CHAPTER II

AN INFINITELY THIN MAGNETIZED PLASMA DISK IN THE
GRAVITATIONAL FIELD OF THE GALACTIC BULGE —
A SELF-CONSISTENT TREATMENT

2.1. Introduction

As was described in the previous chapter, theories of galactic magnetic field suffer from several difficulties. We think that most of the difficulties stems from the fact that they are kinematical in nature and a complete dynamical treatment, we hope, shall remove the difficulties encountered by them. Therefore, it calls for a complete MHD analysis of the phenomena in the galactic disk.
We shall be considering the dynamics of a thin magnetized plasma disk with a central bulge in a self-consistent manner as an eigen-value problem in anticipation that the different magnetic field morphologies will turn out to be the normal mode of the system. The magnetic field morphologies should be the outcome of the dynamics of the system and that they should draw their energy from the gravitational field of the bulge. Rigid rotation is the only permissible solution of an equilibrium induction equation. As was noted in the previous chapter, differential rotation of the galactic disk and finite electrical resistivity of the ionized matter are the paramount condition for a galactic dynamo to function and we hope, dynamical treatment of the problem (i.e. coupling of magnetic force with the gravitational force through motion) will give us the different magnetic field morphologies as the normal modes of the systems.

We first study the magnetized plasma disk with a central bulge with the assumption that the fluid is incompressible. We derive an eigen-value equation for an incompressible plasma and see that fluid is unstable against gravity. Rayleigh-Taylor instability exists in the disk i.e. top-heavy arrangement is that of plasma density gradient supported by magnetic field against gravity. Ultimate source of energy turns out to be the gravitational energy which causes the growth of the magnetic field.

Next, in section 2.7, we consider a more general isothermal equation of state and recast the whole
eigen-value problem in the matrix form and solve numerically the general eigen-value problem. The eigenmodes are considered. Eigenmodes and eigenpatterns have been analyzed.

We discuss the basic magnetohydrodynamic equation in section 2.2. In section 2.3, we consider the equilibrium of the disk. In section 2.4, we describe the normal mode analysis. Section 2.5 discusses the instability and its growth rate. In section 2.6, we recast the eigenvalue equation in the variational form. In section 2.7, we discuss a more general equation of state and derive a density distribution consistent with the equilibrium of the system. Section 2.8 describes the non-dimensionalization of the different quantities. Section 2.9 discusses the Maxwell’s equation and related with it the dispersion relation. In section 2.10 we discuss the general matrix formulation of the problem and finally, section 2.11 describes the numerical results.

2.2. Magnetohydrodynamic Equations

We study here the amplification of the magnetic field perturbation in an infinitesimally thin magnetized plasma disk with a central bulge. One knows that only a fraction of the galactic matter is in ionized form (~3 to 4%) and rest of it consists of neutral component. The ionized and the neutral components in the galactic disk interact via collisions which exchange momentum and energy between them.
It is assumed here that ion-neutral collision is negligible since their frictional time-scale is typically short \((6 \times 10^5 \text{ years}, \text{Spitzer, 1968})\) compared to the dynamical time-scale \((10^6 \text{ years or more})\). Therefore, we consider a disk consisting of one fluid which represents the ionized component.

It is further assumed that the plasma in the disk is charge-neutral, since for a typical value of electrical conductivity \(\sigma \sim 10^{-8} \text{ s}^{-1}\) and \(\xi \sim 1\), time of charge separation \(\sim 10^{-8} \text{ s}\). Because of the charge neutrality

\[
\nabla \cdot \mathbf{J} = 0
\]

i.e. currents must flow either in closed loops or in infinitely long circuits.

We must also assume that the plasma in the disk is infinitely conducting, i.e. \(\sigma \rightarrow \infty\). Infinite conductivity \((\sigma \rightarrow \infty)\) is a good approximation in most of the astrophysical situations. The gigantic scale-length involved in the heavenly processes ensures its validity. Magnetic Reynold number describes the relative importance of whether the magnetic fields are "frozen" in the matter or it diffuses with time. It is defined as

\[
\mathcal{R}_m = \frac{\sigma V}{\mu L}
\]

Where \(L\) is the characteristic scale-length, \(V\) - velocity of the fluid element, \(\sigma\) - the electrical conductivity of the fluid and \(\mu\) - the magnetic permeability of the vacuum. Numerically, if \(\sigma\) is in \(\text{s}^{-1}\), \(V\) in \(\text{cm s}^{-1}\), and \(L\) in \(\text{cm}\), we have

\[
\mathcal{R}_m \sim 1 \times 10^{20} \sigma L V
\]
In cosmic plasmas, $R_m$ nearly always turns out to be large compared to unity. For example, for solar convection zone $R_m \approx 10^8$; for an interstellar HII region $R_m \approx 10^5$; for HII region $R_m \approx 10^6$. On the other hand, in laboratory fluids we seldom have $R_m > 1$; for example, in a vat of mercury about 1 m in size, we might have $R_m \approx 10^{-1}$.

Note that $R_m$ is proportional to the product $\sigma L$. Hence a large value of $R_m$ resulting from enormous linear dimension (large $L$) is equivalent to a large value of electrical conductivity $\sigma$. Thus, cosmic plasma behave in much the same way as would a laboratory fluid with infinite conductivity.

One fluid magnetohydrodynamic equations which describes the dynamics of a magnetized gravitating plasma disk, in the infinite conductivity limit are,

\[ \frac{d\mathbf{v}}{dt} = -\nabla p - \frac{\sigma M}{\mu_0} \mathbf{E} + \mathbf{j} \times \mathbf{B} \]  

(1)

\[ \frac{\partial \mathbf{E}}{\partial t} + \nabla \cdot (\mathbf{E} \mathbf{v}) = 0 \]  

(2)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  

(3)

\[ \nabla \cdot \mathbf{B} = 0 \]  

(4)

\[ \nabla \cdot \mathbf{B} = \mathbf{F} \cdot \mathbf{j} \]  

(6)
\[ \vec{\nabla} \cdot \vec{B} = 0 \]  

(7)

Equation (1) is the momentum balance equation, with \( \rho \) the mass density and \( \vec{v} \) the mean or macroscopic plasma fluid velocity. The plasma is assumed electrically neutral and the particle pressure tensor is assumed to be a scalar. The time derivative here is the total derivative, taken moving along the fluid.

\[ \frac{d}{dt} \vec{v} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \]

Equation (3) is Ohm's law in infinite conductivity limit. It shows that \( \vec{E} \) has no component parallel to \( \vec{B} \). Secondly, \( \vec{V}_\perp \), the part of \( \vec{v} \) is perpendicular to \( \vec{B} \) is given by \( \vec{E} \times \vec{B} \) drift.

Equation (4) is the equation of state, which tells that fluid is incompressible. We assume this to be the case for the present treatment.

Equations (5), (6) and (7) are Maxwell's equation with the neglect of the displacement current.

By taking curl of equations (3) and making use of equation (5), we have the equation of induction in the infinite conductivity limit.

\[ \frac{\partial \vec{B}}{\partial t} - \vec{v} \times \vec{v} \times \vec{B} = 0 \]  

(3')

meaning of equation (3') will become more transparent, if one considers the change of magnetic flux \( \Phi \),
\[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \]  

(8)

\( S \) is a surface that moves along with the fluid velocity and is bounded by a fixed contour. The rate of change of flux of a vector field \( \mathbf{A} \) through a moving surface \( S \) (Smyrnov, 1964), is given by

\[ \frac{d}{dt} \int_S \mathbf{A} \cdot d\mathbf{S} = \int_S \left[ \frac{\partial \mathbf{A}}{\partial t} + \nabla \cdot (\mathbf{v} \times \mathbf{A}) + \mathbf{v} \times (\nabla \times \mathbf{A}) \right] dS \]

Hence,

\[ \frac{d}{dt} \Phi = \int_S \left[ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times \mathbf{v} \times \mathbf{B} \right] \cdot d\mathbf{S} \]  

(9)

Equation (9) states that for a perfectly conducting fluid the magnetic flux through any contour following the material motion remains constant in time. Alfven has described the consequence of (9) by saying that the magnetic lines of force are "frozen" into the fluid.

2.3. Equilibrium for a Flat Disk

Spiral galaxies are known to be highly flattened structures with a thickness of typically \( \approx 400 \) pc and a radius \( R \approx 10 \) kpc, so that \( h/R \approx 0.04 \). They can thus be modelled as infinitesimally thin discs to the lowest order.

A cylindrical system of coordinates \((r, \psi, z)\) is adopted. Since the disk is infinitesimally thin, all the physical quantities are restricted to \( z=0 \) plane, i.e.

\[ \mathbf{J}(r, \psi, z, t) = 0 \]  

\[ \mathbf{J}(r, \psi, z, t) = \mathbf{J}(r, \psi, t) \delta(z) \]
\[ P(\gamma, \varphi, z, t) = \rho(\gamma, \varphi, t) \delta(z) \]

where, \( \sigma(\gamma, \varphi, t), J(\gamma, \varphi, t) \) are the surface matter density and surface current density. \( \rho \) is pressure per unit length in the plane of the disk. Further, 
\[ \mathbf{v} = \begin{pmatrix} v_x, v_y, 0 \end{pmatrix} \]

that, in a thin disk, material is confined to the \( (\gamma, \varphi) \) -plane and hence \( v_z = 0 \). Vorticity of the fluid is perpendicular to the plane of the disk. Therefore, both the \( r \) and \( \varphi \) components of \( \nabla \times \mathbf{v} \) vanish.

\[ (\nabla \times \mathbf{v})_\varphi = 0 \]

and,
\[ (\nabla \times \mathbf{v})_r = 0 \]

From where, one gets (since \( v_z = 0 \))
\[ \frac{\partial v_x}{\partial z} = 0 \]

and,
\[ \frac{\partial v_\varphi}{\partial z} = 0 \]

Therefore,
\[ \mathbf{v} = \begin{pmatrix} v_x(\gamma, \varphi, t), v_\varphi(\gamma, \varphi, t), 0 \end{pmatrix} \]

Momentum balance equation (1) can be written as:
\[ \delta(2) \delta \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p \delta(2) - \sigma \delta(2) \frac{Gn}{\gamma} + \nabla \times \mathbf{B} \delta(2) \]

Which after integration over \( z \) gives:
The continuity equation (2)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot ( \rho \mathbf{u} ) = 0 \]

likewise is

\[ \text{s(2)} \frac{\partial \sigma}{\partial t} + \frac{1}{\gamma} \left[ \frac{\partial}{\partial \gamma} \left( \gamma \sigma \mathbf{u}_\gamma \right) + \frac{\partial}{\partial \varphi} \left( \sigma \mathbf{u}_\varphi \right) \right] = 0 \]

So that, after integration over \( z \), it becomes

\[ \frac{\partial \sigma}{\partial t} + \frac{1}{\gamma} \left[ \frac{\partial}{\partial \gamma} \left( \gamma \sigma \mathbf{u}_\gamma \right) + \frac{\partial}{\partial \varphi} \left( \sigma \mathbf{u}_\varphi \right) \right] = 0 \]

The induction equation is

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{u} \times \mathbf{B} \]

While the incompressibility condition (4) is:

\[ \frac{\partial}{\partial \gamma} \left( \gamma \mathbf{u}_\gamma \right) + \frac{\partial \mathbf{u}_\varphi}{\partial \varphi} = 0 \]

Writing equilibrium induction equation in cylindrical \((r, \varphi, z)\) coordinate

\[ \frac{\partial}{\partial \gamma} \left( \mathbf{B}_{\varphi} \mathbf{V}_0 \right) + \frac{\partial}{\partial \varphi} \left( \mathbf{V}_0 \mathbf{B}_{0z} \right) = 0 \]
and using $\nabla \cdot B = 0$, i.e.

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r B_{\theta r} \right) + \frac{\partial B_{\theta \phi}}{\partial z} = 0$$

Plugging it in the induction equation, one gets:

$$- B_{\theta r} \frac{V_0}{r} + B_{\theta \phi} \frac{dV_\phi}{dz} = 0$$

From where,

$$V_0 = \gamma \Omega_0$$  \hfill (10)

i.e. the only permissible consistent solution of Ohm's law is rigid rotation of the disk. Taking into account the rigid rotation of the disk and integrating the momentum balance equation over $z$, one gets the following equilibrium equation.

$$- \sigma_0 \gamma \Omega_0^2 = - \frac{dP_0}{d\gamma} - \frac{GM}{r^2} \sigma_0 + J_0 \gamma B_{0z}$$

It was seen in the last chapter that the dynamos cannot be sustained without differential rotation. On the other hand as noted above, differential rotation would require a non-zero electrical resistivity for an equilibrium solution. It would then appear that no dynamo equation of magnetic field would be possible with infinite conductivity which leads to a rigidly rotating disk. We shall, however, see that in the present self-consistent dynamical treatment of the problem the gravitational energy is converted into the
magnetic energy through the agency of the Rayleigh-Taylor instability, and the rigid rotation is no constraint. Magnetic perturbation grow even in the rigidly rotating equilibrium disk.

To know the equilibrium magnetic field, one solves the Maxwell's equation for the equilibrium current in the following way

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \]

Using \( \mathbf{B} = \nabla \times \mathbf{A} \), one gets,

\[ \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j} \]  \hspace{1cm} (6')

For azimuthal current,

\[ \left[ \nabla^2 \mathbf{A} \right] \varphi = \int_0^{\infty} J_0 \varphi(r) \delta(z) \]

where \( J_0 \varphi \) is the equilibrium azimuthal current. Solution of the above Poisson equation is (Panofsky and Phillips, 1964).

\[ A_\varphi(r, z) = \int \int dk' dk'' \left( \mathcal{J}_1(kr') \mathcal{J}_1(kr) e^{-ik'z} \right) \]

\[ \varphi(r', z') \]  \hspace{1cm} (11)

With

\[ B_{0\gamma}(\gamma, z) = -\frac{\partial}{\partial z} A_\varphi(\gamma, z) \]

\[ B_{0z}(\gamma, z) = \frac{B_{0\gamma}(\gamma, z)}{\varphi(\gamma, z)} \]

We see that the radial component of the magnetic field changes sign across \( z = 0 \) i.e., \( B_{0\gamma}(\gamma, z = 0) = 0 \). On the other hand, \( B_{0z}(\gamma, z) \) remains continuous throughout. Source of
discontinuity of $B_\theta$ is azimuthal current. This follows from integration of Maxwell's equation over $z$:

$$n J_0 \psi (r) \delta (z) = \frac{\partial B_0 }{\partial z} - \frac{\partial B_0 }{\partial r}$$

Integrating over $z$ from $z = -\epsilon$ to $z = +\epsilon$, then yields

$$2 B_\gamma (r, \pm 0) = \int n J_0 \psi (r)$$

(12)

Since

$$\lim_\varepsilon \int_{-\varepsilon}^{+\varepsilon} B_{0z} (r, z) \, dz = \lim_{\varepsilon \to 0} \left[ B_z (r, z) \frac{\partial z}{\partial z} \right]_{-\varepsilon}^{+\varepsilon} = 0$$

Writing momentum balance equation with $\frac{\partial \phi}{\partial r} = 0$ and azimuthal current and rotational velocity $V_0 = r \Omega$:

$$- \sigma_0 r^2 \Omega = - \frac{dV_0}{d\gamma} - \sigma_0 \frac{G M}{r^2} + 2 \frac{B_0 }{r} \frac{B_0 }{r}$$

where the use has been made of (11). Equation of radial equilibrium, can be rewritten as:

$$\sigma_0 \left( \frac{G M}{r^2} - r ^2 \Omega ^2 \right) = - \frac{dV_0}{d\gamma} + 2 \frac{B_0 }{r} \frac{B_0 }{r}$$

(13)

2.4. Normal Mode Analysis

As we have seen above, the equilibrium disc has an axisymmetric poloidal magnetic field supported by an azimuthal current density. This current density $J_0 \psi$ then
supports the imbalance between the pressure gradient, gravitational force and the inertial force. Such a disc could be Rayleigh-Taylor unstable to global perturbations if there exist inverted density gradients somewhere in the disc. A global mode of magnetic perturbation could thus grow at the expense of the gravitational energy of the system and could be identified with the observed global structure if it has an appropriate azimuthal symmetry. To this end we study the stability of the disc.

We study the stability of this thin magnetized plasma disk by using normal mode analysis.

Now, consider a non-axisymmetric perturbation in the density, pressure, current and velocity over its equilibrium values in the disk. Perturbation in current in turn causes a perturbation in the magnetic field. We assume all the perturbed quantities to be small compared to the corresponding equilibrium quantities, so that a linearization around unperturbed quantities can be carried out. Thus we write the net quantities as

\[\begin{align*}
\sigma (\gamma, \psi, t) &= \sigma_0 (\gamma) + \epsilon \tilde{\sigma} (\gamma, \psi, t) \\
\bar{P} (\gamma, \psi, t) &= \bar{P}_0 (\gamma) + \epsilon \tilde{\bar{P}} (\gamma, \psi, t) \\
\bar{r} (\gamma, \psi, t) &= \bar{r}_0 (\gamma) + \epsilon \tilde{\bar{r}} (\gamma, \psi, t) \\
\bar{J} (\gamma, \psi, t) &= \bar{J}_0 (\gamma) + \epsilon \tilde{\bar{J}} (\gamma, \psi, t) \\
\bar{B} (\gamma, \psi, z, t) &= \bar{B}_0 (\gamma, z) + \epsilon \tilde{\bar{B}} (\gamma, \psi, z, t)
\end{align*}\]

(14)

Where,

\[\tilde{\bar{J}} (\gamma, \psi, t) = (\tilde{J}_\gamma, \tilde{J}_\psi, 0)\]
\[ v'(r, \theta, t) = (\tilde{v}_r, \tilde{v}_\theta, 0) \]

where \( \rho, p, \tilde{v}, J \) and \( B \) represents the surface density, pressure, velocity, surface current density and magnetic field respectively. We recall that the velocity in equilibrium is entirely azimuthal, and corresponding to a rigid rotation.

Substituting (14) in equation (1)-(6) and linearizing and integrating over \( z \), we get the following equations for the perturbed quantities:

\[
\begin{align*}
\sigma_e \left[ \left( \frac{\partial}{\partial t} + \Omega_o \frac{\partial}{\partial \gamma} \right) \tilde{v}_r - \gamma \left( \Omega e \right) \tilde{v}_\gamma \right] - \gamma \Omega_o \frac{\partial}{\partial \gamma} \tilde{v}_r &= - \frac{c \tilde{h}}{\sigma_1} \\
- \frac{\gamma}{\gamma} \frac{GM}{r^2} + \frac{2 B_{01} \tilde{h}_z}{r} + \frac{2 B_{01} B_{02}}{r} \\
\sigma_o \left[ \left( \frac{\partial}{\partial t} + \Omega_o \frac{\partial}{\partial \gamma} \right) \tilde{v}_\gamma + 2 \Omega_o \tilde{v}_r \right] &= - \frac{1}{\gamma} \frac{\partial \tilde{h}}{\partial \gamma} + \frac{2 b_{21} B_{02}}{r} 
\end{align*}
\]

(15)

Induction equation (3) becomes

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \Omega_o \frac{\partial}{\partial \gamma} \right) \tilde{b}_r &= \tilde{v}_r \left( \frac{\partial B_{02}}{\partial \gamma} \right)_{\gamma=0} - B_{01} \frac{\partial \tilde{v}_r}{\partial \gamma} \\
\left( \frac{\partial}{\partial t} + \Omega_o \frac{\partial}{\partial \gamma} \right) \tilde{b}_\gamma &= \tilde{v}_\gamma \left[ \left( \frac{\partial B_{02}}{\partial \gamma} \right)_{\gamma=0} + \frac{\partial}{\partial \gamma} (\tilde{v}_r B_{02}) \right] \\
\left( \frac{\partial}{\partial t} + \Omega_o \frac{\partial}{\partial \gamma} \right) \tilde{b}_z &= - \frac{1}{\gamma} \left[ \frac{\partial}{\partial \gamma} (r \tilde{v}_r B_{02}) + \frac{\partial}{\partial \gamma} (\tilde{v}_\gamma B_{02}) \frac{\partial}{\partial \gamma} \right]
\end{align*}
\]

(16)

where use has been made of the fact

\[ \tilde{j}_\gamma(r) = \sigma_0 \tilde{b}_r \]
\[
\tilde{J}_\gamma (\tau) = -2 \sqrt{n} \tilde{b}_\gamma (\tau, 0)
\]

where tilda denotes the perturbed quantities whereas index 0 means equilibrium quantities.

The above set of partial differential equations (15-16) involves terms with coefficients which are constant with respect to \( \varphi \) and \( t \) but depend on \( r \) explicitly. We can then Fourier analyze with respect to the azimuthal angle \( \varphi \) and time \( t \), and thus seek a solution in the form:

\[
\tilde{A}(\gamma, \varphi, t) = \tilde{A}(\gamma) \exp \left[ i (\omega t - m \varphi) \right]
\]

(17)

Where \( \tilde{A} \) stands for any perturbed quantity. The resulting set of equations can be reduced to one ordinary differential equation in terms of radial velocity \( \tilde{v}_r \), which must be solved as an eigenvalue problem with appropriate boundary conditions.

In equation (17) \( \omega \) is the frequency of the perturbation and in general, is complex; \( m \) is the azimuthal wave number. The frequency \( \omega = \omega_\gamma + i \omega \gamma \) such that the pattern-velocity \( \omega_\gamma = \omega \gamma / m \) and \( \omega \gamma \) represents the temporal growth rate of the perturbation. Using equation (17) along with \( \frac{1}{r} \frac{\partial}{\partial r} (r \tilde{v}_r) = \frac{im}{r} \tilde{v}_\varphi \), set of equations (15-16) can be written in the following form:
Continuity
\[ i \left( (\omega - m_0 c^2) \tilde{\nu} + \tilde{\nu}_y \frac{d \tilde{\nu}}{d \gamma} \right) = 0 \]

Equations of motion:
\[ \tilde{\nu}_y \left[ i \left( \omega - m_0 c^2 \right) \tilde{\nu}_y - 2 \tilde{\nu}_0 \tilde{\nu}_y \right] + \tilde{\nu}_y \left( \frac{\vec{G} N}{\gamma} - \nu \tilde{\nu}_0 \right) = - \frac{d \tilde{\nu}_y}{d \gamma} + \frac{2 \tilde{B}_0 \tilde{b}_2}{\gamma} + \frac{2 \tilde{b}_2 \tilde{B}_0}{\gamma} \]
\[ \tilde{\nu}_y \left[ i \left( \omega - m_0 c^2 \right) \tilde{\nu}_y + 2 \tilde{\nu}_0 \tilde{\nu}_y \right] = \frac{i m \tilde{\nu}_y}{\gamma} + \frac{2 \tilde{b}_2 \tilde{B}_0}{\gamma} \]

Induction equations:
\[ \begin{align*}
\frac{d}{d \gamma} \left( \omega - m_0 c^2 \right) \tilde{b}_y &= - \tilde{\nu}_y \left( \frac{d \tilde{\nu}_y}{d \gamma} \right) + \frac{i m \tilde{\nu}_y \tilde{B}_0}{\gamma} \\
\frac{d}{d \gamma} \left( \omega - m_0 c^2 \right) \tilde{b}_\varphi &= \gamma \tilde{B}_0 \left( \tilde{\nu}_y \frac{d \tilde{\nu}_y}{d \gamma} \right) \\
\frac{d}{d \gamma} \left( \omega - m_0 c^2 \right) \tilde{b}_2 &= - \tilde{\nu}_y \left( \frac{d \tilde{\nu}_y}{d \gamma} \right) - \frac{2 \tilde{B}_0 \tilde{b}_2}{\gamma} 
\end{align*} \]

Pressure \( p \) is eliminated between the \( \gamma \) and \( \varphi \) component of the momentum balance equation. Also using the induction equation (18) to eliminate \( \tilde{b}_y, \tilde{b}_\varphi \) and \( \tilde{b}_2 \) we get the following differential equation for \( \tilde{\nu}_y \),
\[ \begin{align*}
\frac{d^2}{d \gamma^2} \left( \omega - m_0 c^2 \right) \left[ \left( \omega - m_0 c^2 \right)^2 + \frac{2 \tilde{B}_0 \tilde{b}_2}{\gamma} \right] \\
+ \frac{d}{d \gamma} \left( \omega - m_0 c^2 \right) \left[ \left( \omega - m_0 c^2 \right)^2 \left( \frac{2 \tilde{B}_0 \tilde{b}_2}{\gamma} + 1 \right) - \frac{2 \tilde{B}_0 \tilde{b}_2}{\gamma} \right]
\end{align*} \]
\[ - \left( \frac{m}{\gamma} \right)^2 \left( \gamma \dot{\gamma} \right) \left( \omega - m \Omega_0 \right)^2 - \frac{2 \left( \omega - m \Omega_0 \right) \gamma \dot{\gamma} \sigma_0}{m \sigma_0} \frac{d \sigma_0}{d \gamma} + \frac{1}{\sigma_0} \frac{d \sigma_0}{d \gamma} \left( \frac{GM}{\gamma^2} - \gamma \Omega_0^2 \right) - \frac{2 B_{0Y} \frac{dB_{0Y}}{d \gamma}}{\mu_0 \varepsilon_0} \right] = 0 \] (19)

2.5. Discussion

If for simplicity, one takes \( \gamma \dot{\gamma} = \)constant and \( \frac{1}{\sigma_0} \frac{d \sigma_0}{d \gamma} \ll 1 \), then one gets

\[ (\omega - m \Omega_0)^2 = - \frac{1}{\sigma_0} \frac{d \sigma_0}{d \gamma} \left( \frac{GM}{\gamma^2} - \gamma \Omega_0^2 \right) + \frac{2 B_{0Y} \frac{dB_{0Y}}{d \gamma}}{\mu_0 \varepsilon_0} \] (20)

which can be rewritten in terms of pressure gradient and magnetic field equation (13)

\[ (\omega - m \Omega_0)^2 = \frac{1}{\sigma_0^2} \frac{d \sigma_0}{d \gamma} \frac{d \sigma_0}{d \gamma} + \frac{2 B_{0Y} \frac{dB_{0Y}}{d \gamma}}{\sigma_0} \frac{d \sigma_0}{d \gamma} \left( \frac{B_{0Y}^2}{\varepsilon_0} \right) \] (21)

Relation (20) determines the stability of the system. Writing \( \omega = \omega_0 + i \omega_i \), the time dependence of the solution is

\[ e^{i \omega t} = e^{i \omega_0 t} e^{i \omega_i t} \]

and hence perturbation grows and mode is unstable if \( \omega_i < 0 \). From (20), it is clear that, for instability

\[ \frac{2 B_{0Y} \frac{dB_{0Y}}{d \gamma}}{\mu_0 \varepsilon_0} \left( \frac{B_{0Y}^2}{\varepsilon_0} \right) < \frac{1}{\sigma_0} \frac{d \sigma_0}{d \gamma} \left( \frac{GM}{\gamma^2} - \gamma \Omega_0^2 \right) \]
or, from equation (21)

\[
\frac{2 B_{0y} \frac{d}{d\gamma} \left( \frac{B_{0z}}{\sigma_c} \right)}{\gamma} < - \frac{1}{\sigma_c^2} \frac{d\sigma_c}{d\gamma} \frac{dP_c}{d\gamma}
\]

Therefore, growth rate is given by

\[
\left[ -\frac{1}{\sigma_c^2} \frac{d\sigma_c}{d\gamma} \left( \frac{GM}{\gamma} - \gamma \Omega_c^2 \right) + \frac{2 B_{0y} \frac{d}{d\gamma} \left( B_{0z} \right)}{\gamma} \right]^{1/2}
\]

From the above expression, we see that the instability grows faster for large positive density gradient. First term in the expression for the growth rate is density gradient times the imbalance of gravitational and centrifugal forces. As the matter spirals towards the bulge, amount of energy released is proportional to the imbalance of gravitational and centrifugal terms. This energy is converted into magnetic energy by twisting of the field lines by the infalling rotating matter.

Expressing through pressure gradient and magnetic force, from the equilibrium equation, the imbalance of gravitational and centrifugal force, one gets the growth rate as

\[
\left[ \frac{1}{\sigma_c^2} \frac{d\sigma_c}{d\gamma} \frac{d\rho_c}{d\gamma} + \frac{2 B_{0y} \frac{d}{d\gamma} \left( B_{0z} \right)}{\gamma} \right]^{1/2}
\]

i.e. instability grows faster for the steeper pressure gradient. Assuming \( p = \sigma_c^2 \rho_c \) and taking \( \sigma_c^2 = 57 \times 10^3 \text{ cm}^2/\text{s} \) and \( \sigma_c^2 = 11 \times 10^5 \text{ cm}^2/\text{s} \) for HI and HII region respectively (Spitzer, 1968) and corresponding spread of this region to
be \( L_{\text{HI}} = 3 \times 10^{19} \) cm and \( L_{\text{HII}} = 3 \times 10^{20} \) cm, one gets the growth rate
\[
\left[ \text{Im}(\omega - m \omega_0) \right]_{\text{HI region}} \approx \frac{C_S}{R} \approx 6 \times 10^{-7} \gamma^{-1}
\]

Therefore, for over the galactic life time \( \sim 10^{10} \) Yr field amplification is \( \sim \text{Exp}^{60} \). For \( t_{\text{galaxy}} \sim 10^8 \) Yr field amplification is \( \sim \text{Exp}^{0.6} \) times only.
\[
\left[ \text{Im}(\omega - m \omega_0) \right]_{\text{HI region}} \approx 10^{-8} \gamma^{-1}
\]

Over the galactic life time field gets amplified \( \text{Exp}^{100} \) times. In the case of high red-shift (\( Z = 4 \)) when age of the galaxy was \( \sim 10^8 \) Yr, field amplification \( \sim \text{Exp}^{(1-2)} \).

A simplification can be affected in equation (19) by invoking a WKBJ approximation and writing
\[
\tilde{A}(\gamma) = \hat{A}_0(\gamma) \text{Exp} \left[ i \int k(\gamma') d\gamma' \right] \quad (22)
\]

Where \( \hat{A}_0(\gamma) \) is a very slowly varying function, whereas the exponential (phase) part is a rapidly varying function of \( \gamma \), that is, \( k_\gamma \gg 1 \).

Assuming \( S = \gamma \tilde{V}_\gamma \) and expressing \( S \) as in (22), we can write the dispersion relation, after simple algebraic manipulation, in the following form:
\[
(\omega - m \omega_0)^2 \left[ k^2 - i k (\gamma \frac{d}{d\gamma} \ln \rho_0) \right] + \frac{\kappa^2 \rho_0}{m \rho_0} \frac{d \rho_0^2}{d\gamma} = 0
\]
From where
\[
(\omega - \omega_0 \sigma_0)^2 = \left( \frac{2B_{0x} \frac{dB_{0x}}{d\gamma}}{\frac{m}{\sigma_0} \frac{d\sigma_0}{d\gamma}} \right) \frac{k}{k\sigma_0 - \frac{i}{\sigma_0} \frac{d\sigma_0}{d\gamma}}
\]
and from equilibrium
\[
\frac{2B_{0x}}{m} \frac{dB_{0x}}{d\gamma} = \frac{GM}{\gamma^2} \left( \frac{\sigma_0 \frac{d\sigma_0}{d\gamma} - 2}{\sigma_0} \right) + \\
+ \gamma \sigma_0^2 \left( \frac{\sigma_0 \frac{d\sigma_0}{d\gamma} - 1}{\sigma_0} \right) + \frac{\gamma}{\sigma_0} \frac{d^2 p_0}{d\gamma^2} + \\
\gamma \frac{2\gamma B_{0x}^2}{m} \frac{dB_{0x}}{d\gamma}
\]
Therefore,
\[
(\omega - \omega_0 \sigma_0)^2 = - \left\{ \frac{k}{k\sigma_0 - \frac{i}{\sigma_0} \frac{d\sigma_0}{d\gamma}} \right\}^2 \left( \frac{GM}{\gamma^2} - \frac{\sigma_0 \frac{d^2 p_0}{d\gamma^2}}{\sigma_0} - \gamma \frac{2\gamma B_{0x}^2}{m} \frac{dB_{0x}}{d\gamma} \right)
\]
Growth rate is given by
\[
\left[ \left\{ \frac{k}{k\sigma_0 - \frac{i}{\sigma_0} \frac{d\sigma_0}{d\gamma}} \right\} \left( \frac{GM}{\gamma^2} - \frac{\sigma_0 \frac{d^2 p_0}{d\gamma^2}}{\sigma_0} - \gamma \frac{2\gamma B_{0x}^2}{m} \frac{dB_{0x}}{d\gamma} \right) \right]^{1/2}
\]
We see that as \( k \to \infty \), \( \omega \to \infty \), for the short wavelength, instability grows very fast. Magnetic filed growth is caused by the infalling matter towards the center.

The above study indicates that the density gradient of
the ionized gas in the disk and the pressure gradient causes the growth of the magnetic field in the disk. Field draws its energy from the "density and pressure gradients" of the gas. Steeper the change in density of the gas, faster the instability grows. This instability should have been anticipated on the ground that in the galactic disk, the arrangement is "top-heavy" of a magnetized plasma against a gravitational field.

2.6. Variational Technique

We now come to a technique that is often used in stability analysis. The technique is to construct a variational form of the eigen-equation (19) since it is difficult to construct even an approximate solution for it as an eigenvalue problem. We need to use a generalized variational technique, associated with the eigenenergy of a system to the eigenvalue equation. Following Moisewitch (variational principles), we should construct our eigen equation in the form:

\[ L \phi = \lambda M \phi \]  \hspace{1cm} \text{(23)}

where \( L \) and \( M \) denotes differential operators. Introducing adjoint operators

\[ \tilde{L} \tilde{\phi} = \lambda \tilde{M} \tilde{\phi} \]  \hspace{1cm} \text{(24)}

where the adjoint operators \( \tilde{L} \) and \( \tilde{M} \) together with the adjoint solution \( \tilde{\phi} \) is defined in such a way that the
\[
\oint \mathbf{L} S d\tau = \int S \mathbf{L} S d\tau \quad , \int S MS d\tau = \int S \mathbf{M} S d\tau
\]  

(25)

are satisfied, the integration being performed over the volume.

Multiplying both sides of (23) by \(\mathbf{S} \) and integrating over yields

\[
\lambda = \mathcal{I} [S]
\]  

(26)

where,

\[
\mathcal{I} [S] = \frac{\oint SLS d\tau}{\int S MS d\tau}
\]  

(27)

The change in \(\mathcal{I} [S]\) due to infinitesimal variation \(\delta S\) and \(\delta \mathbf{S}\) in \(S\) and \(\mathbf{S}\) respectively is

\[
\delta \mathcal{I} [S] = \mathcal{I} [S] \oint S MS d\tau + \int [S] \oint SMS d\tau + \mathcal{I} [S] \oint S MS d\tau = \oint \delta SLS d\tau + \oint \delta L \delta S d\tau
\]

neglecting \(O(\delta S^2)\) quantities. Provided \(\delta S\) and \(\delta \mathbf{S}\) satisfies appropriate boundary conditions, one can put

\[
\oint \delta L \delta S d\tau = \oint \delta S \mathbf{L} \mathbf{S} d\tau
\]

and

\[
\oint \delta S \mathbf{M} S d\tau = \oint \delta S \mathbf{M} S d\tau
\]

and therefore

\[
\delta \mathcal{I} [S] = \oint \delta S (LS - 2MS) d\tau + \oint \delta S (L S - \mathbf{L} \mathbf{S} - \mathbf{M} S) d\tau
\]

\[
\oint S MS d\tau
\]
We know that $S$ and $S$ are solutions of (23) and (24) respectively,

$$\tilde{S}[S] = 0$$

From the above description of the variational principle, it is clear that we need to reformulate our eigen-equation (19) in the form

$$L_S = \lambda MS$$

Let us denote

$$S = \tau \tilde{\nu}_\gamma, \quad \lambda = (\omega \cdot m - \omega)^2$$

and

$$R = \frac{\partial B_\gamma}{\partial \sigma_0} \frac{d B_\gamma}{d \gamma}$$

and then (19) becomes

$$\frac{d^2 S}{d \gamma^2} \left[ \lambda + R(\gamma) \right] + \frac{1}{\tau} \frac{d S}{d \gamma} \left[ \lambda \left( \frac{\sigma_0}{\sigma_0} \frac{d \sigma_0}{d \gamma} + 1 \right) - R \right] =$$

$$- \left( \frac{\omega \cdot m}{\gamma} \right)^2 S \left[ \lambda - \frac{2 (\omega \cdot \gamma)}{m} \frac{\sigma_0}{\sigma_0} \frac{d \sigma_0}{d \gamma} + \frac{1}{\sigma_0^2} \frac{d \sigma_0}{d \gamma} \left( - \frac{d \log \sigma_0}{d \gamma} + \frac{2 B_0 B_\gamma}{m} \right) \right] - R(\gamma) = 0$$

Denoting by

$$L = \frac{d^2}{d \gamma^2} - \frac{1}{\tau} \frac{d}{d \gamma} - \left( \frac{\omega \cdot m}{\gamma} \right)^2 \left[ \frac{1}{R \sigma_0} \frac{d \sigma_0}{d \gamma} \left( \frac{d \log \sigma_0}{d \gamma} + \frac{2 B_0 B_\gamma}{m} \right) \right]$$

and assuming $\frac{\gamma \frac{d \sigma_0}{d \gamma}}{\sigma_0} \ll 1$, we can rewrite the eigen-equation
in the form (23). Differential operators \( L \) and \( M \) are not self-adjoint and hence equation (24) cannot be satisfied.

To make \( L \) self-adjoint, one multiplies \( L \) by \( 1/r \). Therefore, \( \tilde{L} = \frac{1}{r} \). But \( M/r \) is not self-adjoint operator, and to make it self-adjoint, one assumes that

\[
R(\gamma) = \frac{2 B_0}{m \sigma_0} \frac{d B_0}{d \gamma} = \frac{1}{\gamma^2}
\]

Then,

\[
\tilde{M} = - \left[ - \gamma \frac{d^2}{d \gamma^2} + \frac{d}{d \gamma} - \gamma \left( \frac{w}{\gamma} \right)^2 \right]
\]

and, therefore:

\[
\tilde{L} \tilde{S} = \lambda \tilde{M} \tilde{S}
\]

Equation (25) is automatically satisfied. In our problem, which possesses cylindrical symmetry, (25) can be rewritten as

\[
\int \tilde{S} L \tilde{S} r \, d\gamma = \int \tilde{S} \tilde{L} \tilde{S} r \, d\gamma
\]

and,

\[
\int \tilde{S} \tilde{M} \tilde{S} r \, d\gamma = \int \tilde{S} \tilde{M} \tilde{S} r \, d\gamma
\]

and,

\[
\lambda = \int \tilde{S} \tilde{L} \tilde{S} r \, d\gamma
\]

\[
\int \tilde{S} \tilde{M} \tilde{S} r \, d\gamma
\]

We have to guess a function \( S = f(\gamma, \alpha) \) and from extremum of the functional \( \mathcal{I}[\alpha] \), i.e. \( \frac{\partial \mathcal{I}}{\partial \alpha} = 0 \) one should find \( \alpha \). For \( m=0 \), we have constructed such a solution

\[
\tilde{V}_\gamma = - \gamma \ln \gamma
\]
Plugging this solution in (25), one gets an upper bound on \( \lambda \) i.e. on \( (\omega - \omega_0)^2 \)

\[
\lambda \leq 2.31
\]

or

\[
\epsilon^2 \lambda \leq 4.34 \times 10^{-9} \gamma^{-1}
\]

For axisymmetric mode, a solution of the eigen-equation goes to zero at the origin and at the boundary of the disk (Fig 2.1). An upper bound to the eigenenergy gives the minimum growth rate of the axisymmetric mode.

2.7. A More General Treatment

We study the equilibrium of a thin disk, as described in section 2.3, with a more general equation of state. We have assumed a polytropic equation of the form

\[
P = c \rho \gamma
\]  

(28)

Where, \( C \) and \( \gamma \) are constants. The acoustics velocity in the disk is given by

\[
\epsilon^2 = \frac{\partial P}{\partial \rho} = C \gamma \epsilon \gamma^{-1}
\]

The parameter \( C \) represents the measure of the randomness or "hotness" of the gas and \( \gamma \) is the polytropic index for
two-dimensional systems. Equation (28) can be related to an ordinary polytropic relation for the three dimensional medium of the form:

$$\Pi = k \int \rho \, d^3 \vec{x}$$

Where $\Pi$ and $\rho$ are pressure and volume density respectively and $K$ and $\alpha$ are constants. Hunter (1972) has obtained the relation

$$\gamma = 3 - \frac{2}{\alpha}, \quad C = \frac{\Pi \gamma - \frac{1}{\alpha}}{2^2 - \frac{1}{\alpha}} = \frac{\Gamma(2 - \frac{1}{\alpha})}{\Gamma(3 - \frac{1}{\alpha})}$$

by taking a limit case of an infinitely thin disk. Here $\Gamma$ is a gamma function. Thus, the case with $r=1$ and $r=3$ in equation (28) corresponds to the equation of state (29) with $\alpha = 1$ (isothermal) and $\alpha = \infty$ (incompressible), respectively. From equation of state (28) assuming $\gamma = 1$ (isothermal case), we can rewrite above equation as:

$$\frac{d\omega_v}{dt} + \sigma_v \left( \frac{G_v}{r_s^2} - \frac{v_s^2}{c_s^2} \right) = \frac{2\mu B_{01} B_{02}}{r_s c_s^2}$$

(13')

Solving above equation, one gets the self-consistent expression for the surface density in the disk:
\[ \sigma_{0}(r) = \frac{1}{2n \cos^{2} \xi} \exp \left( \frac{GM}{r \cos^{2} \xi} + \frac{(2 \Omega r)^{2}}{2 \sqrt{r^{2} + \xi^{2}}} \right) \int_{0}^{\infty} e^{-\left( \frac{(2 \Omega r)^{2}}{2 \sqrt{r^{2} + \xi^{2}}} - \frac{GM}{r \cos^{2} \xi} \right)} \right) \times \frac{\mathbf{B}_{0z}}{B_{0z}} \, d\xi \] \hspace{1cm} (30)

Equation (30) can be employed to construct some models with special current density distribution in finite as well as infinite disks. We have considered one such model in equilibrium, before studying their stability against various perturbation.

For regular current distributions, such as

\[ J_{0}^{(r)}(\gamma) = \begin{cases} J_{0} \left( \frac{\gamma}{R} \right) \left( 1 - \frac{\gamma^{2}}{R^{2}} \right) & \gamma \leq R_{d \infty} \\\n & \gamma > R_{d \infty} \end{cases} \]

z-component of the magnetic field is found to change sign along the disk. The circumstance is not occasional: it may be shown (Freedman and Polyachenko, 1984) that, at any regular current distribution, the field \( B_{0z} \) necessarily changes sign at \( r < R_{d} \). This fact happens to be important in the analysis of the stability of such systems. It is very easy to prove that for any regular current, \( B_{0z} \) changes sign. We use for \( J_{0}^{(r)} \) and \( B_{0z} \) somewhat different representations:

\[ J_{0}^{(r)} = \frac{1}{\pi} \sqrt{1-\eta^{2}} \sum_{n=1}^{\infty} \frac{\mathbf{d}_{2n}}{\gamma_{2n}} P_{2n}(\eta) \]

\[ B_{0z} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\mathbf{d}_{2n}}{\gamma_{2n}} P_{2n}(\eta) \]

where,

\[ \gamma_{2n} = \eta(2n-1) \left[ \frac{(2n-1)!!}{(2n)!!} \right]^{2} \]
\[ \eta = \sqrt{1 - \frac{\gamma^2}{R_{\text{disk}}^2}} ; \quad P_{2n}'(\eta) \text{ is the derivative of the Legendre polynomial.} \]

In the expression for current and magnetic field, \( n=0 \) term is not considered, since it would correspond to singular term. This can be proved by rule of contraries, i.e., assume that \( B_{oz} \) does not change sign on the disk, say \( B_{oz} > 0 \). Then integrating \( B_{oz} \) from 0 to 1 over \( d\eta \)

\[ 0 < \int_{0}^{1} B_{oz} \, d\eta = \frac{1}{m} \sum g_{2n} \int_{0}^{1} P_{2n} \, d\eta = -\sum g_{2n} \int_{0}^{1} \frac{\sum_{l=1}^{n} (\eta^2 - l^2) P_l'(\eta)}{2l(2l+1)} \, d\eta \]

Thus, the component \( B_{oz} \) of a self-consistent magnetic field ought to change sign on the disk.

It is clear from equation (30) that integrand

\[ B_{ov} B_{oz} \in -\left( \frac{\partial^2 \xi^2}{\partial \xi^2} + \frac{GM}{\omega^2 \xi} \right) \]

will change sign along the disk making the density unphysical. Therefore regular current distribution can not be taken as equilibrium current and we take the current distribution of the following form (like Yobushita's, 1969 density distribution)

\[ J_{ov}(r) = \begin{cases} \sum_{i=0}^{\infty} \alpha_j \, J_1(k_i r) & ; \, r < R_{\text{disk}} \\ 0 & ; \, r > R_{\text{disk}} \end{cases} \]

where \( k_j \)'s are zero's of the transcendental equation

\[ J_1(k_j R_{\text{disk}}) = 0 \]
to ensure that the surface current density vanishes at the boundary of the disk.

We consider a particular current, with $j=1$ and $a_j=1$. $k = 3.8317$ is the first zero of the Bessel function. Plugging $j_0(\gamma) = J_0(k_0\gamma)$ in the expression (11), one gets the vector potential

$$\mathbf{A}_0(\gamma) = \left( \frac{2 \bar{\gamma}_c}{k_0} \frac{J_1(k_0\gamma)}{J_0(k_0\gamma)} e^{k_0z} \right)$$

From where,

$$\mathbf{B}_{0\gamma} \bigg|_{z=0} = \left( \frac{2 \bar{\gamma}_c}{k_0} \frac{J_1(k_0\gamma)}{J_0(k_0\gamma)} \right)$$

and,

$$\mathbf{B}_{0\gamma} \bigg|_{z=0} = \left( \frac{2 \bar{\gamma}_c}{k_0} \frac{J_1(k_0\gamma)}{J_0(k_0\gamma)} \right)$$

From above expression for $\mathbf{B}_{or}$ and $\mathbf{B}_{oz}$ we see that integrand $\mathbf{B}_{0\gamma} \cdot \mathbf{B}_{0\gamma} \cdot \rho \left( \frac{\Omega^2 \xi^2}{c_n^2} - \frac{G_n}{c_s^2} \right)$ is positive definite throughout the disk i.e.:

$$\int \mathbf{B}_{0\gamma} \cdot \mathbf{B}_{0\gamma} \cdot \rho \left[ \frac{\Omega^2 \xi^2}{c_n^2} - \frac{G_n}{c_s^2} \right] d\xi > 0$$

The formalism for the above current density distribution, may be extended to other current distribution. We have not taken any other current distribution since the computational time required even with this model is large.

2.8. Non-Dimensional Equation

Non-dimensional function $\hat{\mathbf{A}}(\xi), \hat{\mathbf{B}}(\xi), \hat{\mathbf{J}}(\xi), \mathbf{B}(\xi)$ and $\mathbf{V}(\xi)$ are introduced corresponding to the surface density, pressure
current density, magnetic field and velocity as follows:

\[ \mathbf{G}(r) = \left( \frac{M}{2 \pi R^2} \right) \hat{r} (\hat{r} \cdot \mathbf{E}(r)) \]
\[ \mathbf{P}(r) = \left( \frac{GM^2}{2R^3} \right) \hat{r} (\mathbf{E}(r) \times \mathbf{B}(r)) \]
\[ \mathbf{J}(r) = \frac{M}{R^2} \left( \frac{G}{2\pi \mu_0} \right)^{\frac{1}{2}} \hat{r} \times \mathbf{E}(r) \]
\[ \mathbf{B}(r) = \frac{M}{R^2} \left( \frac{GM}{2\pi \mu_0} \right)^{\frac{1}{2}} \hat{r} \times \mathbf{E}(r) \]

(31)

and

\[ \nabla(r) = \left( \frac{GM}{R} \right)^{\frac{1}{2}} \nabla \left( \frac{r}{R} \right) \]

where \( \xi = \frac{r}{R_d} \), \( R_d \) is the radius of the disk and \( M \) mass of the gravitational bulge. The angular velocity is measured in non-dimensional unit of time

\[ \tau = \left( \frac{GM}{R^3} \right) t \]

(32)

Substituting equation (29)-(30) in (1)-(2) and dropping the cap from \( \hat{r} \), \( \hat{\mathbf{P}} \) etc. and writing \( r \) for \( \xi \), \( t \) for \( \tau \), one obtains non-dimensionalized equation

\[ J_{0\phi}(r) = J_1(k_0 r) \]
\[ B_{0\phi}(r) = \frac{J_1(k_0 r)}{2} \]
\[ B_{0\phi}(r) = \frac{J_0(k_0 r)}{2} \]

and

\[ \sigma_{0\phi}(r) = \frac{1}{2} \mathbf{E} \times \mathbf{P} \left( \frac{r^2}{2} \right)^{\frac{1}{2}} \left( \int J_0(k_0 \xi) \xi J_0(k_x \xi) e^{-\left( \xi^2 \frac{1}{4} \right)} d\xi \right) \]
The above density distribution for \( \hat{\rho}_r = 1, K_0 = 3.8 \times 1 \), has been plotted in Fig. 2.2. Using (31) and (32) in equation (13), the non-dimensional radial equilibrium equation of motion becomes:

\[
- \frac{\nu_0^2}{\gamma} = - \frac{1}{\sigma_0} \frac{d\sigma_0}{d\gamma} - \frac{1}{\gamma} + \frac{2B_0 \beta_2}{\hat{\sigma}_0}
\]

The central bulge has been assumed here to be spherical and contribute a term \( r^{-2} \), at the plane of the disk in the above equation.

2.9. Solution of Maxwell's Equation and Dispersion Relation

Maxwell's equation (6') can be rewritten in the following form:

\[
\left\{ \begin{array}{l}
\frac{1}{\gamma} \frac{\partial}{\partial \gamma} \left( \gamma \frac{\partial F}{\partial \gamma} \right) - \left( \frac{m-1}{\gamma} \right)^2 F + \frac{\partial^2 F}{\partial \gamma^2} = - \left( \frac{J_1 \gamma}{i J_0} \right) \hat{\sigma} (x) \\
\frac{1}{\gamma} \frac{\partial}{\partial \gamma} \left( \gamma \frac{\partial Q}{\partial \gamma} \right) - \left( \frac{m+1}{\gamma} \right)^2 Q + \frac{\partial^2 Q}{\partial \gamma^2} = - \left( \frac{J_1 \gamma}{i J_0} \right) \hat{\sigma} (x)
\end{array} \right. \quad (33)
\]

where,

\[
F = A_y + i A_\varphi
\]

\[
Q = A_y - i A_\varphi
\]

and without loss of generality, one can assume \( A_z = 0 \) i.e.

\[
\nabla^2 A_z \equiv 0
\]


Integrating equation (33) over $z$, one gets the "jump" condition:

$$
\frac{\partial F}{\partial z} \bigg|_{z=0} = 2 \left( b_\phi - ib_\gamma \right)
$$

$$
\frac{\partial Q}{\partial z} \bigg|_{z=0} = 2 \left( b_\phi + ib_\gamma \right)
$$

(35)

where use has been made of the fact that radial and azimuthal component of the magnetic field suffers a discontinuity caused by the azimuthal and radial current respectively.

Solutions of equations (33), in the disk ($r<1$) can be written as

$$
F(r, z) = \sum_{m=0}^{\infty} E_m J_{m-1}(kr) e^{-k|z|}
$$

$$
Q(r, z) = \sum_{m=0}^{\infty} D_m J_{m+1}(kr) e^{-k|z|}
$$

Outside the disk ($r>1$), equation (33) becomes Laplace's equation and their solution can be written as

$$
F(r, z) = \sum B_m N_{m-1}(kr) e^{-k|z|}
$$

$$
Q(r, z) = \sum B_m N_{m+1}(kr) e^{-k|z|}
$$

Plugging the disk solution for $F(r, z)$ and $Q(r, z)$ in (35), multiplying both sides of the 1st and 2nd equation by $J_{m-1}(k'/r)$ and $J_{m+1}(k'/r)$ respectively and integrating over
rdr from 0 to 1, one gets:

\[ E_m(k) = -\frac{2}{\kappa} \int_0^1 \left( b\varphi - i b\gamma \right) J_{m-1}(k\gamma) \varphi d\gamma \]

and

\[ D_m(k) = -\frac{2}{\kappa} \int_0^1 \left( b\varphi + i b\gamma \right) J_{m+1}(k\gamma) \varphi d\gamma \]

Where

\[ M = \frac{1}{2} \left( J_m(k) \right)^2 ; \quad M_1 = \frac{1}{2} \left( J_{m+1}(k) \right)^2 \]

From (34), one gets

\[ A_\gamma = \frac{F + Q}{2} ; \quad A_\varphi = \frac{F - Q}{2} \]

and therefore, one can write the solutions of Poisson's equation (33) in terms of vector potential in the disk as follows:

\[ A_\gamma = \frac{1}{2} \sum_{k=0}^{\infty} \left[ D_m J_{m+1}(k\gamma) + E_m J_{m-1}(k\gamma) \right] e^{-k\gamma} \]

\[ A_\varphi = \frac{i}{2} \sum_{k=0}^{\infty} \left[ D_m J_{m+1}(k\gamma) - E_m J_{m-1}(k\gamma) \right] e^{-k\gamma} \]

outside the disk (r>1)

\[ A_\gamma = \frac{-2im}{2} \sum_{k=0}^{\infty} B_m \left[ N_{m+1}(k\gamma) + N_{m-1}(k\gamma) \right] e^{-k\gamma} \sum_{k=0}^{\infty} b_m \frac{N_{m+1}(k\gamma) + N_{m-1}(k\gamma)}{k} \]

\[ A_\varphi = \frac{-2i}{2} \sum_{k=0}^{\infty} B_m \left[ N_{m+1}(k\gamma) - N_{m-1}(k\gamma) \right] \]

From the above expressions, magnetic fields inside and outside the disk are for r<1

\[ B_\gamma = \frac{2}{2} \sum_{k=0}^{\infty} \left[ D_m J_{m+1}(k\gamma) - E_m J_{m-1}(k\gamma) \right] \]
\[ b_{\phi} \mid_{r=0} = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{\Delta_m}{k} \left[ D_m T_{m-1}(kr) + E_m T_{m-1}(kr) \right] \]

\[ b_{\phi} \mid_{r=1} = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{\Delta_m}{k} \left( D_m E_m \right) T_m(kr) \]

for \( r > 1 \)

\[ b_{r} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Delta_m}{k} \left[ N_{m-1}(kr) - N_m(kr) \right] \]

\[ b_{r} \mid_{r=0} = -\frac{1}{2} \sum_{k=0}^{\infty} B_m N_m(kr) \]

\[ b_{\phi} \mid_{r=0} = 2 \sum_{k=0}^{\infty} B_m N_m(kr) \]

Inside and outside magnetic fields should be matched at the boundary of the disk, i.e. \( r=1 \). Matching \( b_r, b_\phi \) and \( b_{\phi} \) respectively, one gets

\[ D_m T_{m+1}(kr) - E_m T_{m-1}(kr) = 2 B_m N_{m+1}(kr) - \frac{\Delta_m}{k} N_m(kr) \]

\[ D_m T_{m+1}(kr) + E_m T_{m-1}(kr) = \frac{\Delta_m}{k} B_m N_m(kr) \]

\[ (D_m + E_m) T_m(kr) = 2 B_m N_m(kr) \]

Expressing from the first two equations \( C_m \) and \( A_m \) in terms of \( B_m \) as:

\[ D_m = B_m \frac{N_{m+1}(kr)}{T_{m+1}(kr)} \]

\[ E_m = B_m \frac{\frac{\Delta_m}{k} N_m(kr) - N_{m+1}(kr)}{T_{m-1}(kr)} \]
and plugging it in the 3rd equation of equation (38); one gets a dispersion relation:

\[
N_{m+1}(k) \sigma_{m-1}(k) \sigma_m(k) + N_{m-1}(k) \sigma_{m+1}(k) \sigma_m(k) - 2 N_m(k) \sigma_{m+1}(k) \sigma_{m-1}(k) = 0
\]

From where, one gets the following dispersion relations for different m-modes: \( m = 0 \)

\[ J_1(k) = 0 \]

\( m = 1 \)

\[ N_2(k) J_0(k) J_1(k) + N_2(k) J_0(k) J_1(k) - 2 N_1(k) J_2(k) \sigma(k) = 0 \]

\( m = 2 \)

\[ N_3(k) J_1(k) J_2(k) + N_2(k) J_2(k) J_2(k) - 2 N_2(k) J_2(k) \sigma(k) = 0 \]

The zeros of the above equations have been tabulated in Table 2.1.

2.10. Global Eigenvalue Equation

The set of MHD equations (1)-(7) can be written in a more appropriate form to allow some simplifications in equations and consequent reduction in the computing time. We add and subtract \( r \) and \( \varphi \) component of equation of motion to get the following two equations involving the quantities \( \tilde{u}_r + \tilde{u}_\varphi \) and \( \tilde{u}_r - \tilde{u}_\varphi \):

\[ i \nabla \left( \tilde{u}_r + i \tilde{u}_\varphi \right) + 2 i \left( \tilde{u}_r + i \tilde{u}_\varphi \right) + \frac{\hat{\sigma}}{\hat{\sigma}_V} \left( \frac{i}{r^2} \right) = \]

\[ = - \frac{i}{\hat{\sigma}_V} \left( \frac{d \tilde{b}_r}{d y} + \frac{m}{r} \tilde{b}_r \right) + \frac{2 B_0^2}{\sigma_0} \left[ \tilde{b}_r + i \tilde{b}_\varphi \right] + \frac{2 B_0^2}{\sigma_0} \tilde{b}_2 \]
\[ i\dot{\tilde{V}}(\tilde{v}_r - i\tilde{v}_\gamma) = -2i(\tilde{v}_r - i\tilde{v}_\gamma) = -\frac{1}{\sigma_0} \left( \frac{d\tilde{p}}{d\gamma} - \frac{m_n}{\gamma} \tilde{p} \right) + \frac{2B_{02}}{\sigma_0} (\tilde{b}_r - i\tilde{b}_\phi) + \frac{2B_{02}}{\sigma_0} \tilde{b}_z \]

Similarly, adding and subtracting \( r \) and \( \gamma \) component of the induction equation, one gets the quantities involving \( \tilde{b}_r + i\tilde{b}_\phi \) and \( \tilde{b}_r - i\tilde{b}_\phi \)

\[ i\dot{\tilde{V}}(\tilde{b}_r + i\tilde{b}_\phi) = -\frac{B_{02}}{\gamma} (\tilde{v}_r + i\tilde{v}_\gamma) - \tilde{v}_r \frac{dB_{02}}{d\gamma} + iB_{02} \left( \frac{d\tilde{v}_r}{d\gamma} + \frac{m_n}{\gamma} \tilde{v}_r \right) \]

\[ i\dot{\tilde{V}}(\tilde{b}_r - i\tilde{b}_\phi) = -\frac{B_{02}}{\gamma} (\tilde{v}_r - i\tilde{v}_\gamma) - \tilde{v}_r \frac{dB_{02}}{d\gamma} - iB_{02} \left( \frac{d\tilde{v}_r}{d\gamma} - \frac{m_n}{\gamma} \tilde{v}_r \right) \]

where,

\( \sqrt{\gamma} = \omega - m\Omega \)

Therefore, after rearranging the equations, as pointed out above, one gets the following set of equations:

\[ i\ddot{\tilde{V}} - \frac{\gamma}{\gamma - 1} \left[ \frac{d}{d\gamma} \left( \gamma \tilde{v}_r \dot{\tilde{v}} \right) - \frac{1}{\sigma_0} \tilde{v}_r \dot{\tilde{v}} \right] = 0 \]

\[ i\dot{\tilde{V}}(\tilde{v}_r + i\tilde{v}_\gamma) + 2i(\tilde{v}_r + i\tilde{v}_\gamma) + \frac{\sigma}{\sigma_0} (\gamma - 1) = 0 \]

\[ = -\frac{1}{\sigma_0} \left( \frac{d\tilde{p}}{d\gamma} + \frac{m_n}{\gamma} \tilde{p} \right) + \frac{2B_{02}}{\sigma_0} (\tilde{b}_r + i\tilde{b}_\phi) + \frac{2B_{02}}{\sigma_0} \tilde{b}_z \]

\[ i\dot{V}(\tilde{v}_r - i\tilde{v}_\gamma) - 2i(\tilde{v}_r - i\tilde{v}_\gamma) = -\frac{1}{\sigma_0} \left( \frac{d\tilde{p}}{d\gamma} - \frac{m_n}{\gamma} \tilde{p} \right) + \frac{2B_{02}}{\sigma_0} (\tilde{b}_r - i\tilde{b}_\phi) + \frac{2B_{02}}{\sigma_0} \tilde{b}_z \]
\[ i \mathcal{V} \left( \hat{b}_r + i \hat{b}_\varphi \right) = - \frac{B_{0r}}{r} \left( \hat{\nu}_r + i \hat{\nu}_\varphi \right) - \hat{\nu}_r \frac{dB_{0r}}{dr} + \left( \frac{d \hat{\nu}_\varphi}{dr} - \frac{m}{r} \hat{\nu}_r \right) \]

\[ i \mathcal{V} \left( \hat{b}_r - i \hat{b}_\varphi \right) = - \frac{B_{0r}}{r} \left( \hat{\nu}_r - i \hat{\nu}_\varphi \right) - \hat{\nu}_r \frac{dB_{0r}}{dr} - \left( \frac{d \hat{\nu}_\varphi}{dr} - \frac{m}{r} \hat{\nu}_r \right) \]

We expand the radial and azimuthal component of the magnetic field as in equation (36) while the density and the quantities \( \hat{\nu}_r + i \hat{\nu}_\varphi \) and \( \hat{\nu}_r - i \hat{\nu}_\varphi \) are written as:

\[ G(\gamma) = \sum_{K=0}^{\infty} A_m J_m(kr) \]

\[ \hat{\nu}_r + i \hat{\nu}_\varphi = i \sum_{K=0}^{\infty} B_m J_{m+1}(kr) \]

\[ \hat{\nu}_r - i \hat{\nu}_\varphi = i \sum_{K=0}^{\infty} C_m J_{m-1}(kr) \]

Which yields for the radial part of the perturbed components \( \hat{\nu}_r \) and \( \hat{\nu}_\varphi \):

\[ \hat{\nu}_r = \frac{c}{2} \sum_{K=0}^{\infty} B_{m+1} J_{m+1}(kr) + C_m J_{m-1}(kr) \]

\[ \hat{\nu}_\varphi = \frac{i}{2} \sum_{K=0}^{\infty} B_{m+1} J_{m+1}(kr) - C_m J_{m-1}(kr) \]

From (36),
\[ \tilde{b}_m + i \tilde{b}_m = -i \sum_{k=0}^{\infty} k E_{m} \tilde{J}_{m-1}(k \gamma) \]  

\[ \tilde{b}_m - i \tilde{b}_m = i \sum_{k=0}^{\infty} k D_{m} J_{m+1}(k \gamma) \]  

Plugging (41) and (36') in equation (39)-(40), one gets an infinite set of equations as follows:

\[ \sqrt{\sum_{k=0}^{\infty} A_m J_m(k \gamma) + \sum_{k=0}^{\infty} B_m \left[ J_{m+1}(k \gamma) \left( \frac{1}{2} \frac{d \delta \nu}{d \gamma} - \frac{w_0 \delta \nu}{\gamma} \right) \right] + \frac{k \sigma_0}{2} J_m(k \gamma) \] 

\[ \sum_{k=0}^{\infty} C_m \left[ J_{m-1}(k \gamma) \left( \frac{1}{2} \frac{d \delta \nu}{d \gamma} + \frac{\delta \nu}{\gamma} \right) \right] + \frac{k \sigma_0}{2} J_{m-1}(k \gamma) = 0 \]  

\[ -(\sqrt{\sum_{k=0}^{\infty} B_m J_{m+1}(k \gamma) + \sum_{k=0}^{\infty} A_m \left[ \left( \frac{1}{2} \frac{d \delta \nu}{d \gamma} + \frac{\delta \nu}{\gamma} \right) \frac{d \delta \nu}{d \gamma} \right] \right] \] 

\[ \times J_m(k \gamma) + \frac{d \delta \nu}{d \gamma} \frac{J_{m-1}(k \gamma)}{\sigma_0^2} + \sum_{k=0}^{\infty} E_m \frac{ik}{\sigma_0} B_{o2} J_{m-1}(k \gamma) \] 

\[ - \sum_{k=0}^{\infty} \frac{ik}{\sigma_0} B_{o2} (D_m + E_m) J_m(k \gamma) = 0 \]  

\[ -(\sqrt{\sum_{k=0}^{\infty} C_m J_{m+1}(k \gamma) + \sum_{k=0}^{\infty} A_m \left( \frac{d \delta \nu}{d \gamma} - \frac{2m}{\gamma} \right) J_m(k \gamma) \right] \] 

\[ + \left[ \frac{d \delta \nu}{d \gamma} - \frac{2m}{\gamma} \right] J_m(k \gamma) + \sum_{k=0}^{\infty} \frac{ik}{\sigma_0} D_m \left[ 2 B_{o2} J_{m+1}(k \gamma) + \right. \] 

\[ \left. + B_{o2} J_m(k \gamma) \right] - \sum_{k=0}^{\infty} \frac{ik}{\sigma_0} B_{o2} E_m J_m(k \gamma) = 0 \]  

\[ \sqrt{\sum_{k=0}^{\infty} k E_m J_{m-1}(k \gamma) + \frac{i B_{o2} \sum_{k=0}^{\infty} k E_m J_{m+1}(k \gamma) + \frac{c}{2} \frac{d \delta \nu}{d \gamma} \sum_{k=0}^{\infty} B_{o2} J_{m+1}(k \gamma) + \right} \] 

\[ \sum_{k=0}^{\infty} C_m \left[ J_{m-1}(k \gamma) \left( k J_m(k \gamma) - \frac{1}{2} J_{m+1}(k \gamma) \right) \right] - \sum_{k=0}^{\infty} C_m \left[ k J_{m-1}(k \gamma) + \frac{1}{2} J_{m-2}(k \gamma) \right] = 0 \]  

(46)
\[ - \frac{\nu}{\gamma} \sum_{k=0}^{\infty} k D_m \tau_{m,1} (k \gamma) + \frac{2 \nu}{\gamma} \sum_{k=0}^{\infty} c_m \tau_{m-1} (k \gamma) + \frac{2}{\gamma} \frac{d \nu}{d \tau} \sum_{k=0}^{\infty} \left[ B_m \right] \\
\times \left[ J_{m+1} (k \gamma) + c_m \tau_{m-1} (k \gamma) \right] + \frac{2 \nu}{\gamma} \sum_{k=0}^{\infty} \left[ B_m \right] \left[ k J_{m} (k \gamma) - \frac{2 m+1}{\gamma} J_{m+1} (k \gamma) \right] = 0 \quad (47) \]

Now, multiplying the above equations (43), (44), (45), (46) and (47) by \( \gamma J_m (k \gamma), \gamma J_{m+1} (k \gamma), \gamma J_{m-1} (k \gamma), \gamma J_{m+1} (k \gamma), \gamma J_{m-1} (k \gamma), \gamma J_{m-2} (k \gamma) \), respectively and integrating over the disk, one gets the following algebraic equations:

\[ m \sum_{l=0}^{\infty} A_m^l + \sum_{k, l=0}^{\infty} B_k^l \beta_{kl} + \sum_{k, l=0}^{\infty} c_m^k \chi_{kl} = \omega \sum_{l=0}^{\infty} A_m^l \]

\[ (m-2) \sum_{l=0}^{\infty} B_m^l - \sum_{k, l=0}^{\infty} A_m^k \alpha_{kl} + \sum_{k, l=0}^{\infty} E_m^k \epsilon_{kl} + \sum_{k, l=0}^{\infty} D_m^k \zeta_{kl} = \omega \sum_{l=0}^{\infty} B_m^l \]

\[ (m+2) \sum_{l=0}^{\infty} C_m^l + \sum_{k, l=0}^{\infty} A_m^k \alpha_{kl} (1) + \sum_{k, l=0}^{\infty} D_m^k \epsilon_{kl} (1) + \sum_{k, l=0}^{\infty} E_m^k \zeta_{kl} (1) = \omega \sum_{l=0}^{\infty} C_m^l \]

\[ m \sum_{l=0}^{\infty} E_m^l + \sum_{k, l=0}^{\infty} B_m^k \beta_{kl} (1) + \sum_{k, l=0}^{\infty} c_m^k \chi_{kl} (1) = \omega \sum_{l=0}^{\infty} E_m^l \]

\[ m \sum_{l=0}^{\infty} D_m^l + \sum_{k, l=0}^{\infty} B_m^k \beta_{kl} (2) + \sum_{k, l=0}^{\infty} c_m^k \chi_{kl} (2) = \omega \sum_{l=0}^{\infty} D_m^l \]

Where:

\[ \beta_{kl} = - \frac{1}{M_1} \int_0^1 \nu d\tau J_m (k \gamma) \left\{ J_{m+1} (k \gamma) \left( \frac{1}{2} \frac{d \omega}{d \tau} - \frac{m \omega}{\gamma} \right) + \frac{k \omega}{2} J_m (k \gamma) \right\} \]

\[ \chi_{kl} = - \frac{1}{M_2} \int_0^1 \nu d\tau J_m (k \gamma) \left\{ J_{m-1} (k \gamma) \left( \frac{1}{2} \frac{d \omega}{d \tau} + \frac{m \omega}{\gamma} \right) + \frac{k \omega}{2} J_m (k \gamma) \right\} \]
\[
\alpha_{kl} = \frac{1}{M_2} \int_0^1 \gamma d\gamma J_{m1}(k\gamma) \left[ \left\{ \left( \frac{1}{2} - 1 \right) \frac{\partial \Omega}{\partial \nu} + \frac{\gamma (\gamma - 1)}{M_0^2} \frac{\partial \Omega}{\partial \nu} \right\} J_m(k\gamma) + \frac{\gamma \Omega}{M_0^2} J_{m-1}(k\gamma) \right]
\]
\[
\delta_{kl} = \frac{i k}{M_2} \int_0^1 \frac{\gamma d\gamma}{M_2} J_{m1}(k\gamma) \left[ 2 B_{\nu 2} J_{m-1}(k\gamma) - B_{\nu 2} J_{m}(k\gamma) \right]
\]
\[
\sigma_{kl} = \frac{-i k}{M_2} \int_0^1 \frac{\gamma d\gamma}{M_2} J_{m1}(k\gamma) B_{\nu 2}(r) J_m(k\gamma)
\]
\[
\alpha^{(1)}_{kl} = \frac{\gamma}{M_2} \int_0^1 \gamma d\gamma J_{m1}(k\gamma) \left[ k J_{m-1}(k\gamma) + \left\{ \frac{(\gamma - 1) \partial \nu}{\partial \nu} - 2 \right\} J_m(k\gamma) \right] \frac{\Omega}{M_0^2}
\]
\[
\delta^{(1)}_{kl} = -\frac{i k}{M_2} \int_0^1 \gamma d\gamma J_{m1}(k\gamma) \left[ 2 B_{\nu 2} J_{m-1}(k\gamma) + B_{\nu 2} J_m(k\gamma) \right]
\]
\[
\sigma^{(1)}_{kl} = \frac{-i k}{M_2} \int_0^1 \frac{\gamma d\gamma}{M_2} J_{m1}(k\gamma) B_{\nu 2} J_m(k\gamma)
\]
\[
\beta^{(1)}_{kl} = \frac{-i}{k M_2} \int_0^1 \gamma d\gamma J_{m1}(k\gamma) \left[ J_{m+1}(k\gamma) \left\{ \frac{k B_{\nu 2} + \frac{3 B_{\nu 2}}{\nu^2} \frac{\partial B_{\nu 2}}{\partial \nu} + \frac{k B_{\nu 2}}{\nu} \right\} J_m(k\gamma) \right]
\]
\[
\chi^{(1)}_{kl} = \frac{i}{2 k M_2} \int_0^1 \gamma d\gamma J_{m1}(k\gamma) \left[ \left( \frac{B_{\nu 2}}{\nu} - \frac{d B_{\nu 2}}{d \nu} \right) J_{m-1}(k\gamma) + k B_{\nu 2} J_m(k\gamma) \right]
\]
\[
\psi^{(2)}_{kl} = \frac{i}{2 k M_2} \int_0^1 \gamma d\gamma J_{m1}(k\gamma) \left[ \frac{d B_{\nu 2}}{d \nu} - \frac{(2m+1) B_{\nu 2}}{\nu} J_{m}(k\gamma) + k B_{\nu 2} J_m(k\gamma) \right]
\]
\[
\chi^{(2)}_{kl} = \frac{i}{2 k M_2} \int_0^1 \gamma d\gamma J_{m1}(k\gamma) \left[ \frac{d B_{\nu 2}}{d \nu} - \frac{(2m+1) B_{\nu 2}}{\nu} J_{m-1}(k\gamma) - k B_{\nu 2} J_m(k\gamma) \right]
\]

The set of algebraic equations (48) can conveniently be written in a matrix form:
which defines an eigenvalue problem with \( \omega \) as the eigenfrequency for the allowed mode of oscillations of the disk in question and \( \mathbf{M} \), the corresponding eigenfunctions. In general, \( \mathbf{M} \) is complex, infinite, nonsymmetric matrix, where submatrices \( \mathbf{B}, \mathbf{K}, \mathbf{C} \) are infinite matrices themselves. \( \mathbf{X} \) is an infinite column matrix representing the eigenfunctions. We may note here that the nonsymmetric matrix, \( \mathbf{M} \), cannot be reduced to a symmetric form by any transformations.

One can attempt to solve the eigenvalue problem (49) by using a sufficiently large, truncated form of \( \mathbf{M} \) and \( \mathbf{X} \). However, while truncating, the convergence of the eigenvalue and eigenfunctions has to be ensured. An analytic solution of
the problem is not possible and use has to be made of numerical methods in order to obtain the eigenspectrum. Complex, nonsymmetric matrices allow complex conjugate pairs of eigenvalues. Complex eigenvalues are associated with complex eigenfunctions, hence, introduce radial phase shifts in the perturbation maxima at various distances from the centre of the disk. Thus, in general, the complex eigenmodes yield spiral-like features.

Let us consider the magnetic field perturbation:

\[
\vec{b}_r + i \vec{b}_\theta = \text{Re} \left[ f(r) e^{i(\omega t - m\phi)} \right]
\]

where,

\[
f(r) = \sum_{k=0}^{\infty} E_j J_{m-1}(k\gamma) = \sum_{k=0}^{\infty} (E_{j, \text{real}} + E_{j, \text{imag}}) \times J_{m-1}(k\gamma) = H_\gamma + i H_\theta
\]

Therefore,

\[
\vec{b}_r + i \vec{b}_\theta = \sqrt{(H_\gamma^2 + H_\theta^2)} \cos (\omega t + m\phi + \phi(r))
\]

where,

\[
H_\gamma = \sum_{k=0}^{\infty} E_{j, \text{real}} J_{m-1}(k\gamma)
\]

and

\[
\phi(r) = \tan^{-1} (H_\theta/H_\gamma)
\]

If the amplitude part \[(H_\gamma^2 + H_\theta^2)^{1/2}\] is reasonably constant over the entire disk, the phase part at any instant of time, traces out a regular spiral for the maxima, described by

\[
m\phi + \phi(r) = \text{Constant}
\]

and other spiral branches at angles shifted by \(\Delta \phi = \frac{\Delta \gamma}{m_1}\) for
m>0. In almost all the cases, it is found that the amplitude part does not behave smoothly with r, so there are significant undulations. To study the allowed patterns in a more convenient manner, we have plotted the \( \tilde{b}_\gamma \) and \( \tilde{b}_\varphi \) given by equation (36) as contour plots and three-dimensional plots. The field-perturbations at a large number of mesh points in and outside the disk have been generated using the eigenfunction in the equation (49).

Here, one may note that if the expansions in the equation (48) for the perturbed quantity retain n terms, the resulting eigenvalue problem would admit the normal mode with a maximum of \((n-1)\) modes in the range \(0<r<1\) and the subsequent short-wavelength modes would not appear. By increasing the number of the terms in the expansion, the values of the existing eigenmodes would be refined, as well as, few more new short wavelength modes would be obtained. However, before the results for a particular mode are confidently discussed, its convergence with increasing dimension of the eigenmatrix in the equation (49) has to be checked.

2.11. Results and Discussions

We have carried out an analysis of the eigenvalue problem of a magnetized, gravitating, finite disk. A particular current density has been considered. However, in the formalism developed, any current density profile can be constructed in terms of a Bessel series and corresponding
magnetic field can be obtained by using equation (11).

The rotational profile is shown in figure (2.3) for different angular velocity. There is no differential rotation throughout the disk and the material revolves around the center as a solid body. Equilibrium magnetic fields $B_r$ and $B_z$ calculated using the equation (11) are shown in figure 2.4 along with the rotational velocity, for a fixed angular velocity. Equilibrium density distribution is calculated self-consistently from the radial equilibrium in the disk and is shown in figure 2.2.

2.11.1. The Eigen-Modes in Disk-Bulge System

The eigenvalue problem (49) has been solved with suitable truncations of the infinite dimensional matrix $N$. The convergence of the eigen-frequencies for $m=1$ and $m=2$ modes, as the dimension of the matrix increases, is shown in table (2.2). The relative changes in real and imaginary parts of the eigenvalue $\omega$, are defined by

$$\Delta \omega_y = \begin{vmatrix}
\omega_y & -\omega_y \\
(\theta,\nu) & (\theta,\nu)
\end{vmatrix}$$

and
\[
\Delta \omega_i = \left| \omega_i \left( \varphi \right) - \omega_i \left( \varphi \varphi \right) \right| / \omega_i \left( \varphi \varphi \right)
\]

where \( g < p \) and \( p \) and \( q \) are the size of the truncated matrix.

Using data from table (2.2), we see for the leading mode that

\[
\Delta \omega_y = 0.23528180 \times 10^4 ; \Delta \omega_i = 0.1476263 \times 10^3 \quad \text{for } m=1
\]

\[
\Delta \omega_y = 0.53609618 \times 10^3 ; \Delta \omega_i = 0.33129514 \times 10^3 \quad \text{for } m=2
\]

thus, the relative error is not so small i.e. the convergence is not very good. As the dimension of the matrix is increased to larger values, numerical errors associated with the eigenvalue routine become significant. In view of these considerations we have adopted \( n_t = 75 \) for \( m=1 \) and \( m=2 \). Thus, the modes with the number of nodes larger than 15 has not been incorporated in our studies.

Figures (2.4), (2.5), (2.6) and (2.7) show the contour and the eigenpatterns for the radial and azimuthal components of the magnetic field for \( m=1 \) mode and for the fastest growth rate. Figures (2.8) and (2.9) compares eigenpattern for radial magnetic field as obtained from 60x60 and 75x75 matrix for \( m=1 \) mode. Figures (2.10) and (2.11) show the eigenpattern for azimuthal magnetic field
for $m=1$ mode with $60 \times 60$ and $(75 \times 75)$ matrix. Clearly pattern
does changes and hence, convergence is not satisfactory.

We find the dominant unstable mode for $m=1$. Growth rate
decreases as radial wavelength of perturbation increases.
Dominant mode corresponds to the global structure of the
magnetic field.

The pattern-frequency and growth rates for the
principle (or the fastest growing) mode, with $m=1$ and $m=2$
are listed in table (2.3). In all the cases, the growth rates
are small as compared to the pattern frequencies given by
$\Omega_p \approx - \frac{\Omega_\ast}{m}$, and hence explosive instabilities are absent
in the disk.

Let us consider some typical calculations for $m=1$ mode,
for allowed pattern velocities are of spiral patterns and
amplification period. Considering a typical disk with a mass,
$m = 2.10^{11} M_\odot$ and radius, $R = 10^8$ kpc, the normalization
coefficients are

$$\left[ \frac{\pi G M_\odot}{R} \right]^{\frac{1}{2}} = 595 \text{ km/sec for velocity}$$

$$\left[ \frac{\pi G M}{R^3} \right]^{\frac{1}{2}} = 74.4 \text{ km/sec-kpc (} \approx 7.8 \times 10^{-8} \text{ yr}^{-1} \text{) for angular velocity}$$

Here $G = 0.45145 \times 10^{-38} \text{ kpc}^3/\text{M}_\odot\text{-Sec}^2$

The circular velocity at a distance of 8 kpc from the
center for the disk is

74.43 km/sec-kpc.

The growth rate $\omega_1$ as for the fastest mode is:
\[ \Omega_I = 0.718324 \times 10^4 \times 7.8 \times 10^{-8} \text{ yr}^{-1} \]
\[ = 0.56 \times 10^{-3} \text{ yr}^{-1} \]

Therefore, \( P_{m+2} \) (amplification period \( \times 10^{+9} \text{ years} \)) =
\[ = 5.6 \times 10^{5} \text{ y}_e^{m+2} \]

As is evident from the amplification period, the magnetic field increases over the life-time of the galaxy by a large amount (~10^5). It points out towards the fact that the linear analysis carried out above might not be accurate enough and one has to achieve the better convergence to be confident about the growth rates.

The pattern frequency and growth rate for the fastest growing mode with \( m=2 \), are listed in table (2.3). In all the cases the growth rates are significantly small as compared to the pattern frequencies \( \sqrt{2} \). Figures (2.12), (2.13) and (2.14) correspond to the contour and 3-D plots of the radial and azimuthal component of the magnetic field respectively.

We would like to remark that eigenvalue problems in the form, discussed above, remove several difficulties pointed in Chapter I and also, provide a natural explanation of the origin and persistence of different spiral modes in a gravitating magnetized disk.
Figure 2.2

RADIAL DISTANCE vs. DENSITY

Equilibrium Density VS. Radial Distance

n = 1
EQUILIBRIUM BR, BZ, V

Fig. (2.3)
Radial Field Contour

75x75 matrix, \( k = 11.70600 \)

A: -2.309509
B: -1.796285
C: -1.283061
D: -0.769836
E: -0.25661
F: 0.25661
G: 0.769836
H: 1.283061
I: 1.796285
J: 2.309509

Fig. (2-4)
Radial Field Component

75x75 matrix, $k = 11.70600$

Fig (2.5)
Azimuthal Field Component

75x75 matrix, \( k = 11.70600 \)

Figure (2.7)
Radial Field Component

75x75 matrix, \( k = 11.70600 \)

Fig. (2.9)
Azimuthal Field Component

60x60 matrix ; k = 11.70 600

Fig. (2-10)
75x75 meter \( |x| \leq 15, \; |y| \leq 6000 \)

Azimuthal Field Component
Radial Component Plot

Fig. 2.13
Azimuthal Component Plot

Fig. (2.14)
Aizimuthal Component Contour (Corresponding to fig. 2.14)

75x75 matrix; k = 13.17637

A: -0.00012
B: -0.00009
C: -0.00007
D: -0.00004
E: -0.00001
F: 0.00001
G: 0.00004
H: 0.00007
I: 0.00009
J: 0.00012

X AXIS
Y AXIS
Table 2.1
First Fifteen Zeros of Dispersion Relation

<table>
<thead>
<tr>
<th>m=1</th>
<th>m=2</th>
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<tr>
<td>1.841184</td>
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<tr>
<td>8.536316</td>
<td>9.969468</td>
</tr>
<tr>
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<td>24.31133</td>
<td>25.82604</td>
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<td>27.45705</td>
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<tr>
<td>30.60192</td>
<td>32.12733</td>
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<tr>
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<tr>
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<td>38.42266</td>
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<tr>
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<td>44.71455</td>
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<tr>
<td>46.31960</td>
<td>47.85964</td>
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### Table 2.2
Principal Mode for Two Truncated Matrices

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<th></th>
<th>(60x60)</th>
<th>(75x75)</th>
<th>(60x60)</th>
<th>(75x75)</th>
</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>3288.49</td>
<td>2353.46</td>
<td>4763.50</td>
<td>3025.87</td>
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<tr>
<td>m=2</td>
<td>866.19</td>
<td>328.57</td>
<td>1131.00</td>
<td>799.50</td>
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</tbody>
</table>

### Table 2.3
Pattern Velocity and Growth Rate for (75x75) Matrix

<p>| | | |</p>
<table>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>m=1</td>
<td>-0.23531469 E+04</td>
<td>0.302587 E+04</td>
</tr>
<tr>
<td>m=2</td>
<td>-0.16428610 E+03</td>
<td>0.799500 E+03</td>
</tr>
</tbody>
</table>