega_c} \partial_\eta \tilde{\phi}} \]  

(3.13)

where \( V_1 \) is now the ion fluid velocity, \( V_1 = \beta V_i - u \). The denominator in equation (3.13) arises due to finite Larmor radius effects. Note that, in chapter II this term had been ignored giving only the dominant numerator. Then the ion equations become,

\[ \partial_\eta (\nu_i \eta) - \frac{c K_n}{B} \eta \partial_\eta \tilde{\phi} - \partial_\eta \left[ \frac{\eta \frac{c}{B \omega_c i} V_i \partial_\eta \tilde{\phi}^2}{1 + \frac{c \beta \omega_c i}{\omega_c} \partial_\eta \tilde{\phi}^2} \right] = 0 \]

\[ V_i \partial_\eta V_i = -\frac{e \beta^2}{M} \partial_\eta \tilde{\phi} \left( 1 + \frac{c}{B \omega_c i} \partial_\eta \tilde{\phi}^2 \right) \]

Solving these, we get the following results,

\[ \eta (\eta) = \frac{e^{-K_n V_i / \nu_i} + \bar{K}}{V_i} \left( 1 + \frac{c}{B \omega_c i} \partial_\eta \tilde{\phi}^2 \right) \]  

(3.13)

\[ \bar{W} = \frac{M}{2} V_i^2 + e \beta^2 \bar{\phi} + \frac{M c^2 \beta^2}{e B^2} (\partial_\eta \tilde{\phi})^2 \]  

(3.14)

where \( \bar{W} \) and \( \bar{K} \) are arbitrary constants. \( \bar{W} \) is the parallel kinetic energy of fluid ions, with contributions from the polarisation drift. Using (3.14) to eliminate \( V_1 \) in (3.13) we use the following boundary conditions to determine \( \bar{K} \) and \( \bar{W} \).

at \( \eta = 0 \), \( \tilde{\phi} = \tilde{\phi}_0 \), \( \partial_\eta \tilde{\phi} = \epsilon \), \( \partial_\eta \tilde{\phi}^2 = 0 \)

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and \[ V_i = - \mathcal{u} \] (3.15)

Then, using quasineutrality and equation (3.12) we can write in normalised variables,

\[
\partial^2_{\eta} \phi = \frac{\eta_c (\eta)}{\mathcal{e}_x \mathcal{p} \left( - \kappa \eta V_i / \lambda_i \right) + \mathcal{K}} \frac{V_i (\mathcal{W})}{\mathcal{e}_x \mathcal{p} \left( - \kappa \eta V_i / \lambda_i \right) + \mathcal{K}}
\] (3.16)

The electron density has been normalised to its value at, \[ \phi = \phi_0 \].

Equation (3.16) describes the most probable steady state potential of a drift wave with electron trapping effects. When \( \mathcal{C} = 0 \), the fluid limit for \( n (\eta) \) is recovered, since there is no trapping.

In chapter II we had studied the one dimensional fluid case in the presence of temperature gradients. Also, only the linear part of the polarisation drift was retained there in one dimension. So, both electrons and ions have been treated more exactly. It has been solved together with the conditions (3.15), numerically. Equation (3.16) can be written in terms of an effective potential,

\[
\partial^2_{\eta} \phi = - d \phi \mathcal{F}_{eff} (\phi)
\] (3.18)

The concept of defining an effective potential has been used in the study of shocks in plasmas (Tidman and Krall 1971). For an ion acoustic shock, this is called the Sagdeev
potential. It has the characteristic shape given below.

\[ \Phi_{\text{eff}} \]

\[ \phi_0 \quad \phi_i \quad \phi_{\text{max}} \rightarrow \phi \]

\[ \partial_\eta \phi / = \varepsilon \]

\[ \phi_o \]

The equation (3.18) gives the description of a 'pseudoparticle' moving in the potential \( \Phi_{\text{eff}} \). The space variable becomes the effective time variable. The motion of such a pseudoparticle in \( \Phi_{\text{eff}} \) gives the properties of \( \phi(\eta) \). In our case the RHS. of (3.18) is not a function of \( \phi \), alone but also of \( \partial_\eta \phi \). So it is not possible to write down an analytic form for \( \Phi_{\text{eff}} \). However, we have drawn an analogy with this description to explain the boundary conditions used. \( \phi_o \) is the turning point on the trajectory of such a pseudoparticle. At this point it is given an energy \( \varepsilon \) and we look for oscillatory solutions of the system. This means specifying the value of \( \partial_\eta \phi \) at \( \phi_o \). \( \eta \) acts as the time variable. Obviously, for larger values of \( \varepsilon \), amplitudes will also be larger. In a single period the pseudoparticle will bounce up to some value \( \phi_i \) and bounce back. The limit on \( \varepsilon \) will come from,

\[ u^2 + 2 \partial_\eta^2 \phi_{\text{max}} + (\partial_\eta \phi) \left( \frac{\phi - \phi_{\text{max}}}{\phi_{\text{max}}} \right)^2 = 0 \]

for a given value of \( u \). This is similar to the condition on the Mach number of an ion acoustic soliton. Since \( \phi_o \) is a
turning point,

\[ \partial \eta \Phi |_{\Phi_0} = \varepsilon, \quad \partial^2 \eta \Phi |_{\Phi_0} = 0 \]

The effect of increasing \( \varepsilon \) can be seen in Fig. (3.1). Curve (1) is the fluid electron limit and corresponds to the solution of the one dimensional problem in equation (2.13), when \( K_T = 0 \). Note that the introduction of trapped electrons has lowered the amplitude for the same set of boundary conditions and values of \( K_n, u, \varepsilon \). The minimum of the potential has been fixed at \( \Phi = 0 \).

\( \Upsilon \) is the temperature of the particles in the hole region. The distribution function for \( \Upsilon > 0 \) is shown in fig. (3.2). As \( \Upsilon \) becomes larger the dip becomes deeper. As a result a larger number of particles traveling close to \( u' \) are now available in the trapping region. So it can be expected that saturation should take place at larger amplitudes. This can be seen from the fig. (3.3). As \( \Upsilon \) becomes larger, the amplitude increases, together with the wavelength. The wavelengths of these periodic structures are typically of the order of a few ion Larmor radii \( (\rho_i) \), as expected from the results of the one dimensional treatment of chapter II.

3.4 A 1-D rectangular hole:

It would be interesting to see analytically how this maximum entropy state differs from that studied by Dupree (1982). Simple approximate solutions for an isolated hole may be obtained by using the rectangular hole approximation.
Fig. 3.1 : Potential showing effects of $\varepsilon$:

(1) The fluid limit of electrons
(2) $K_n = -0.1$, $u/\Theta = 1.2$, $\phi_e = 0.25$, $\varepsilon = 0.3$
(3) $\varepsilon = 0.1$
Fig. 3.2: The complete distribution function for $\zeta > 0$. The dip in the trapping region deepens with
Fig. 3.3: Potential showing effect of

\[ K_n = -0.1, \quad u/\Theta = 1.2, \quad \phi_0 = 0.36, \quad \varepsilon = 0.11; \]

(1) \( \zeta = 0.1 \), \hspace{1cm} (2) \( \zeta = 0.23 \)
(Dupree 1982). The shape of the hole boundary is taken to be a rectangle instead of the one described by equation (3.10). When the hole is shallow, the hole depth can be approximated by a constant. The description of such a hole can be given by,

\[
f (V, \eta) = \tilde{f} e^{\frac{K \eta V}{\Delta V}} + f_o (u)
\]

when, \(-\Delta \eta / 2 \leq \eta \leq \Delta \eta / 2\)

\(-\Delta V / 2 \leq V \leq \Delta V / 2\)

\[
= f_o (u) \quad \text{otherwise}. \quad (3.19)
\]

for a hole moving with velocity \(u\), and located at \(\eta = 0\). The scale lengths in phase space, \(\Delta \eta, \Delta V\), will be determined now from entropy maximisation.

For the sake of simplicity we shall treat ions linear, giving,

\[
\frac{n_i}{n_0} = \left( \frac{e^2}{\mathcal{L} \cdot \nu} - \frac{V d}{\mathcal{L} \cdot \partial \eta} \right) \frac{e^{\tilde{f}}}{T e}
\]

\( (3.20) \)

Equations (3.19) and (3.20) describe a special subset of hole solutions of the drift kinetic equation described in section (3.3). The parallel motion nonlinearity and FLR
effects studied in chapter II and section (3.3) are neglected. Electron trapping effects are retained upto the $\sqrt{\varphi}$ dependence of velocity limits of the trapping region. Higher order effects are ignored. The trapping parameter, $\zeta$, is replaced by $\tilde{f}$.

We now use the maximum entropy property of the hole to obtain the scale lengths $\Delta \eta$ and $\Delta V$, subject to constraints that $M_0$, $P_0$ and $T_0$ as given by equation (3.7), are kept constant.

Using equation (3.17) and the definition of entropy given by equation (3.5) we get the expression,

$$\sigma = \frac{M_0 \tilde{f}}{m F_0 (u)}$$  

(3.21)

for the entropy.

Using (3.19) and (3.20), we get the following equation for the potential $\tilde{\phi}(\eta)$,

$$\left( \partial^2 \eta - \lambda^2 \right) \tilde{\phi} = \frac{2 T e \tilde{f}}{\varepsilon} \frac{x \varepsilon}{K \eta} \sin \h (\frac{K \eta \Delta V}{2 \varepsilon})$$

where, $A^{-2} = (1 + V_d u) / l_s^2$  

(3.22)

Then using standard Greens function methods we can solve equation (3.22) to give the hole potential to be,

$$\tilde{\phi}(\eta) = - \frac{2 T e \tilde{f} x \varepsilon}{\varepsilon L \xi K \eta} \sin \h (\frac{K \eta \Delta V}{2 \varepsilon})$$

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\[ \frac{1}{2} \left[ 1 - e^{-\Delta \eta/2} \cosh \left( \frac{\eta}{\lambda} \right) \right]. \]  

(3.23)

The hole mass, momentum and energy can be evaluated,

\[ M_o = 2 \pi m \int f' \Delta \eta \frac{\kappa e}{\kappa_n} \sinh \left( \frac{\kappa_n \Delta V}{2 \Delta e} \right) \]

\[ P_o = \frac{M_o \Delta e}{\kappa_n} \left[ \frac{\kappa_n \Delta V}{2 \Delta e} \coth \left( \frac{\kappa_n \Delta V}{2 \Delta e} \right) - 1 \right] \]

\[ T_o = \frac{\kappa_n^2 M_o}{2 \Delta e} \left[ \frac{\kappa_n^2 \Delta V^2}{4 \Delta e^2} + \frac{\kappa_n P_o}{M_o \Delta e} \right] \]

\[ + \frac{2 \pi \kappa T_e}{\kappa_n^2} \int f' (\kappa e) \Theta' \sinh^2 \left( \frac{\kappa_n \Delta V}{2 \Delta e} \right) \left( \Delta \eta - \frac{1}{2} \lambda + \frac{1}{2} \Delta \eta / \lambda \right) \]

In the limit of \( \kappa_n = 0 \), these quantities revert to those calculated by Dupree (1982). Using these and maximising \( \delta \) in (3.21), we get the following relation,

\[ \frac{M_o}{2} (\Delta V)^2 \sinh^2 \left( \frac{\kappa_n \Delta V}{2 \Delta e} \right) = Q_h \Theta' (0) g \left( \frac{4 \eta}{\lambda} \right) \]

(3.24)

where,

\[ Q_h = g M_o / m \]

and \[ g(y) = \int dy \left[ \frac{1}{y} \frac{(y + e^{-y} - 1)}{(1 - e^{-y/2})} \right] \]

\[ \Phi'(0) = \Phi'(\eta = 0) \]

In the limit of large scale length density gradients or small \( \kappa_n \),

\[ (\Delta V)^2 \sim \Phi'(0) \]

recovering the parallel trapping limit of Dupree (1982).
There are corrections to this result due to the finite $K_n$. For a specified value of $\theta^2$ and $\tilde{\phi}(0)$ equation (3.22) can be solved for $\Delta V$. The potential $\tilde{\phi}(\gamma)$ in the hole region is plotted in fig.(3.4). It has the right sign for trapping electrons and is an isolated hole solution unlike in the previous section, where we had looked for oscillatory solutions. $\Delta \gamma$ typically varies from a few times $I_0$ to some fraction of $K_n^{-1}$. Outside the hole region the $\tilde{\phi}(\gamma)$ decays with the e-folding length $\lambda$. Note that, the solutions studied in section (3.3) can be regarded as a periodic array of such isolated holes.

3.5 **Ion trapping effects:**

In section (3.3) we have taken into consideration the nonlinear fluid ion response. This was a good description as long as $u'$, the parallel hole velocity, was close to the electron thermal velocity. However, typically it lies anywhere between the ion and electron thermal velocities. If $T_i = T_e$, then as $u'$ gets closer to the sound speed $C_s$, parts of the ion distribution will be affected by the resonance and get trapped by the wave field. Also as pointed out by Dupree(1982), the essential basis for this formalism is that, whatever happens in the nonlinear regime need not be predicted in a perturbative way only on the basis of the linear theory. So, it could very well happen that there are stationary nonlinear modes propagating close to the ion thermal velocity, even though linear theory does not predict them. In this parameter regime, we shall take into account
Fig. 3.4 : The one dimensional rectangular hole potential.

\[ \Delta \eta = 2.208, K_n = -0.1, u/\Theta = 1.2, \Theta = 0.02 \]
the ion trapping effects. In the ion DKE, perpendicular velocities would consist of the ExB and the polarisation drift as given by equation (3.13). $V_\perp$ is replaced by the phase space velocity $V = \theta V_\parallel - u$. Then the steady state ion DKE in one dimension becomes

$$
\frac{V \partial \eta f}{1 + \frac{c}{\beta \omega_c} \partial^2 \eta \phi} - \frac{e \theta^2}{M} \partial \eta \phi \partial \nu f = \frac{c \kappa}{\beta} \partial \eta \phi f
$$

$$
+ V \partial \eta \left[ \frac{c}{\beta \omega_c \gamma \theta^2 \phi} \partial^2 \eta \phi \right] f.
$$

The general solution for this equation is,

$$
f(V, \eta) = g(E_i) \exp \left( - \frac{K V}{\alpha_i} \right) (1 + \frac{c}{\beta \omega_c} \partial^2 \eta \phi),
$$

where

$$
\alpha_i = \omega_c \theta^2,
$$

$$
E_i = \frac{M}{2} V^2 + e \theta^2 \phi \partial^2 \eta \phi + \frac{M c^2 \theta^2}{2 B^2} \partial^2 \eta \phi^2,
$$

as in equation (3.14). The general distribution function to be determined, therefore, can be written as

$$
f_{hs} = g_s(E_s) \exp \left( - \frac{K V}{\alpha_s} \right) (1 + \gamma_s \partial^2 \eta \phi),
$$

where 's' denotes the species.

$$
E_e = \frac{m}{2} V^2 + e \theta^2 \phi , \quad \alpha_e = \omega_c \theta^2 , \quad \gamma_e = 0.
$$

$$
\alpha_i = \omega_c \theta^2 , \quad \gamma_i = \frac{c}{\beta \omega_c}.
$$

The total entropy of the system to be maximised is,

$$
S = \sum_s \eta_s \int d\eta dV \left( f_h \ln f_h - F_{os} \ln F_{os} \right).
$$
We now impose constraints for the maximisation. It turns out that it is not sufficient to say that the total mass, momentum and energy be constant. All the Lagrange multipliers do not get determined self consistently in that case. This is obvious since instead of a one species case as in the last section, we are now dealing with a two species system whose masses and charges are different. In order to take into account these we must introduce two additional constraints. Apart from saying that \( M_0, P_0, T_0 \) as defined by,

\[
\begin{bmatrix}
M_0 \\
P_0 \\
T_0
\end{bmatrix} = \sum_s \eta_s \int d\gamma d\nu \left( f_{hS} - f_{OS} \right) \begin{bmatrix}
m_s \\
m_s \nu \\
E_s
\end{bmatrix}
\]

be constant, we also say that the total current and charge of the system should remain unaltered, giving the following two additional constraints:

\[
\begin{bmatrix}
Q_0 \\
J_0
\end{bmatrix} = \sum_s \eta_s \int d\gamma d\nu \left( f_{hS} - f_{OS} \right) \begin{bmatrix}
q_s \\
q_s \nu
\end{bmatrix}
\]

These additional constraints regarding constancy of total charge and current were not necessary in the last section. This is because, for a single species, constancy of total mass and charge and constancy of total momentum and current are degenerate constraints. Now introducing the Lagrange multipliers \( a, b, -C^{-1}, d \) and \( e \) for \( M_0, P_0, T_0, J_0 \) and \( Q_0 \) respectively and carrying out the maximisation
processes, we obtain,

\[ f_{hs} = \exp \left[ -am_s - b \nu + \frac{E_s}{c} - d q_s \nu - e q_s - 1 \right] \]

Or

\[ g_s (E_s) = \exp \left[ \frac{E_s}{c} + \nu \left( \frac{k_n}{\alpha_s} - b - d q_s \right) - a m_s - e q_s - \ln \left( 1 + \gamma_s \frac{q_s}{\phi} \right) \right] \]

It is to be noted that the variations in the electron and ion trapped distributions are taken to be independent of each other, as are the variations in their respective velocity boundaries. Knowing that \( g_s \) is a function of \( E_s \) alone, at all times, we get the following algebraic equations for the Lagrange multipliers.

\[ b m_s + d q_s = \frac{k_n}{\alpha_s} \]

\[ a m_s + e q_s = - \ln \left( 1 + \gamma_s \frac{q_s}{\phi} \right) \]

Thus, \( a, b, d \) and \( e \) are determined completely, fixing the values of the hole mass, momentum, current and charge. \( \gamma \), the hole temperature remains a parameter. Then apart from the electron distributions in equations (3.10) and (3.11), we get the trapped and untrapped distributions of ions to be,

\[ f_{iu} (V, \eta) = \exp \left\{ - \frac{1}{T_i} \left[ \pm \left( E_i + \frac{c k_n \nu}{\alpha_i} \right)^{1/2} - \eta \right] \right\} \]

when \( V > V_{i+}, \ V < V_{i-} \)

and,

\[ \text{72} \]
\[ f_{ni}(\nu, \eta) = f_{oi}(\nu) \exp \left( \frac{E_i}{c} + \frac{\kappa n}{a_i} \nu \right) \]

where,

\[ V_{i\pm} = -\frac{\nu}{M^2 a_i^2} \pm \sqrt{\frac{2 e \theta^2}{M} (\psi - \bar{\phi}) + \frac{2 M e^2 \theta^2}{e B^2} \left( \frac{e^2}{2} \hat{\theta} \dot{\phi}^2 \right) + \frac{2 \kappa_n^2}{M^2 a_i^2}} \]

\( \psi \) is related to the maximum of the potential.

In the limit of \( \zeta \to 0 \), we recover the cold fluid limit of \( n_1(\eta) \) given in the previous section.

Using these equations and quasineutrality we can write the following equation for the potential \( \phi \),

\[ \frac{d^2}{d\eta^2} \phi = f(\phi) - 1 \]

This equation was studied numerically in various parameter regimes. In case of the fluid ions it was seen that as \( u \) increases the amplitude of the potential decreases. This can be understood from the fact that as \( u \) increases the parallel resonance shifts further down the tail of the distribution. So lesser number of particles are available for trapping, even when \( \zeta \) is kept constant. However, for \( u' \) close to \( C_s \) it was found that the amplitude of the potential in the trapped ion case was smaller than the fluid ion case for the same set of initial conditions. This is shown in fig. (3.5). The decrease is small. This can be understood again because the ion distribution function sampled by the resonance is very small, the effect of trapped ions is proportionately less. As \( u' \) increases beyond \( C_s \) this effect becomes negligible and the fluid description holds good.
Fig. 3.5: Trapped ion potential.

$K_n = -0.1$, $u/\theta = 1.2$, $\phi_0 = 0.24$,
$\varepsilon = 0.11$, $\zeta = 0.1$

(1) Fluid ions, (2) Kinetic ions
In the next section we shall consider a further modification in the form of an equilibrium temperature gradient. This is motivated by the interesting physics of the one-dimensional fluid case studied in chapter II.

3.6 Temperature gradient effects:

In this section the maximum entropy state of a magnetised, inhomogeneous plasma with an equilibrium electron temperature gradient, due to phase space trapping in one dimension will be set up. In chapter II, the stationary solutions of the one-dimensional fluid drift wave were studied in the presence of an electron temperature gradient (equation 2.13). It was shown that the scalar nonlinearity due to this term is important in certain parameter regimes and is, in fact, the most dominant nonlinear term. There, the electron response was taken to be adiabatic and given by the Boltzmann relation. However, as pointed out earlier, as the amplitude of the modes become large enough to trap particles, this is no longer a good description. In the following calculation, it will be modified to incorporate the parallel trapping effects. So, we are now applying the formalism for entropy maximisation, developed in the earlier sections, to a known case of interest, to see the difference in the properties of the final steady states.

In section (2.3) the parallel nonlinearity of fluid ions was also taken into account. In this section we shall continue to treat ions as a fluid and use the equation (3.14)
to give their steady state response. So, their parallel fluid nonlinearity and the polarisation drift effects are also included.

As before, if we assume that there is an equilibrium temperature gradient in the x-direction, such that, when \( k_x \ll k_y \), i.e. in one dimension,

\[
\partial_x f/f = \kappa \eta + T \kappa T \partial_T f
\]

Then in place of equation (3.3) we now get the following 1-dim. form for the DKE of electrons.

\[
\nabla \eta f + \frac{T c \kappa T}{B} \eta \phi \tilde{ \eta } \partial_T f + \frac{e\theta^2}{m} \eta \phi \tilde{ \eta } \partial_T f
\]

\[
= \frac{c \kappa \eta}{B} \eta \phi \tilde{ \eta } f . \tag{3.25}
\]

For \( \kappa_T = 0 \), this reduces to equation (3.3). We assume that temperature \( T \) appears only as the normalisation of the energy \( E \), therefore,

\[
\partial_T f = \partial_{(E/T)} f \cdot \partial_T \left( \frac{E}{T} \right)
\]

\[
= - \partial_E f \cdot \frac{E}{T} .
\]

Substituting in (3.25) we obtain,

\[
\nabla \eta f - \frac{c \kappa T}{B} \eta \phi \tilde{ \eta } \partial_E f + \frac{e\theta^2}{m} \eta \phi \tilde{ \eta } \partial_T f
\]

\[
= \frac{c \kappa \eta}{B} \eta \phi \tilde{ \eta } f . \tag{3.26}
\]

Apart from the parallel nonlinearity in the third term, there is an additional nonlinear term due to the temperature.
gradient. This term is similar to the scalar nonlinearity in the equation (2.9) and studied by Petviashvili (1977). In fact, if the parallel nonlinearity is neglected then, the trapping effect would vanish. The velocity average of equation (3.26) would then reduce to the Boltzmann relation for electrons with a temperature gradient. Then the equations (2.9), (2.10) would be recovered. It can be seen further that \( \dot{E} \) is no longer a constant of motion. Changing the variables,

\[
\tilde{f}(\eta, \gamma) = f(\gamma, \dot{E})
\]

equation (3.26) becomes,

\[
\partial_\gamma \tilde{f} - \frac{c K_T}{B} \frac{E}{V(\dot{E})} \partial_\gamma \tilde{\varphi} \partial_\dot{E} f = \frac{c K_T}{B} \partial_\gamma \tilde{\varphi} \tilde{f}
\]

Then, the new equation for the characteristics is,

\[
\partial_\eta \dot{E} = \frac{c K_T}{B} \partial_\gamma \tilde{\varphi} \frac{E}{V(\dot{E})},
\]

where

\[
V(\dot{E}) = \left[ \frac{2}{m} (\dot{E} - e \theta \tilde{\varphi}) \right]^{1/2}
\]

This equation can be solved perturbatively for the new constant of motion, \( \tilde{E} \),

\[
\tilde{E} = E (1 + \frac{K_T}{\alpha_e} \dot{V})
\]

Then to lowest order in \( K_T f_3 \), the form of distribution function is,

\[
f(V, \gamma) = g(\tilde{E}) e^{\frac{\theta}{\alpha_e} \left( \frac{K_T V}{\alpha_e} \right)}
\]

Treating ions as a nonlinear fluid as in equation (3.14)
and following the procedure for entropy maximisation given in section (3.3), we get an altered trapped electron distribution function:

$$f_h(V, \eta) = f_0(u) \exp \left[ \frac{E}{c} + \frac{k_n V}{\alpha e} \right]$$

when,

$$\left[ \frac{1}{c} + \frac{k_n V}{\alpha e} \right] = 0$$

$$\Rightarrow V_\pm = -\frac{k_n c}{2 \alpha e V_{bo}} \pm \left( \frac{c^2 k_n^2}{4 \alpha e^2 V_{bo}^2} + \frac{2 e \theta^2}{m} \right)^{1/2}$$

where $V_{bo}$ is the positive boundary value in the limit $K_T = 0$.

Again, making the untrapped distributions continuous at the boundaries in velocity space, we can write,

$$f_u(V, \eta) = \exp \left[ -\frac{1}{Te} \left\{ \pm \left( \frac{c}{\alpha e} + \frac{c k_n V}{\alpha e} \right)^{1/2} - u \right\}^2 \right]$$

when, $V < V_-$, $V > V_+$

Using these distributions and the expression for $n_1(\eta)$ in equation (3.14) we can now solve for $\tilde{\phi}(\eta)$. These distributions reduce to the fluid limit when $c = 0$. The nonlinearity due the $\nabla T_e$ in equation (2.14) has now been modified by the presence of the trapped particles. Fig. (3.6) shows the results of the numerical calculations and the difference between the fluid electron case of section (2.3) and the effect trapping. As $K_T$ increases both the amplitude and wavelength increase. However, the amplitude of the trapped electron case is lower than that of the solution of equation (2.14) for the same set of parameters. The difference between the amplitudes for successive values of $K_T$
Fig. 3.6 : Temperature gradient case.

\[ K_n = -0.1, \ u/\Theta = 1.2, \ \phi = 0.25, \]
\[ \varepsilon = 0.11, \ \tau = 0.1: \]

(1) \[ K_T = 0.2, \] (2) \[ K_T = 0.1. \]
is small since only first order terms in (K_T / S) have been retained.

Thus we have successfully applied the formalism of maximum entropy stationary states to the interesting case of one dimensional structures with equilibrium temperature gradients. It should be remembered, however, that the fluid one dimensional solutions were unstable to two dimensional perturbations. This analysis is not attempted here for the trapped electron case.

3.7 Conclusion

In this chapter we extended the theory of phase space holes given by Dupree (1982) for an unmagnetised, homogeneous plasma to the case of a magnetised, inhomogeneous plasma. Electron and ion trapping effects were considered to build the most probable BGK modes in one dimension. In chapter II we had studied the stationary solutions of the fluid drift wave and found that they are vortices in two dimensional physical space. The solutions studied here are phase space structures in one dimension. The fluid solutions were essentially a result of the ExB and polarisation drift nonlinearities of ions. Therefore it would seem that in two dimensions there could be a coupling between these two effects in some parameter regime. The ExB nonlinearity would come into play when k_x ~ k_y and then the system can no longer be reduced to a one dimensional form.
It would be appropriate at this point to explore the regime of existence of such states. These steady states may be regarded as saturated states of the drift instability. As the wave grows beyond the linear limit, the amplitude would become large enough for the resonant electrons near $\omega/k$ to be trapped in the potential of the wave. In the limit of sufficient density of trapped electrons the instability is quenched and a finite amplitude steady state established. In the limit $k_x \ll k_y$ used here the existence of such states is possible, even though the ExB nonlinearity has been ignored. However Ott et al. (1976) have examined the stability of drift waves with electrostatically trapped electrons in the presence of collisions. It appears that collisions can induce a detrapping of particles from the wave fields to give rise to a net growth. So the stability of these one dimensional phase space holes against collisions may restrict its stationarity to time periods smaller than the collisional detrapping time scales. Their stability to one and two dimensional perturbations will also have to be explored since the one dimensional fluid structures of chapter II were shown to be unstable to these.

In the next chapter we shall go on to the more complex problem of setting up two dimensional phase space holes with contributions from both the ExB and parallel phase space nonlinearity.