Chapter 2

Iterative approximations for accretive operators

In this chapter we have studied the iterative approximation for the solutions of accretive operator equations using the Bruck iteration as well as the viscosity approximation technique in different space structures.

We recall some basic definitions.
Let $E$ be a real Banach space with dual $E^*$. The normalized duality mapping from $E$ to $2^{E^*}$ is defined by

$$J(x) := \{x^* \in E^* : (x, x^*) = \|x\|^2, \|x\| = \|x^*\|\},$$

where $(.,.)$ denotes the generalized duality pairing between the elements of $E$ and $E^*$.

A mapping $A: \mathcal{D}(A) \subseteq E \to E$ is said to be accretive [20] if for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$(Ax - Ay, j(x - y)) \geq 0.$$
An element $x^* \in D(A)$ is called a zero of $A$, if $Ax^* = 0$.

Let $K$ be a nonempty subset of $E$. The mapping $T : K \to K$ is called pseudocontractive [26] if for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$(Tx - Ty, j(x - y)) \leq \|x - y\|^2.$$ 

An element $x^* \in K$ is a fixed point of $T$, if $Tx^* = x^*$.

It is observed that, $A$ is accretive if and only if $I - A$ is pseudocontractive [20, Proposition 1] and thus a zero of $A$ is a fixed point of $T := I - A$.

We now recall some examples of accretive operators.

**Example 2.0.1.** [51] Let $\mathbb{R} = (-\infty, \infty)$ with usual norm and $A : [0, 1] \to \mathbb{R}$ be defined by $Ax = \frac{x^2}{2} - 1$. Then for $x, y \in [0, 1]$,

$$(Ax - Ay, j(x - y)) = |Ax - Ay| |x - y|$$

$$= \frac{1}{2} |x - y|^2 \geq 0.$$

Hence, $A$ is accretive.

**Example 2.0.2.** Let $H$ be a Hilbert space and let $f : H \to (-\infty, \infty]$ be proper and convex. Define a multivalued mapping $\partial f$ on $H$ by

$$\partial f(x) = \{x^* \in H : f(y) \geq f(x) + (x^*, y - x), y \in H\},$$

for all $x \in H$. Such $\partial f$ is called the subdifferential of $f$ and it is an example of accretive operator (see [148, Theorem 4.6.6]).

In the past few decades, considerable research efforts have been made on the methods of finding approximate solutions (when they exist) of the equation $Ax = 0$, for an accretive operator $A$, see [28, 30, 83, 134, 150]. The main tool for approximation of zeros or fixed points of nonlinear mappings remains iterative technique.
In the case when the operator \( A \) is strongly accretive or equivalently, \( T \) is strongly pseudocontractive, the Mann iteration scheme [106] has been successfully employed to approximate the corresponding zero or fixed point. All efforts to use the Mann scheme to approximate a fixed point of a Lipschitz pseudocontractive map defined on a compact convex subset of a Hilbert space into itself and consequently to approximate a solution of an \( m \)-accretive operator, proved unsuccessful. This led to the introduction of other iterative processes which proved successful for this class of nonlinear maps under the above setting. Some of these schemes have been extended to Banach spaces (see e.g. [140] for the details on the iterative approximation of such mappings). Nevanlinna [120] observed that Lipschitz continuity of the operator can be weakened in two directions: (i) by assuming that the operator satisfies a linear growth condition, (ii) by assuming only continuity.

2.1 Bruck iteration for iterative solution of accretive operators

Bruck [28] studied the iteration process

\[
x_{n+1} = x_n - \lambda_n Ax_n - \lambda_n \theta_n(x_n - z),
\]

for approximating solutions of the equation \( Ax = 0 \) in Hilbert space setting, where \( A \) is a maximal monotone operator. Here the requirement is that the sequences \( \{\lambda_n\} \) and \( \{\theta_n\} \) of nonnegative real numbers satisfy the conditions: \( \{\theta_n\} \) is decreasing, \( \lim_{n \to \infty} \theta_n = 0 \) and there exists a strictly increasing sequence \( \{n(i)\}_{i=1}^{\infty} \) of positive integers such that

\[
\lim \inf_{i} \theta_{n(i)} \sum_{j=n(i)}^{n(i+1)} \lambda_i > 0, \quad \lim_{i} [\theta_{n(i)} - \theta_{n(i+1)}] \sum_{j=n(i)}^{n(i+1)} \lambda_i = 0, \\
\lim_{i} \sum_{j=n(i)}^{n(i+1)} \lambda_i^2 = 0.
\]
Later on, Reich [132] studied the recursion formula (2.1.1) for Lipschitz accretive operator on uniformly convex Banach spaces with slightly stronger conditions on the iteration parameters. Nevanlinna [120] gave a global version of the result of Bruck [28], still in Hilbert spaces, by assuming that $A$ satisfies a linear growth condition of the form

$$
\|Ax\| \leq c(1 + \|x\|), \quad \forall \, x \in \mathcal{D}(A). \tag{2.1.2}
$$

Chidume and Zegeye [52] extended the result in more general uniformly smooth Banach space for an $m$-accretive operator satisfying linear growth condition (2.1.2).

Ishikawa [80] introduced the Ishikawa iterative scheme and proved its convergence to a fixed point of a Lipschitz pseudo-contractive self-map $T$ of a compact convex subset $K$ of a Hilbert space. But it was not known whether or not this method converges, to a fixed point of $T$ in Banach spaces. In this connection, Chidume [54] modified the iteration (2.1.1) and proved that it not only converges to a zero of Lipschitz accretive operator in a real reflexive Banach space with uniformly Gateaux differentiable norm but also seems simpler than the Ishikawa scheme. The modified iteration follows as

$$
x_{n+1} = x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1). \tag{2.1.3}
$$

In this continuation, numerous convergence results have been proved on the iterative methods for approximating zeros of Lipschitz accretive-type nonlinear mappings. Also, many authors have proved convergence theorems under the assumption that these operators have bounded range.

A natural generalization of the class of Lipschitz mappings and the class of mappings with bounded range is the class of generalized Lipschitz mappings.
Definition 2.1.1. A mapping $T : E \to E$ is said to be generalized Lipschitz [32] if there exists $L > 0$, such that

$$\|Tx - Ty\| \leq L(1 + \|x - y\|), \quad \forall \ x, y \in E,$$

whereas, the mapping $T$ is said to be Lipschitz if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall \ x, y \in E.$$

The following example [32] shows that the class of generalized Lipschitz mappings properly includes the class of Lipschitz mappings and that of mappings with bounded range.

Example 2.1.2. Let $E = (-\infty, +\infty)$ and $T : E \to E$ by

$$Tx = \begin{cases} 
    x - 1; & \text{if } x \in (-\infty, -1), \\
    x - \sqrt{1 - (x + 1)^2}; & \text{if } x \in [-1, 0) \\
    x + \sqrt{1 - (x - 1)^2}; & \text{if } x \in [0, 1] \\
    x + 1; & \text{if } x \in (1, +\infty) 
\end{cases}$$

Clearly $T$ is generalized Lipschitz mapping which is not Lipschitz and whose range is not bounded.

For the more general class of generalized Lipschitz accretive operators, it was not known whether or not the scheme (2.1.3) studied by Chidume and Zegeye [54] converges to its zero. Thus answering this question, Chidume and Ofoedu [47] modified the iteration (2.1.3) for the larger class of generalized Lipschitz accretive mappings in reflexive Banach spaces modifying many of the previous results.

Most of the results for approximation of accretive operators have been established with the assumption of Lipschitz continuity. In our result, we have proved the strong
convergence of the modified iteration formula given by Chidume and Zegeye [47] for
a uniformly continuous accretive operator in a real reflexive Banach space, without
any Lipschitz condition on the operator with weaker conditions on the parameters.
The modified scheme [47] follows as

\[ x_{n+1} = x_n - \lambda_n \alpha_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad \forall \ n \geq 1. \]

Similar result is proved for uniformly continuous pseudocontractive mappings.

Before proceeding to our main results, we present some definitions and results
required in the proof.

**Definition 2.1.3.** [148] The norm in Banach space \( E \) is said to be uniformly
Gateaux differentiable if for each \( y \in S_1(0) := \{ x \in E : \| x \| = 1 \} \), the limit

\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\]

exists finite, uniformly for \( x \in S_1(0) \).

If \( E \) has a uniformly Gateaux differentiable norm, then the duality mapping is
norm-weak* uniformly continuous on bounded subsets of \( E \) [148, Theorem 4.3.6].

**Definition 2.1.4.** [70] Let \( E \) be a Banach space with \( E^* \) be its dual and \( E^{**} = L(E^*, \mathbb{R}) \), its second dual. If \( x \in E \) is fixed, then the relation \( (x, x^*) \) defines a
continuous linear functional on \( E^* \); thus \( x \) is associated in a natural way with an
element \( x^{**} \) of \( E^{**} \). The mapping \( x \to x^{**} \) is called the canonical (or natural)
embedding of \( E \) in \( E^{**} \). This embedding is always a linear isometry. If it is also
surjective then \( E \) is said to be Reflexive and then \( E = E^{**} \).

**Definition 2.1.5.** [148] Let \( \mathbb{N} \) denotes the set of all natural numbers and \( \mu \) be a
continuous linear functional on \( l^\infty \) satisfying \( \| \mu \| = 1 = \mu(1) \). Then \( \mu \) is said to be
a mean on \( \mathbb{N} \) if and only if

\[
\inf \{ a_n : n \in \mathbb{N} \} \leq \mu(a) \leq \sup \{ a_n : n \in \mathbb{N} \},
\]
for every \(a = (a_1, a_2, \cdots) \in l^\infty\). Here \(\mu_n(a_n)\) is used instead of \(\mu(a)\). A mean \(\mu\) on \(\mathbb{N}\) is called a Banach limit if

\[
\mu_n(a_n) = \mu_n(a_{n+1}),
\]

for every \(a = (a_1, a_2, \cdots) \in l^\infty\).

In the sequel, we shall need the following results.

**Lemma 2.1.6.** [148, Lemma 4.5.4] Let \(C\) be a nonempty closed convex subset of a Banach space \(E\) with a uniformly Gateaux differentiable norm. Let \(\{x_n\}\) be a bounded sequence of \(E\) and let \(\mu\) be a mean on \(\mathbb{N}\). Let \(z \in C\). Then

\[
\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2
\]

if and only if

\[
\mu_n (y - z, j(x_n - z)) \leq 0, \forall y \in C,
\]

where \(j\) is the duality mapping of \(E\).

**Lemma 2.1.7.** [143] Let \(\alpha\) be a real number and \((x_1, x_2, \cdots) \in l^\infty\) be such that \(\mu_n x_n \leq \alpha\), for all Banach limits. If \(\limsup_{n \to \infty} (x_{n+1} - x_n) \leq 0\), then \(\limsup_{n \to \infty} x_n \leq \alpha\).

**Lemma 2.1.8.** [1] Let \(\{a_n\}_{n=1}^\infty\) be a sequence of nonnegative real numbers satisfying the following relation:

\[
a_{n+1} \leq a_n - \varphi(a_{n+1}) \alpha_n + \sigma_n, \ n \geq 0,
\]

where (i) \(0 < \alpha_n < 1\); (ii) \(\sum_{n=1}^\infty \alpha_n = \infty\); (iii) \(\varphi : [0, +\infty) \to [0, +\infty)\) is a strictly increasing function with \(\varphi(0) = 0\). Suppose that \(\sigma_n = o(\alpha_n)\). Then \(a_n \to 0\) as \(n \to \infty\).
Lemma 2.1.9. [127] Let \( E \) be a real Banach space, then for all \( x, y \in E \), there exists \( j(x+y) \in J(x+y) \) such that

\[
\|x + y\|^2 \leq \|x\|^2 + 2(y, j(x+y)).
\]

We now state and prove our results.

Theorem 2.1.10. * Let \( E \) be a real Banach Space and \( A : E \to E \) be a uniformly continuous accretive operator such that \( N(A) := \{x \in E : Ax = 0\} \neq \emptyset \). Suppose \( \{\alpha_n\}_{n=1}^{\infty}, \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\theta_n\}_{n=1}^{\infty} \) be real sequences in \((0,1)\) satisfying the conditions

\[
\lim_{n \to \infty} \lambda_n = 0, \quad \alpha_n = o(\theta_n). \tag{2.1.4}
\]

Let the sequence \( \{x_n\}_{n=1}^{\infty} \) be generated iteratively from arbitrary \( x_1 \in E \), by

\[
x_{n+1} = x_n - \lambda_n \alpha_n A x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall \ n \geq 1. \tag{2.1.5}
\]

Then \( \{x_n\}_{n=1}^{\infty} \) is bounded.

Proof. We first recall that \( \alpha_n = o(\theta_n) \) means that \( \frac{\alpha_n}{\theta_n} \to 0 \) as \( n \to \infty \).

Let \( x^* \in N(A) \).

If \( x_n = x^*, \forall \ n \geq 1 \), then the theorem is proved.

So let, \( N_0 \) be the first smallest integer such that \( x_{N_0} \neq x^* \). So that there exists \( N_0 \geq N_0 \) and \( r > 0 \) be sufficiently large such that \( x_{N_0} \in \overline{B}_r(x^*) = B := \{x \in E : \|x - x^*\| \leq r\}, \ x_1 \in B_{\frac{r}{2}}(x^*) \).

In order to prove that \( \{x_n\}_{n=1}^{\infty} \) is bounded, it is sufficient to show by induction that \( x_n \in B = \overline{B}_r(x^*) \), for all integers \( n \geq N_0 \).

Now by our construction, \( x_{N_0} \in B \).

So we next assume that \( x_n \in B \), for some \( n > N_0 \) and we shall prove that \( x_{n+1} \in B \).

Let if possible, this is not true, i.e. \( x_{n+1} \notin B \), so that \( \|x_{n+1} - x^*\| > r \).

Thus by (2.1.5) and Lemma 2.1.9, we have

\[
\|x_{n+1} - x^*\|^2 = \|x_n - x^* - \lambda_n(\alpha_n Ax_n + \theta_n(x_n - x_1))\|^2 \\
\leq \|x_n - x^*\|^2 - 2\lambda_n(\alpha_n Ax_n + \theta_n(x_n - x_1), j(x_{n+1} - x^*)) \\
\leq \|x_n - x^*\|^2 - 2\lambda_n\alpha_n(Ax_{n+1}, j(x_{n+1} - x^*)) - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 \\
+ 2\lambda_n\alpha_n(Ax_{n+1} - Ax_n, j(x_{n+1} - x^*)) + 2\lambda_n\theta_n(x_{n+1} - x_n, j(x_{n+1} - x^*)) \\
+ 2\lambda_n\theta_n(x_1 - x^*, j(x_{n+1} - x^*)) \\
\leq \|x_n - x^*\|^2 - 2\lambda_n\alpha_n(Ax_{n+1}, j(x_{n+1} - x^*)) - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 \\
+ 2\lambda_n\theta_n\|x_1 - x^*\|\|x_{n+1} - x^*\| + 2\lambda_n\theta_n\|x_{n+1} - x_n\|\|x_{n+1} - x^*\| \\
+ 2\lambda_n\alpha_n\|Ax_{n+1} - Ax_n\|\|x_{n+1} - x^*\|. \quad (2.1.6)
\]

Since $A$ is a bounded operator, so we can define

\[
M_0 := \sup\{|x - Ax| : \|x - x^*\| \leq 4r\} \quad (2.1.7)
\]

and by uniform continuity of $A$, for given $\epsilon_0 > 0$, there exists $\delta > 0$ such that

\[
\|Ax - Ay\| < \epsilon_0, \text{ whenever } \|x - y\| < \delta. \quad (2.1.8)
\]

Define

\[
\gamma_0 := \frac{1}{2} \min\left\{1, \frac{\delta}{M_0 + \frac{3}{2}r}\right\} \quad (2.1.9)
\]

and let

\[
\lambda_n \leq \frac{r}{8(M_0 + \frac{3}{2}r)}, \quad \frac{\alpha_n}{\theta_n} \leq \frac{r}{4\epsilon_0}, \forall n \geq N_0. \quad (2.1.10)
\]
Also we have

\[
\|x_{n+1} - x_n\| \leq \lambda_n \alpha_n \|x_n - Ax_n\| + \lambda_n \theta_n \|x_n - x_1\|
\]
\[
\leq \lambda_n \left[ \alpha_n M_0 + \theta_n \|x_n - x^*\| + \theta_n \|x_1 - x^*\| \right]
\]
\[
\leq \lambda_n \left[ \alpha_n M_0 + \frac{3}{2} \theta_n r \right]
\]
\[
\leq \gamma_0 [M_0 + \frac{3}{2} r]
\]
\[
\leq \delta,
\]

which implies that

\[
\|Ax_{n+1} - Ax_n\| < \epsilon_0.
\] (2.1.12)

Again since \(A\) is accretive, so

\[
(Ax_{n+1}, j(x_{n+1} - x^*)) \geq 0.
\] (2.1.13)

Hence using the above estimates (2.1.9)-(2.1.13), equation (2.1.6) becomes

\[
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + 2\lambda_n \alpha_n \epsilon_0 \|x_{n+1} - x^*\| - 2\lambda_n \theta_n \|x_{n+1} - x^*\|^2
\]
\[
+ 2\lambda_n^2 \theta_n \left[ \alpha_n M_0 + \theta_n \|x_n - x^*\| + \theta_n \|x_1 - x^*\| \right] \|x_{n+1} - x^*\|
\]
\[
+ 2\lambda_n \theta_n \|x_{n+1} - x^*\| \|x_1 - x^*\|,
\]

Since \(\|x_{n+1} - x^*\| > \|x_n - x^*\|\) by our assumption, thus

\[
0 \leq 2\lambda_n \alpha_n \epsilon_0 \|x_{n+1} - x^*\| - 2\lambda_n \theta_n \|x_{n+1} - x^*\|^2
\]
\[
+ 2\lambda_n^2 \theta_n \left[ \alpha_n M_0 + \theta_n \|x_n - x^*\| + \theta_n \|x_1 - x^*\| \right] \|x_{n+1} - x^*\|
\]
\[
+ 2\lambda_n \theta_n \|x_{n+1} - x^*\| \|x_1 - x^*\|,
\]
or,

\[
2\lambda_n \theta_n \|x_{n+1} - x^*\| \leq 2\lambda_n \alpha_n \epsilon_0 + 2\lambda_n^2 \theta_n \left[ \alpha_n M_0 + \theta_n \|x_n - x^*\| + \theta_n \|x_1 - x^*\| \right] \\
+ 2\lambda_n \theta_n \|x_1 - x^*\|
\]

\[
\leq 2\lambda_n \alpha_n \epsilon_0 + 2\lambda_n \theta_n \left[ \alpha_n M_0 + \theta_n r + \theta_n \frac{r}{2} \right] + 2\lambda_n \theta_n \frac{r}{2}
\]

\[
\leq 2\lambda_n \alpha_n \epsilon_0 + 2\lambda_n \theta_n \left( \alpha_n M_0 + \frac{3}{2} \theta_n r \right) + 2\lambda_n \theta_n \frac{r}{2},
\]

where \( x_n \in B \) and \( x_1 \in B_{\frac{r}{2}}(x^*) \).

This implies using (2.1.10) that

\[
\|x_{n+1} - x^*\| \leq \frac{\alpha_n}{\theta_n} \epsilon_0 + \lambda_n \left( M_0 + \frac{3}{2} r \right) + \frac{r}{2}
\]

\[
\leq \frac{r}{4} + \frac{r}{2} + \frac{r}{8}
\]

\[
< r,
\]

which is a contradiction of our assumption that \( x_{n+1} \notin B \).

Hence, \( x_n \in B \) for all \( n \geq N_0 \), which implies that \( \{x_n\}_{n=1}^{\infty} \) is bounded.

\[ \square \]

**Remark 2.1.11.** Since \( \{x_n\}_{n=1}^{\infty} \) is bounded, so let \( \varphi : E \to \mathbb{R} \) be defined by

\[
\varphi(y) = \mu_n \|x_n - y\|^2, \forall \ y \in E,
\]

(2.1.14)

where \( \mu_n \) is the Banach limit.

Then \( \varphi(y) \) is convex and continuous and \( \varphi(y) \to \infty \) as \( \|y\| \to \infty \). Since \( E \) is reflexive with uniformly Gateaux differentiable norm, thus, there exists \( z \in E \) such that \( \varphi(z) = \inf_{y \in E} \varphi(y) \) [148]. So the set

\[
E_{\text{min}} = \{ z \in E : \varphi(z) = \inf_{y \in E} \varphi(y) \} \neq \emptyset.
\]

Thus, \( E_{\text{min}} \) is non-empty, closed, convex and bounded by the convexity and continuity of \( \varphi(y) \) [145].
Theorem 2.1.12.† Let $E$ be a real reflexive Banach space with uniformly Gateaux differentiable norm and let $A : E \to E$ be a uniformly continuous accretive operator with $\mathcal{N}(A) \neq \emptyset$. Let $x_1 \in E$ be fixed and $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by

$$x_{n+1} = x_n - \lambda_n (\alpha_n Ax_n + \theta_n (x_n - x_1)), \quad n \geq 1, \quad (2.1.15)$$

where $\{\lambda_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty} \in (0, 1)$, satisfy the conditions (2.1.4) of Theorem 2.1.10 with $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$. Suppose that $E_{\min} \cap \mathcal{N}(A) \neq \emptyset$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique element $x^* \in \mathcal{N}(A)$.

Proof. The existence of a solution of the equation $Ax = 0$, for a continuous accretive operator $A$ follows from [107, pp 312] and the uniqueness is obvious from the definition of accretivity. Let $x^*$ be the unique solution and $x^* \in E_{\min} \cap \mathcal{N}(A)$.

Using Lemma 2.1.6

$$\mu_n \|x_n - x^*\|^2 = \min_{y \in E} \mu_n \|x_n - x_1\|^2$$

implies that

$$\mu_n (x_1 - x^*, j(x_{n+1} - x^*)) \leq 0. \quad (2.1.16)$$

On the other hand, as by Theorem 2.1.10, $\{x_n\}_{n=1}^{\infty}$ and hence $\{Ax_n\}_{n=1}^{\infty}$ is bounded, thus using the condition $\lim_{n \to \infty} \lambda_n = 0$, we have as $n \to \infty$

$$\|x_{n+1} - x_n\| \leq \lambda_n (\|Ax_n\| + \theta_n \|x_n - x_1\|) \to 0.$$

Thus, from the norm-weak* uniform continuity of the duality mapping, we have

$$\lim_{n \to \infty} ((x_1 - x^*, j(x_{n+1} - x^*)) - (x_1 - x^*, j(x_n - x^*))) = 0. \quad (2.1.17)$$

Thus using the relations (2.1.16) and (2.1.17), clearly the sequence $\{(x_1 - x^*, j(x_n - x^*))\}$ satisfies the conditions of Lemma 2.1.7, so that

†Int. J. Pure Appl. Math. 56(2009), no.3, 383-391
\[
\limsup_{n \to \infty} \langle x_1 - x^*, j(x_n - x^*) \rangle \leq 0.
\]

Taking \( \epsilon_n = \max\{\langle x_1 - x^*, j(x_{n+1} - x^*) \rangle, 0\} \), we get
\[
\lim_{n \to \infty} \epsilon_n = 0 \text{ and } (x_1 - x^*, j(x_{n+1} - x^*)) \leq \epsilon_n.
\] (2.1.18)

Finally, we show that \( x_n \to x^* \).

Using (2.1.15) and Lemma 2.1.9 again, we have
\[
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\lambda_n (\alpha_n Ax_n + \theta_n (x_n - x_1), j(x_{n+1} - x^*)) \\
\leq \|x_n - x^*\|^2 - 2\lambda_n \theta_n (x_{n+1} - x^*, j(x_{n+1} - x^*)) \\
+ 2\lambda_n \theta_n (x_{n+1} - x_n, j(x_{n+1} - x^*)) - 2\lambda_n \alpha_n (Ax_n, j(x_{n+1} - x^*)) \\
+ 2\lambda_n \theta_n (x_1 - x^*, j(x_{n+1} - x^*)) \\
\leq \|x_n - x^*\|^2 - 2\lambda_n \theta_n \|x_{n+1} - x^*\|^2 + 2\lambda_n \theta_n \|x_{n+1} - x_n\| \|x_{n+1} - x^*\| \\
+ 2\lambda_n \theta_n \langle x_1 - x^*, j(x_{n+1} - x^*) \rangle + 2\lambda_n \|Ax_n\| \|x_{n+1} - x^*\| \\
\leq \|x_n - x^*\|^2 - 2\lambda_n \theta_n \|x_{n+1} - x^*\|^2 + (2\lambda_n^2 \theta_n + 2\lambda_n \alpha_n) L + 2\lambda_n \theta_n \epsilon_n,
\]

for some constant \( L > 0 \).

Thus
\[
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\lambda_n \theta_n \|x_{n+1} - x^*\|^2 + \rho_n,
\] (2.1.19)

where \( \rho_n := 2\lambda_n [(\lambda_n \theta_n + \alpha_n) L + \theta_n \epsilon_n] \).

Obviously \( \lim_{n \to \infty} \frac{\rho_n}{\lambda_n \theta_n} = 0 \), by the condition (2.1.4) and (2.1.18), which implies that \( \rho_n = o(\lambda_n \theta_n) \).

Thus, using Lemma 2.1.8, \( \{x_n\}_{n=1}^\infty \) converges strongly to the unique solution \( x^* \) of the equation \( Ax = 0 \).

\[\square\]

**Remark 2.1.13.** Our result modifies the corresponding results of [47, 145] and the references therein, to uniformly continuous accretive operators with weaker condi-
tions on the parameters. Also the results of [28, 54, 134] and many others are extended to a more general reflexive Banach space.

Next we prove the strong convergence theorem for Pseudocontractive mappings.

**Theorem 2.1.14.** Let $E$ be a real reflexive Banach space with uniformly Gateaux differentiable norm. Let $K$ be a nonempty, closed, convex subset of $E$ and $T : K \to K$ be a uniformly continuous pseudocontractive mapping such that $\mathcal{F}(T) = \{x \in E : Tx = x\} \neq \emptyset$. Let $x_1 \in E$ be fixed and $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by

$$x_{n+1} = (1 - \lambda_n \alpha_n)x_n + \lambda_n(\alpha_n Tx_n - \theta_n(x_n - x_1)),$$

where $\{\lambda_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty} \in (0, 1)$ satisfy the conditions (2.1.4) of Theorem 2.1.10 with $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$. Suppose that $K_{\text{min}} \cap \mathcal{F}(T) \neq \emptyset$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $x^* \in \mathcal{F}(T)$, where $K_{\text{min}}$ is as the set $E_{\text{min}}$ defined in Remark 2.1.11.

**Proof.** We first observe that $T$ is pseudocontractive if and only if $A := I - T$ is accretive [20]. Again, $x^* \in K_{\text{min}} \cap \mathcal{F}(T)$ implies $Tx = x$, which again implies $Ax = 0$. Thus $K_{\text{min}} \cap \mathcal{F}(T) \neq \emptyset$. Hence, replacing $T$ by $I - A$ in (2.1.15), then boundedness of $\{x_n\}_{n=1}^{\infty}$ follows from Theorem 2.1.10 and rest of the result follows from Theorem 2.1.12.

$\Box$

We now prove our result in arbitrary Banach spaces, giving a strong convergence theorem for the zeros of uniformly continuous accretive operator in arbitrary Banach spaces without any Lipschitz condition on the operator and implementing weaker conditions on the parameters using the iteration scheme [47],

$$x_{n+1} = x_n - \lambda_n \alpha_n Ax_n - \lambda_n \theta_n(x_n - x_1), \quad \forall n \geq 1.$$

Similar result for uniformly continuous pseudocontractive map is also proved.
Our result follows as

**Theorem 2.1.15.** Let \( E \) be arbitrary Banach space and let \( A : E \to E \) be a uniformly continuous accretive operator. Let \( x_1 \in E \) be fixed and \( \{x_n\}_{n=1}^{\infty} \) be generated iteratively by

\[
x_{n+1} = x_n - \lambda_n (\alpha_n Ax_n + \theta_n (x_n - x_1)), \quad n \geq 1,
\]

where \( \{\lambda_n\}_{n=1}^{\infty}, \{\theta_n\}_{n=1}^{\infty} \in (0, 1) \) satisfy the conditions

\[
\lim_{n \to \infty} \lambda_n = 0, \quad \alpha_n = o(\theta_n),
\]

with,

\[
\lambda_n \theta_n < 1, \quad \sum_{n=1}^{\infty} \lambda_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty.
\]

Then \( \{x_n\}_{n=1}^{\infty} \) converges strongly to the unique element \( x^* \in N(A) \).

**Proof.** The existence of a solution of the equation \( Ax = 0 \), for a continuous accretive operator \( A \) follows from [107, pp 312] and the uniqueness is obvious from the definition of accretivity. Let \( x^* \in N(A) \) be the unique solution. The boundedness of the sequence \( \{x_n\} \) follows from Theorem 2.1.10.

Set

\[
M_1 := 2 \sup_n \|x_1 - x^*\| \|x_{n+1} - x^*\|,
\]

\[
M_2 := 2 \sup_n \|Ax_n\| \|x_{n+1} - x_1\|,\]

\[
M_3 := 2 \sup_n \|x_n - Ax_n\| \|x_{n+1} - x^*\|.
\]

Using (2.1.20) and Lemma 2.1.9 again, we have

\[
\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2 \lambda_n \alpha_n \|Ax_{n+1} - x^*\|^2 - 2 \lambda_n \theta_n \|x_{n+1} - x^*\|^2
\]

\[
+ 2 \lambda_n \theta_n \|x_{n+1} - x_1\| \|Ax_{n+1} - j(x_{n+1} - x^*)\| + 2 \lambda_n \theta_n \|x_{n+1} - x^*\| \|x_{n+1} - x^*\|
\]

\[
+ 2 \lambda_n \alpha_n \|Ax_{n+1} - Ax_n - j(x_{n+1} - x^*)\|
\]

\[
\leq \|x_n - x^*\|^2 - 2 \lambda_n \theta_n \|x_{n+1} - x^*\|^2 + 2 \lambda_n \theta_n \|x_{n+1} - x^*\| \|x_{n+1} - x^*\|
\]

\[
+ 2 \lambda_n \theta_n \|Ax_{n+1}\| + \theta_n \|x_n - x_1\| \|x_{n+1} - x^*\|
\]

\[
+ 2 \lambda_n \alpha_n \|Ax_{n+1} - Ax_n\| \|x_{n+1} - x^*\|.
\]

Thus

\[ \|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\ell\theta_n\|x_{n+1} - x^*\|^2 + \ell\theta_n M_2\|Ax_{n+1} - Ax_n\| + \ell_2 \theta_n M_3 + \ell \theta_n M_1. \]  

(2.1.23)

Next we claim that \( \inf\{\|x_{n+1} - x^*\|; n \geq 0\} = 0 \).

Let if possible, this is not true, i.e. let \( \inf\{\|x_{n+1} - x^*\|; n \geq 0\} = \delta > 0 \).

Then \( \|x_{n+1} - x^*\| > \delta, \forall n \geq 0 \).

Again since

\[ \|x_{n+1} - x_n\| \leq \ell_n[\alpha_n\|Ax_n\| + \theta_n\|x_n - x_1\|] \rightarrow 0 \text{ as } n \rightarrow \infty, \]

so that by uniform continuity of \( A \), there exists \( N_0 > 0 \) such that

\[ \|Ax_{n+1} - Ax_n\| < \frac{\delta^2}{M_2}, \forall n \geq N_0. \]

Hence, for all \( n \geq N_0 \), (2.1.23) becomes

\[ \|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2\ell\theta_n\delta^2 + \ell n\theta_n M_3 + \ell \theta_n M_1, \]

\[ \ell n\theta_n\delta^2 \leq (\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) + \ell n(\delta^2 + M_1) + \ell_2 \theta_n M_3, \]

i.e.

\[ \delta^2 \sum_{i=1}^{n} \ell_i \theta_i \leq \sum_{i=1}^{n} (\|x_i - x^*\|^2 - \|x_{i+1} - x^*\|^2) + \sum_{i=1}^{n} \ell_i(\delta^2 + M_1 + M_3), \]

which implies that \( \sum_{n=1}^{\infty} \ell_n \theta_n < \infty \), since \( \sum_{n=1}^{\infty} \ell_n < \infty \), which is a contradiction since \( \sum_{n=1}^{\infty} \ell_n \theta_n = \infty \). Hence our claim is true.

Thus, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that \( \lim_{n \rightarrow \infty} \|x_{n_j} - x^*\| = 0 \).
Let $\epsilon > 0$ be given. Since $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and $A$ is uniformly continuous, so we can choose an integer $N_1 > 0$ such that $\forall \ n \geq N_1$

$$\|Ax_{n+1} - Ax_n\| \leq \frac{\epsilon}{M_2}.$$ 

Now choose an integer $N_2 > N$, such that $\|x_{n_i} - x^*\| < \epsilon$, $\forall \ i \geq N_2$.

Fix $i_* \geq N_2$. Then

$$\|x_{n_{i_*}} - x^*\| < \epsilon.$$ 

We next claim that $\|x_{n_{i_*}+m} - x^*\| < \epsilon$, $\forall \ m = 1, 2, \cdots$.

We prove it by induction on $m$. So we first show that $\|x_{n_{i_*}+1} - x^*\| < \epsilon$.

Suppose this is not true i.e. $\|x_{n_{i_*}+1} - x^*\| \geq \epsilon$.

Then since

$$\|x_{n+1} - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 - 2\lambda_n \theta_n \|x_{n+1} - x^*\|^2 + \lambda_n M_3 \|Ax_{n+1} - Ax_n\| + \lambda_n^2 \theta_n M_3 + \lambda_n M_1,$$

thus

$$\|x_{n_{i_*}+1} - x^*\|^2 \leq \|x_{n_{i_*}} - x^*\|^2 - 2\lambda_n \theta_n \|x_{n_{i_*}} - x^*\|^2 + \lambda_n^2 \theta_n \|x_{n_{i_*}} - x^*\|^2 + \lambda_n^2 \theta_n M_3 + \lambda_n M_1,$$

or

$$2\lambda_n \theta_n \|x_{n_{i_*}} - x^*\|^2 \leq \|x_{n_{i_*}} - x^*\|^2 - \|x_{n_{i_*}} - x^*\|^2 + \frac{3}{2} \epsilon^2 \lambda_n.$$ 

This implies that

$$2\epsilon^2 \sum_{j=1}^{n_{i_*}} \lambda_j \theta_j \leq (\|x_{n_{j+1}} - x^*\|^2 - \|x_{n_{j}} - x^*\|^2) + \frac{3}{2} \epsilon^2 \sum_{j=1}^{n_{i_*}} \lambda_j < \infty,$$

since $\sum_{n=1}^{\infty} \lambda_n < \infty$, which is again a contradiction.

Hence, the claim holds for $m = 1$. Assume now it holds for $m = k$. Following the similar arguments as above, we can show that the claim holds for $m = k + 1$ also.

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Hence it is true for all the values of $m$. This implies that $\{x_n\}$ converges strongly to $x^*$ as $n \to \infty$. \hfill \Box

\textbf{Remark 2.1.16.} Our result modifies the corresponding results of \cite{43, 47, 145} and the references therein, to uniformly continuous accretive operators with weaker conditions on the parameters. Also the results of \cite{28, 54, 134} and many others are extended to a more general arbitrary Banach space.

Next we prove the strong convergence for Pseudocontractive mappings.

\textbf{Theorem 2.1.17.} Let $E$ be arbitrary Banach space and $T : E \to E$ be a uniformly continuous pseudocontractive mapping such that $\mathcal{F}(T) = \{x \in E : Tx = x\} \neq \emptyset$. Let $x_1 \in E$ be fixed and $\{x_n\}_{n=1}^\infty$ be generated iteratively by

$$x_{n+1} = (1 - \lambda_n \alpha_n)x_n + \lambda_n(\alpha_n Tx_n - \theta_n(x_n - x_1)), \quad n \geq 1,$$

where $\{\lambda_n\}_{n=1}^\infty$, $\{\theta_n\}_{n=1}^\infty \in (0, 1)$ satisfy the conditions of Theorem 2.1.15. Then $\{x_n\}_{n=1}^\infty$ converges strongly to some $x^* \in \mathcal{F}(T)$.

\textbf{Proof.} We first observe that $T$ is pseudocontractive if and only if $A = I - T$ is accretive \cite{20}. Hence, replacing $A$ by $I - T$ in (2.1.20), then boundedness of $\{x_n\}_{n=1}^\infty$ follows from Theorem 2.1.10 and rest of the result follows from Theorem 2.1.15. \hfill \Box

\section{2.2 Viscosity approximation method}

Moudafi \cite{118} proposed the viscosity approximation method of finding a fixed point of a nonexpansive mapping in a Hilbert space, defined as

$$x_{n+1} = \frac{\epsilon_n}{1 + \epsilon_n} f(x_n) + \frac{1}{1 + \epsilon_n} T(x_n). \quad (2.2.1)$$
Xu [160] further studied the continuous version of (2.2.1) for nonexpansive mappings as

\[ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(x_n). \]

Recently, Takahashi [149] considered the sequence generated by the algorithm

\[ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J_{t_n}x_n, \quad n \geq 0, \tag{2.2.2} \]

and proved its strong convergence for the accretive operators in a reflexive Banach space with a uniformly Gateaux differentiable norm.

On the other hand, Martinet [108] introduced the \textit{proximal point algorithm}. In 1976, using the proximal point algorithm, Rockafellar [137] proved the weak convergence of the sequence \( \{x_n\} \) generated by

\[ x_{n+1} = J_{r_n}x_n, \]

for all \( n \in \mathbb{N} \), \( \{r_n\} \subset (0, \infty) \) satisfying \( \lim \inf_{n \to \infty} r_n > 0 \), to an element of \( A^{-1}0 = \{x \in D(A) : 0 \in Ax\} \), where \( A \subset H \times H \) be a maximal monotone operator in the Hilbert space \( H \) and \( J_r \) is the resolvent of \( A \), given by \( J_r = (I + rA)^{-1} \) for all \( r > 0 \).

In 1967, Halpern [73] introduced an explicit iterative process

\[ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \forall n \geq 0, \]

where \( \{\alpha_n\} \subset [0, 1] \), for a nonexpansive mapping \( T \) in the framework of Hilbert spaces. Kamimura and Takahashi [85] proved the strong convergence for a monotone operator in a Hilbert space to \( P_{A^{-1}0}(x) \), where \( P_{A^{-1}0} \) is the metric projection from
on to $A^{-1}0$, and the sequence $\{x_n\}$ is defined as: $x_1 = x \in H$, 

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{r_n}x_n,$$  \hspace{1cm} (2.2.3)

for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy (C1) $\lim_{n \to \infty} \alpha_n = 0$, and (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} r_n = \infty$.

In 2005, Kim and Xu [92] extended the result of Kamimura and Takahashi [84, 85] for accretive operators in the framework of uniformly smooth Banach space using the iteration (2.2.3).

Motivated by the iterative sequences (2.2.2) and (2.2.3), we have proposed the following composite viscosity iterative sequence

$$y_n = \alpha_n x_n + (1 - \alpha_n)J_{t_n}x_n,
\begin{align*}
x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n)y_n, \quad n \geq 0,
\end{align*}$$  \hspace{1cm} (2.2.4)

$x_0 \in C$ chosen arbitrarily, $f$ is an $\alpha$-contraction, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$. Under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, we prove that $\{x_n\}$ defined by (2.2.4) converges to a zero of the accretive operator in a reflexive Banach space.

In our result, we have used this composite iteration scheme for the proximal point algorithm to prove a strong convergence theorem by the viscosity approximation method in a reflexive Banach space.

We recall some definitions needed to present our result.

**Definition 2.2.1.** [148] Let $U = \{x \in E : \|x = 1\|\}$. The norm of $E$ is said to be Gateaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$  \hspace{1cm} (2.2.5)
exists. In this case, \( E \) is said to be *smooth*. The norm of \( E \) is said to be *uniformly Gateaux differentiable* if for each \( y \in U \), the limit (2.2.5) is attained finite, uniformly for \( x \in U \).

If \( E \) is smooth, then the duality mapping \( J \) is single-valued [148, Theorem 4.3.1]. In the sequel, we shall denote the single-valued normalized duality map by \( J \). Further, if the norm of \( E \) is uniformly Gateaux differentiable, then \( J \) is uniformly norm to weak* continuous on each bounded subset of \( E \) [148, Theorem 4.3.6].

**Definition 2.2.2.** Let \( C \) be a closed convex subset of \( E \). A mapping \( T : C \to C \) is said to be *nonexpansive* if \( \|Tx - Ty\| \leq \|x - y\| \), for all \( x, y \in C \). Let \( \mathcal{F}(T) \) denotes the set of all fixed points of \( T \).

A mapping \( f : C \to C \) is called *\( a \)-contractive* if there exists \( a \in [0,1) \) such that

\[
\|f(x) - f(y)\| \leq a \|x - y\|
\]

for every \( x, y \in C \).

**Definition 2.2.3.** An operator \( A \subset E \times E \) with domain \( \mathcal{D}(A) = \{z \in E : Az \neq \emptyset\} \) and range \( \mathcal{R}(A) = \bigcup \{Az : z \in \mathcal{D}(A)\} \) is called *accretive* if for each \( x_i \in \mathcal{D}(A) \) and \( y_i \in Ax_i, i = 1,2 \), there exists \( j(x_1 - x_2) \in J(x_1 - x_2) \) such that

\[
(y_1 - y_2, j(x_1 - x_2)) \geq 0.
\]

For an accretive operator \( A \), the *resolvent* \( J_r : R(I + rA) \to \mathcal{D}(A) \) is defined by \( J_r = (I + rA)^{-1} \). For each \( r > 0 \), \( J_r \) is a nonexpansive single-valued mapping and \( A^{-1}0 = \mathcal{F}(J_r) \), where \( A^{-1}0 = \{u \in \mathcal{D}(A) : 0 \in Au\} \) and \( \mathcal{F}(J_r) = \{x \in \mathcal{D}(A) : J_r x = x\} \) (see for details [91, pp 141]).

**Definition 2.2.4.** [148] A closed convex subset \( C \) of a Banach space \( E \) is said to have *normal structure* if for each bounded closed convex subset \( K \) of \( C \) which contains at least two points, there exists an element \( x \) of \( K \) which is not a diametral
point of $K$, where a point $x \in K$ is a diametral point of $K$ if $r_x(K) = \delta(K)$. Here $\delta(K)$ is the diameter of $K$ defined as $\delta(K) = \sup\{\|x - y\| : x, y \in K\}$ and $r_x(K) = \sup\{\|x - y\| : y \in K\}$.

A closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure [148].

**Definition 2.2.5.** [134] Let $C \subset E$ be nonempty. Let $D$ be a subset of $C$ and let $P$ be a retraction of $C$ onto $D$, i.e., $Px = x$ for each $x \in D$. Then $P$ is said to be **sunny** if for all $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$, $P(Px + t(x - Px)) = Px$. A subset $D$ of $C$ is said to be a **sunny nonexpansive retract** of $C$, if there exits a sunny nonexpansive retraction $P$ of $C$ onto $D$.

If $E$ is smooth and $P$ is a retraction of $C$ onto $D$, then $P$ is sunny and nonexpansive if and only if for each $x \in C$ and $z \in D$,

$$(x - Px, J(z - Px)) \leq 0.$$  

**Theorem 2.2.6.** [149] Let $E$ be a reflexive Banach space with a uniformly Gateaux differentiable norm and let $C$ be a nonempty closed convex subset of $E$ which has normal structure. Let $A \subset E \times E$ be an accretive operator with $A^{-1}0 \neq \emptyset$ satisfying $\overline{D(A)} \subset C \subset \bigcap_{t > 0} R(I + tA)$ and let $J_t$ be the resolvent of $A$ for $t > 0$. Let $f$ be an $\alpha$-contractive mapping of $C$ into itself. Then the following holds:

(i) for $t > 0$, $J_tf$ has a unique fixed point $u_t$ in $C$;

(ii) if $t \to \infty$, then the net $\{u_t\}$ converges strongly to $u \in A^{-1}0$, where $u = P_{A^{-1}0}fu$ and $P_{A^{-1}0}$ is a sunny nonexpansive retraction of $C$ onto $A^{-1}0$.

We shall make use of the following lemmas.

**Lemma 2.2.7.** *(The Resolvent Identity)*[7] For $\lambda, \mu > 0$, there holds the identity,

$$J_\lambda x = J_\mu \left( \frac{\mu}{\lambda} x + (1 - \frac{\mu}{\lambda}) J_\lambda x \right)$$  

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Lemma 2.2.8. [159]. Let $\sum_{n=0}^{\infty} \alpha_n$ be a sequence of non-negative real numbers satisfying the condition

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n \sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \in (0, 1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ be such that (i) $\lim_{n \to \infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) either $\limsup_{n \to \infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$. Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

Now we present our result.

Theorem 2.2.9. Let $E$ be a reflexive Banach space with a uniformly Gateaux differentiable norm and let $C$ be a nonempty closed convex subset of $E$ which has normal structure. Let $A \subset E \times E$ be an accretive operator with $A^{-1}0 \neq \emptyset$ satisfying, $\overline{D(A)} \subseteq C \subset \bigcap_{t>0} \mathcal{R}(I + tA)$ and let $J_t$ be resolvent of $A$ for $t > 0$. Let $f : C \to C$ be an a contractive map. Let $\{x_n\}$ be a sequence in $C$ defined by $x_0 \in C$,

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n, \quad y_n = \alpha_n x_n + (1 - \alpha_n)J_{t_n}x_n, \quad n \geq 0, \tag{2.2.6}$$

where $\{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset [0, 1], \{t_n\} \subset (0, \infty)$ satisfy the conditions

(i) $\sum_{n=0}^{\infty} \beta_n = \infty, \lim_{n \to \infty} \beta_n = 0$,

(ii) $t_n \geq \epsilon, \forall n$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.

Then $\{x_n\}$ converges strongly to $u \in A^{-1}0$, where $u = Pf(u)$ and $P$ is a sunny nonexpansive retraction of $C$ onto $A^{-1}0$.

Proof. We first prove that $\{x_n\}$ is bounded.

For this, let $z \in A^{-1}0$ and put $M = \max\{\|x_0 - z\|, 1/(1 - a)\|f(z) - z\|\}$, where $a \in [0, 1)$ is the constant appearing in the definition of $a$-contractive mapping. Then
it is obvious that \(\|x_0 - z\| \leq M\). Suppose that \(\|x_n - z\| \leq M\), for some \(n \in \mathbb{N}\). Then we have

\[
\|y_n - z\| = \|\alpha_n x_n + (1 - \alpha_n)J_{t_n}x_n - z\|
\]
\[
= \|\alpha_n(x_n - z) + (1 - \alpha_n)(J_{t_n}x_n - z)\|
\]
\[
\leq \alpha_n \|x_n - z\| + (1 - \alpha_n) \|J_{t_n}x_n - z\|
\]
\[
\leq \alpha_n \|x_n - z\| + (1 - \alpha_n) \|x_n - z\|
\]
\[
= \|x_n - z\|.
\]

Thus

\[
\|x_{n+1} - z\| = \|\beta_n f(x_n) + (1 - \beta_n) y_n - z\|
\]
\[
= \|\beta_n(f(x_n) - z) + (1 - \beta_n)(y_n - z)\|
\]
\[
\leq \beta_n \|f(x_n) - z\| + (1 - \beta_n) \|y_n - z\|
\]
\[
\leq \beta_n \|f(x_n) - f(z)\| + \beta_n \|f(z) - z\| + (1 - \beta_n) \|y_n - z\|
\]
\[
\leq a \beta_n \|x_n - z\| + \beta_n \|f(z) - z\| + (1 - \beta_n) \|f_n - z\|
\]
\[
= (1 - (1 - a) \beta_n) \|x_n - z\| + \beta_n(1 - a) \|f(z) - z\| / (1 - a)
\]
\[
\leq (1 - \beta_n(1 - a))M + \beta_n(1 - a)M
\]
\[
= M.
\]

So that

\[
\|x_n - z\| \leq M, \quad \forall \ n \in \mathbb{N},
\]

and hence \(\{x_n\}\) is bounded and also \(\{f(x_n)\}\) and \(\{y_n\}\) are bounded.
From definition of $x_{n+1}$ and condition (i),

$$
\|x_{n+1} - y_n\| = \|\beta_n f(x_n) + (1 - \beta_n)y_n - y_n\|
= \beta_n \|f(x_n) - y_n\|
\to 0. \quad (2.2.7)
$$

Next we show that

$$
\|x_{n+1} - x_n\| \to 0. \quad (2.2.8)
$$

In order to prove (2.2.8), we first calculate $(y_{n+1} - y_n)$, where from (2.2.6)

$$
y_n = \alpha_n x_n + (1 - \alpha_n)J_{t_n}x_n,
$$

$$
y_{n-1} = \alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1})J_{t_{n-1}}x_{n-1}.
$$

Hence

$$
\|y_n - y_{n-1}\| \leq \alpha_n \|x_n - x_{n-1}\| + (1 - \alpha_n)\|J_{t_n}x_n - J_{t_{n-1}}x_{n-1}\|
+ |\alpha_n - \alpha_{n-1}| \|x_{n-1} - J_{t_{n-1}}x_{n-1}\|. \quad (2.2.9)
$$

From Lemma 2.2.7 (Resolvent Identity)

$$
J_{t_n}x_n = J_{t_{n-1}} \left( t_{n-1} \frac{t_n}{t_n} + (1 - t_{n-1}) \frac{t_{n-1}}{t_n} \right) x_n.
$$

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Now if \( t_{n-1} \leq t_n \), then using above identity

\[
\|J_{t_n}x_n - J_{t_{n-1}}x_{n-1}\| = \|J_{t_{n-1}} \left( \frac{t_{n-1}}{t_n} + (1 - \frac{t_{n-1}}{t_n})J_{t_n} \right) x_n - J_{t_{n-1}}x_{n-1}\|
\]
\[
\leq \left\| \frac{t_{n-1}}{t_n}x_n + (1 - \frac{t_{n-1}}{t_n})J_{t_n}x_n - x_{n-1} \right\|
\]
\[
= \left\| \frac{t_{n-1}}{t_n}(x_n - x_{n-1}) + (1 - \frac{t_{n-1}}{t_n})(J_{t_n}x_n - x_{n-1}) \right\|
\]
\[
\leq \|x_n - x_{n-1}\| + \left( \frac{t_n - t_{n-1}}{\epsilon} \right) \|J_{t_n}x_n - x_{n-1}\|
\]
\[
\leq \|x_n - x_{n-1}\| + \left( \frac{t_n - t_{n-1}}{\epsilon} \right) \|J_{t_n}x_n - x_{n-1}\|.
\]

Putting (2.2.10) into (2.2.9) we have

\[
\|y_n - y_{n-1}\| \leq \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - J_{t_{n-1}}x_{n-1}\|
\]
\[
+ (1 - \alpha_n) \left( \|x_n - x_{n-1}\| + \left( \frac{t_n - t_{n-1}}{\epsilon} \right) \|J_{t_n}x_n - x_{n-1}\| \right)
\]
\[
= \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - J_{t_{n-1}}x_{n-1}\|
\]
\[
+ (1 - \alpha_n) \left( \frac{t_n - t_{n-1}}{\epsilon} \right) \|J_{t_n}x_n - x_{n-1}\|
\]
\[
\leq \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1} - J_{t_{n-1}}x_{n-1}\|
\]
\[
+ \left( \frac{t_n - t_{n-1}}{\epsilon} \right) \|J_{t_n}x_n - x_{n-1}\|
\]
\[
\leq \|x_n - x_{n-1}\| + M_1( |\alpha_n - \alpha_{n-1}| + |t_n - t_{n-1}|),
\]

where \( M_1 \) is a constant such that \( M_1 > \max \left\{ \|x_{n-1} - J_{t_{n-1}}x_{n-1}\|, \frac{\|J_{t_n}x_n - x_{n-1}\|}{\epsilon} \right\} \).

On the other hand by (2.2.6),

\[
x_{n+1} - x_n = (1 - \beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(f(x_{n-1}) - y_{n-1})
\]
\[
+ \beta_n(f(x_n) - f(x_{n-1})),
\]

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Therefore
\[
\|x_{n+1} - x_n\| \leq (1 - \beta_n)\|y_n - y_{n-1}\| + (\beta_n - \beta_{n-1})\|f(x_{n-1}) - y_{n-1}\|
\]
\[+ a\beta_n\|x_n - x_{n-1}\|. \tag{2.2.12}
\]

Substituting (2.2.11) into (2.2.12)
\[
\|x_{n+1} - x_n\| \leq (1 - \beta_n)(\|x_n - x_{n-1}\| + M_1(\|\alpha_n - \alpha_{n-1}\| + |t_n - t_{n-1}|))
\]
\[+ a\beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1}) - y_{n-1}\|
\]
\[= (1 - (1 - a)\beta_n)\|x_{n+1} - x_n\| + (1 - \beta_n)M_1(\|\alpha_n - \alpha_{n-1}\| + |t_n - t_{n-1}|)
\]
\[+ |\beta_n - \beta_{n-1}|\|f(x_{n-1}) - y_{n-1}\|
\]
\[\leq (1 - (1 - a)\beta_n)\|x_{n+1} - x_n\| + M_2(\|\alpha_n - \alpha_{n-1}\| + |\beta_n - \beta_{n-1}|)
\]
\[+ |t_n - t_{n-1}|, \tag{2.2.13}
\]

where $M_2$ is a constant such that $M_2 > \max\{M_1, \|f(x_{n-1}) - y_n\|\}$.

Similarly we can prove (2.2.13) if $t_{n-1} \geq t_n$.

From the conditions (i)- (iii), $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} (\|\alpha_{n+1} - \alpha_n\| + |\beta_{n+1} - \beta_n| + |t_{n+1} - t_n|) < \infty$.

Hence applying Lemma 2.2.8, we obtain
\[
\|x_{n+1} - x_n\| \to 0. \tag{2.2.14}
\]

We next prove that $\|x_n - J_{t_n}x_n\| \to 0$.

From (2.2.6)
\[
\|J_{t_n}x_n - y_n\| = \|\alpha_n J_{t_n}x_n - \alpha_n x_n\| = \alpha_n \|J_{t_n}x_n - x_n\|,
\]
\[
\| J_t x_n - x_n \| \leq \| J_t x_n - y_n \| + \| y_n - x_{n+1} \| + \| x_{n+1} - x_n \|
\]

\[
= \alpha_n \| J_t x_n - x_n \| + \| y_n - x_{n+1} \| + \| x_{n+1} - x_n \|
\]

i.e. \( (1 - \alpha_n) \| J_t x_n - x_n \| \leq \| y_n - x_{n+1} \| + \| x_{n+1} - x_n \| \).

By (2.2.8) and (2.2.14), we obtain

\[
\| J_t x_n - x_n \| \to 0. \tag{2.2.15}
\]

So that

\[
\| x_{n+1} - J_t x_n \| \leq \| x_{n+1} - y_n \| + \| y_n - J_t x_n \|
\]

\[
= \beta_n \| f(x_n) - y_n \| + \alpha_n \| J_t x_n - x_n \|
\]

\[
\to 0.
\]

For any \( t > 0 \), consider \( u_t \in C \) such that \( u_t = J_t f u_t \).

Theorem 2.2.6 implies that \( u_t \to u \) as \( t \to \infty \), where \( u = Pf_u \) and \( P \) is a sunny nonexpansive retraction of \( C \) onto \( A^{-1}0 \).

For such a \( u \in C \), let us prove the following inequality:

\[
\limsup_{n \to \infty} (f(u) - u, J(x_n - u)) \leq 0. \tag{2.2.16}
\]

Since \( E \) has a uniformly Gateaux differentiable norm, so \( J \) is uniformly continuous on any bounded subset of \( E \), so we have

\[
\lim_{n \to \infty} | (f(u) - u, J(x_{n+1} - u)) - (f(u) - u, J(J_t x_n - u)) | = 0
\]

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and hence (2.2.16) is equivalent to

$$\limsup_{n \to \infty} (f(u) - u, J(J_{t_n} x_n - u)) \leq 0.$$  \hfill (2.2.17)

So we now prove (2.2.17).

Let $\epsilon > 0$, since $u_t \to u$ and $J$ is uniformly continuous on any bounded set, so $\exists \ t_0 > 0$ such that

$$|(f(u) - u, J(J_{t_n} x_n - u)) - (f(u) - u, J(J_{t_n} x_n - u))| < \epsilon/3 \quad \forall \ t \geq t_0, \ n \in \mathbb{N}.$$

Further, since $f(u_t) - u_t \to f(u) - u$ and $\{u_t\}, \{J_{t_n} x_n\}$ are bounded, so $\exists s_0 > 0$ such that

$$|(f(u) - u, J(J_{t_n} x_n - u_t)) - (f(u_t) - u_t, J(J_{t_n} x_n - u_t))| < \epsilon/3, \quad \forall \ t \geq s_0, \ n \in \mathbb{N}.$$

On the other hand, since $A_t f(u_t) \in AJ_t f(u_t) = Au_t$ and $A_{t_n} x_n \in AJ_{t_n} x_n$, so we have

$$0 \leq (A_{t_n} x_n - A_t f(u_t), J(J_{t_n} x_n - u_t))$$
$$= (A_{t_n} x_n - (f(u_t) - u_t)/t, J(J_{t_n} x_n - u_t)),$$

hence

$$(f(u_t) - u_t, J(J_{t_n} x_n - u_t)) \leq t(A_{t_n} x_n, J(J_{t_n} x_n - u_t)).$$

Also, since

$$\|A_{t_n} x_n\| = \|x_n - J_{t_n} x_n\| \to 0 \quad n \to \infty,$$

so, for a fixed $t \geq \max\{t_0, s_0\}, \exists n_0 \in \mathbb{N}$ such that

$$(f(u_t) - u_t, J(J_{t_n} x_n - u_t)) < \epsilon/3, \quad \forall \ n \geq n_0.$$
Hence for any \( n \geq n_0 \)

\[
(f(u) - u, J(J_{t_n}x_n - u)) = (f(u) - u, J(J_{t_n}x_n - u)) - (f(u) - u, J(J_{t_n}x_n - u_t)) \\
+ (f(u) - u, J(J_{t_n}x_n - u_t)) - (f(u_t) - u_t, J(J_{t_n}x_n - u_t)) \\
+ (f(u_t) - u_t, J(J_{t_n}x_n - u_t)) \\
< \epsilon,
\]

and so

\[
\limsup_{n \to \infty} (f(u) - u, J(J_{t_n}x_n - u)) \leq \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, so

\[
\limsup_{n \to \infty} (f(u) - u, J(J_{t_n}x_n - u)) \leq 0
\]

and hence

\[
\limsup_{n \to \infty} (f(u) - u, J(x_n - u)) \leq 0.
\]

Next we prove that \( x_n \to u \).

By (2.2.6) and Lemma 2.1.9, we have

\[
\|y_n - u\|^2 = \|(1 - \alpha_n)(J_{t_n}x_n - u) + \alpha_n(x_n - u)\|^2 \\
\leq (1 - \alpha_n)^2 \|J_{t_n}x_n - u\|^2 + 2\alpha_n((x_n - u), J(y_n - u)) \\
\leq (1 - \alpha_n)^2 \|x_n - u\|^2 + 2\alpha_n \|x_n - u\| \|y_n - u\| \\
\leq (1 - \alpha_n)^2 \|x_n - u\|^2 + \alpha_n [\|x_n - u\|^2 + \|y_n - u\|^2]
\]

or

\[
(1 - \alpha_n) \|y_n - u\|^2 \leq ((1 - \alpha_n)^2 + \alpha_n) \|x_n - u\|^2 \\
\leq (1 - \alpha_n)^2 \|x_n - u\|^2
\]

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so,

$$\|y_n - u\|^2 \leq \|x_n - u\|^2.$$  

Hence

$$\|x_{n+1} - u\|^2 = \|(1 - \beta_n)(y_n - u) + \beta_n(f(x_n) - u)\|$$

$$\leq (1 - \beta_n)^2 \|y_n - u\|^2 + 2\beta_n(f(x_n) - u, J(x_{n+1} - u))$$

$$\leq (1 - \beta_n)^2 \|x_n - u\|^2 + 2\beta_n(f(x_n) - f(u), J(x_{n+1} - u))$$

$$+ 2\beta_n(f(u) - u, J(x_{n+1} - u))$$

$$\leq (1 - \beta_n)^2 \|x_n - u\|^2 + 2\beta_n a \|x_n - u\| \|x_{n+1} - u\|$$

$$+ 2\beta_n(f(u) - u, J(x_{n+1} - u))$$

$$\leq (1 - \beta_n)^2 \|x_n - u\|^2 + \beta_n a(\|x_n - u\|^2 + \|x_{n+1} - u\|^2)$$

$$+ 2\beta_n(f(u) - u, J(x_{n+1} - u)),$$

or

$$\|x_{n+1} - u\|^2 \leq \left(1 - \beta_n\right)^2 \|x_n - u\|^2 + \frac{2\beta_n}{1 - a\beta_n} (f(u) - u, J(x_{n+1} - u))$$

$$= \left(1 - \frac{2\beta_n + a\beta_n}{1 - a\beta_n}\right) \|x_n - u\|^2 + \frac{2\beta_n}{1 - a\beta_n} \|x_n - u\|^2$$

$$+ \frac{2\beta_n}{1 - a\beta_n} (f(u) - u, J(x_{n+1} - u))$$

$$\leq (1 - \frac{2(1-a)\beta_n}{1 - a\beta_n}) \|x_n - u\|^2 + \frac{2(1-a)\beta_n}{1 - a\beta_n} \left\{ \frac{M\beta_n}{2(1-a)} + \frac{1}{1-a} (f(u) - u, J(x_{n+1} - u)) \right\},$$

where $M = \sup_n \|x_n - u\|^2$.

Putting $\gamma_n = \frac{2(1-a)\beta_n}{1 - a\beta_n}$, we get that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\lim_{n \to \infty} \gamma_n = 0$.

Hence

$$\|x_{n+1} - u\|^2 \leq (1 - \gamma_n) \|x_n - u\|^2 + \gamma_n \left\{ \frac{M\beta_n}{2(1-a)} + \frac{1}{1-a} (f(u) - u, J(x_{n+1} - u)) \right\}.$$
Using (2.2.16) and applying Lemma 2.2.8, we get that

$$\|x_n - u\| \to 0.$$  

So, we conclude that \( \{x_n\} \) converges strongly to \( u = Pf(u) \). \( \square \)

Remark 2.2.10. If we take \( \beta_n = 0 \) in our Theorem 2.2.9, then we get the required result of Takahashi [149, Theorem 4.1].

Remark 2.2.11. On taking \( f(x_n) \) to be a constant \( u \) in our result, where \( f: C \to C \) is a contraction map, we get the theorem of Qin and Su [129, Theorem 2.2].