Chapter 2

Invariant Best Simultaneous Approximations in Metric Spaces

2.1 Introduction

In this chapter, we establish some results on best simultaneous approximation and invariant best simultaneous approximation involving weak contractive conditions, in the setting of convex metric spaces [152].

Fixed point theory has gained impetus, due to its wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations, theory of nonlinear integral equations, game theory, mathematical economics, control theory, and so forth. For example, in theoretical economics, such as general equilibrium theory, a situation arises where one need to know whether the solution to the system of equations necessarily exists; more specifically, under what condition will a solution necessarily exist. The mathematical analysis of this question usually relies on fixed point theorem. Hence finding necessary and sufficient conditions for the existence of fixed points is an interesting aspect.

Fixed point theorems have been used in many instances in best approximation theory. It is pertinent to say that in best approximation theory, it is
viable, meaningful, and potentially productive to know whether some useful properties of the function being approximated is inherited by the approximating function. The idea of applying fixed point theorems to approximation theory was initiated by Meinardus [101]. Meinardus introduced the notion of invariant approximation in normed linear spaces.

The study of fixed points for multivalued contraction and nonexpansive maps using the Hausdorff metric was initiated independently by Markin [100] and Nadler [109]. Later, an interesting and rich fixed point theory for such maps has been developed.

The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric properties play a key role in metric fixed point problems, see for example [57] and references mentioned therein. These results mainly rely on geometric properties of Banach spaces. These results were the starting point for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory. Takahashi [152] introduced the notion of convexity in metric spaces. Afterward, Beg [10], Beg and Abbas [11], Chang and Kim [25], Ćirić[36], Ding [42], Shimizu and Takahashi [143] and many other authors have studied fixed point theorems in convex metric spaces. Applying fixed point theorems, useful results have been established in approximation theory.

We start by briefly introducing of some relevant definition from existing literature for the convenience of later reference.

**Definition 2.1.1.** [152] Let $(X, d)$ be a metric space $I$ denotes the unit interval $[0, 1]$. A continuous mapping $W : X \times X \times I \to X$ is said to be a convex structure on $X$ if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \quad (2.1.1)$$

The metric space $X$ together with a convex structure $W$ is called a convex
metric space. Obviously \( W(x, x, \lambda) = x \).

Notice that a Banach space and each of its convex subsets are, of course, simple examples of convex metric spaces. But a Fréchet space is not necessarily a convex metric space. There are many convex metric spaces which cannot be imbedded in any Banach space. We give one preliminary example here.

**Example 2.1.1.** [152] We consider a linear space \( L \) which is also a metric space with the following properties:

1. For \( x, y \in L \), \( d(x, y) = d(x - y, 0) \);
2. For \( x, y \in L \) and \( \lambda(0 \leq \lambda \leq 1) \),

\[
d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0).
\]

**Definition 2.1.2.** A convex metric space \((X, d)\) is said to satisfy Property (I), if for all \( x, y, p \in X \) and \( \lambda \in [0, 1] \),

\[
d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y).
\]

**Definition 2.1.3.** [152] A subset \( K \) of a convex metric space \( X \) is said to be convex if \( W(x, y, \lambda) \in K \) for all \( x, y \in K \) and \( \lambda \in I \). This notion of convexity is closely related to that given by K. Menger ([110]).

**Definition 2.1.4.** [72] The set \( K \) is said to be starshaped if there exists \( p \in K \) such that \( W(x, p, \lambda) \in K \) for all \( x \in K \) and \( \lambda \in I \).

Evidently starshaped subsets of \( X \) contain all convex subsets of \( X \) as a proper subclass.

**Definition 2.1.5.** (Best simultaneous approximations) Let \( C \) be a subset of a metric space \((X, d)\). Given any bounded subset \( F \) of \( X \), define

\[
\delta(F, C) = \inf_{x \in C} \sup_{y \in F} d(y, x).
\]
an element \( c^* \in C \) is said to be a best simultaneous approximation (b.s.a.) to \( F \) if
\[
\sup_{y \in F} d(y, k^*) = \delta(F, C).
\]
Many author have studied relative Chebyshev centres in normed linear space (see [137]). This concept is closely related to the best simultaneous approximation and was first introduced in 1962 by A. L. Garkavi [54].

**Definition 2.1.6.** Let \((X, d)\) be a metric space and \( G \) a nonempty subset of \( X \). Suppose \( A \in B(X) \), the set of nonempty bounded subsets of \( X \), then we write
\[
r_G(A) = \inf_{g \in G} \sup_{a \in A} d(a, g)
\]
\[
\text{cent}_G(A) = \{g_0 \in G : \sup_{a \in A} d(a, g_0) = r_G(A)\}.
\]
The number \( r_G(A) \) is called the Chebyshev radius of \( A \) w.r.t \( G \) and an element \( y_0 \in \text{cent}_G(A) \) is called a best simultaneous approximation of \( A \) w.r.t \( G \). If \( A = \{x\} \), then \( r_G(A) = d(x, G) \) and \( \text{cent}_G(A) \) is the set of all best approximations of \( x \) out of \( G \).

**Definition 2.1.7.** A point \( x \in X \) is called fixed point of mapping \( T \) if \( Tx = x \). The set of fixed points of \( T \) is denoted by \( F(T) \).

**Definition 2.1.8.** A point \( x \in M \) is a coincidence point (common fixed point) of \( f \) and \( T \) if \( fx = Tx \) \((x = fx = Tx)\). The set of coincidence points of \( f \) and \( T \) is denoted by \( C(f, T) \).

**Definition 2.1.9.** The pair \( f, T \) is called commuting if \( Tf x = fTx \) for all \( x \in M \).

**Definition 2.1.10.** The map \( f \) defined on a \( q \)--starshaped set \( M \) is called affine if \( f(W(x, y, \lambda)) = W(fx, fy, \mu)) \) for all \( x, y \in M \), and affine with respect to \( p \) if \( f(W(x, q, \lambda)) = W(fx, fq, \lambda)) \) for all \( x \in M \) and \( \lambda \in [0, 1] \).
Clearly, for linear space every linear mapping is affine. An affine mapping with respect to a point need not be affine [127].

2.2 Best simultaneous approximation in convex metric spaces

Lemma 2.2.1. Let $C$ be a subset of a metric space $(X,d)$ and $F$ a bounded subset of $X$. Then the function $\phi : C \to R$ defined by

$$\phi(c) = \sup_{f \in F} d(f,c) \quad (2.2.1)$$

is lower semi continuous.

Proof: The supremum of a family of continuous function is lower semi continuous.(Rudin [135])

Lemma 2.2.2. [111] Let $C$ be a convex subset of a convex metric space $(X,d)$ and $F$ a bounded subset of $X$. If $c_1^*,c_2^* \in C$ are best simultaneous approximation to $F$ then $W(c_1^*,c_2^*,\lambda)$ is also a best simultaneous approximation in $C$ to $F$ for every $\lambda \in I$.

Lemma 2.2.3. [111] Let $C$ be a convex subset of a strictly convex metric space $(X,d)$ and $F$ be a subset of $X$ which is remotal w.r.t. $C$. Then there exists at most one best simultaneous approximation in $C$ to $F$.

Theorem 2.2.4. Let $(X,d)$ be a strictly convex metric space, and $C$ a weakly compact, convex subset of $X$. Then there exists a unique best simultaneous approximation from the elements of $C$ to given any bounded subset $F$ of $X$.

Proof. By definition $c \in C$ is said to be a best simultaneous approximation to $F$ if,

$$\delta(F,C) = \sup_{y \in F} d(y,c).$$
From Lemma 2.2.1 the function \( \phi \) defined by

\[
\phi(c) = \sup_{f \in F} d(f, c)
\]
is lower semi continuous function on \( C \). Since \( C \) is weakly compact subset of \( X \), \( \phi \) attains its infimum at \( c \) in \( C \), say. Therefore,

\[
\delta(F, C) = \sup_{y \in F} d(y, c).
\]

Uniqueness can be obtained directly Lemma 2.2.2 and Lemma 2.2.3.

**Corollary 2.2.5.** [61] Let \( C \) be a closed, bounded and convex subset of a uniformly convex Banach space \( X \). Then for any compact subset \( F \) of \( X \) there exists a unique best simultaneous approximation to \( F \) from the elements of \( C \).

Since a uniformly convex Banach space is strictly convex and reflexive. Also, \( C \) is weakly compact so the result follows from Theorem 2.2.4.

### 2.3 Invariant best simultaneous approximation in convex metric space

The well-known Banachs contraction mapping principle is one of the most useful results in nonlinear analysis. In a metric space setting it can briefly be stated as follows:

**Theorem 2.3.1.** Let \((X, d)\) be a complete metric space and \( T : X \to X \) a strict contraction, i.e., a map satisfying

\[
d(Tx, Ty) \leq a \, d(x, y) \quad \text{for all } x, y \in X,
\]

where \( 0 \leq a < 1 \) is constant. Then \( T \) has a unique fixed point in \( X \).
Theorem 2.3.1, has many applications in solving nonlinear functional equations but suffers from one drawback - the contractive condition (2.3.1) forces $T$ to be continuous throughout $X$. In order to remove this drawback, in 1968 Kannan [81] obtain a fixed point theorem for mappings $T$ that need not be continuous by considering instead of 2.3.1 the following contractive condition:

$$d(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)] \quad \text{for all } x, y \in X \quad (2.3.2)$$

Following Kannans fixed point theorem, a lot of paper were devoted to obtaining fixed point theorem for various type of contractive type conditions that do not require the continuity of $T$, see for example, the one obtained by Chatterjea [31]. One of the most general contraction condition obtained in this way, for which the Picard iteration still converge to the unique fixed point, was given by Ćirić [33] in 1974, by considering the contractive condition

$$d(Tx,Ty) \leq h.\max d(x,y).d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx) \quad (2.3.3)$$

for all $x, y \in X$ and some constant $0 \leq h < 1$. On the other hand, the Banach contraction mapping principle has been extended in another direction by Jungck [73] to obtain the following common fixed point theorem.

**Theorem 2.3.2.** Let $X$ be a complete metric space. let $S$ and $T$ be a continuous self map on $X$ and $S$ and $T$ commutes. Further $S$ and $T$ satisfy $T(X) \subset S(x)$ and there exist a constant $\lambda \in (0,1)$ such that for every $x, y \in X$

$$d(Tx,Ty) \leq \lambda d(Sx,Sy) \quad (2.3.4)$$

Then $S$ and $T$ have a unique common fixed point.

Other common fixed point results for the Kannan, Chatterjea and Zamfirescu contractive conditions have been recently obtained in [152], while the corresponding common fixed point result for Ćirićs fixed point theorem has been
derived by Das and Naik [39]. Due to fact that all these theorems required
S and T be commuting mappings, an extension of this result to weakly com-
muting maps is obtained by Subrahmanyam [150] in the following form.

**Theorem 2.3.3.** Let \((X, d)\) be a complete metric space. let \(f : X \rightarrow X\) be
a continuous map and \(g : X \rightarrow X\) a map such that \(g(x) \subseteq f(x)\), \(g\) and \(f\)
weakly commute and there exist \(k \in (0, 1)\) with \(d(gx, gy) \leq k(fx, fy)\) for all
\(x, y \in X\). Then \(f\) and \(g\) have a common fixed point.

To prove our main result we employ a technique due to Tarafdar for two
maps which are not necessarily commutative. Define \(\Phi = \varphi : [0, 1] \rightarrow [0, 1]\),
and each \(\varphi \in \Phi\) satisfies the following conditions:
(a) \(\varphi\) is continuous on \([0, 1]\),
(b) \(\varphi\) is non-decreasing. Using Theorem 2.3.3, we establish common fixed
point generalization for the maps which are not necessarily commutative.

**Theorem 2.3.4.** Let \(M\) be a nonempty \(q\)-starshaped subset of a convex metric
space \((X, d)\) with property (I). Let \(T\) and \(I\) be self maps on \(M\). Suppose that
\(T\) is \(I\)-nonexpansive, \(I(M) = M, q \in F(I)\), \(I\) is nonexpansive and affine with
respect to \(q\). If \(T\) and \(I\) satisfies the following :
\[
d(ITx, TIx) \leq \frac{1}{\varphi(\lambda)} d(W(Tx, q, \lambda), Ix)
\]  
(2.3.5)
for each \(x \in M\), and \(\lambda \in (0, 1)\). Then \(T\) and \(I\) have a common fixed point
provided one of the following condition holds:
(i) \(clT(M)\) is compact;
(ii) \(T\) is compact;
(iii) \(wclT(M)\) is weakly compact, \(I\) is weakly continuous, \(I - T\) is demiclosed
at 0.

**Proof.** Choose a sequence \(\{k_n\}\) of real numbers such that \(0 < k_n < 1\) and
\(k_n \rightarrow 1\) as \(n \rightarrow \infty\). For each \(n\), define \(T_n : M \rightarrow M\) as :
\[ T_n(x) = W(Tx, q, k_n) \text{ for each } x \in M \]

\( T_n \) is well defined since \( M \) is q-starshaped. It also from the definition of \( T_n \) and I-nonexpansiveness of \( T \) that

\[
d(T_n x, T_n y) = d(W(Tx, q, k_n), W(Ty, q, k_n)) \\
\leq \varphi(k_n) d(Tx, Ty) \\
\leq \varphi(k_n) d(Ix, Iy).
\]

Hence \( T_n \) is also I-nonexpansive. As \( I \) is affine with respect to \( q \) and \( q \in F(I) \), so \( IT_n x = IW(Tx, q, k_n) = W(ITx, q, k_n) \) and \( T_n I x = W(TIx, q, k_n) \). Since \( T \) and \( I \) satisfy (2.3.5), so for each \( n \),

\[
d(IT_n x, T_n I x) = d(W(ITx, q, k_n), W(TIx, q, k_n)) \\
= \varphi(k_n) d((ITx, TIx)) \\
\leq \varphi(k_n) \frac{1}{\varphi(k_n)} (d(W(Tx, q, k_n)), Ix) \\
= d(T_n x, Ix),
\]

which implies that \( T_n \) and \( I \) are weakly commuting for each \( n \). Since \( clT(M) \) is compact and \( T \) is continuous, \( cl(T_n(M)) \) is also compact. A comparison of our hypotheses with that of Theorem 2.3.3 tells that we can apply it to \( M \) as a subset of \( X \) to conclude that there exist \( x_n \in M \) such that

\[ x_n = T_n x_n = I x_n \text{ for each } n \geq 1. \]

(i) As \( clT(M) \) is compact and \( \{x_n\} \) is a sequence in \( M \), so \( \{x_n\} \) has a convergent subsequence \( \{x_{n_j}\} \) such that \( x_{n_j} \to x_0 \in M \). As \( I \) and \( T \) are continuous and \( x_{n_j} = I x_{n_j} = T_{n_j} x_{n_j} = W(Tx_{n_j}, q, k_{n_j}) \) so it follows that \( x_0 = Tx_0 = Ix_0 \).

(ii) As \( T \) is compact and \( x_n \) is bounded, so \( \{Tx_n\} \) has a subsequence \( \{Tx_{n_i}\} \) such that \( \{Tx_{n_i}\} \to x_0 \in M \). Now we have that \( x_{n_i} = I x_{n_i} = T_{n_i} x_{n_i} = W(Tx_{n_i}, q, k_{n_i}) \) as \( \lim_{i \to \infty} \) and \( T \) and \( I \) are continuous, we get \( x_0 = Tx_0 = Ix_0 \).
(iii) The weak compactness of \( \text{wcl}T(M) \) implies that \( \text{wcl}T_n(M) \) is weakly compact and hence complete due to completeness of \( X \). Due to weak compactness of \( \text{wcl}T(M) \) implies that, there is a subsequence \( x_{n_j} \) of \( x_n \) converging weakly to \( y \in M \). Since \( I \) is weakly continuous and \( x_{n_j} = Ix_{n_j} \), so \( Iy = y \). From \( Ix_{n_j} = T_{n_j}x_{n_j} = W(Tx_{n_j}, q, k_{n_j}) \), we obtain \( Ix_{n_j} - Tx_{n_j} = W(Tx_{n_j}, q, k_{n_j}) - Tx_{n_j} \to 0 \) as \( j \to \infty \). Since \( I - T \) is demiclosed at 0, therefore \((I - T)(y) = 0 \) implies that \( Iy = Ty \) as desired.

**Remark 1.** The maps \( T \) and \( I \) satisfying (2.3.5) are known as 1-subcommutative.

**Corollary 2.3.5.** ([44]-Theorem 1) Suppose \( C \) is a compact, starshaped subset of a Banach space \( E \) and \( T \) a nonexpansive self mapping of \( C \). Then \( T \) has a fixed point in \( C \).

**Corollary 2.3.6.** ([18]-Theorem 3) Let \((X, d)\) be a convex metric space satisfying property (I) and \( E \) be a closed and \( q \)-starshaped subset of \( X \). If \( T \) is a nonexpansive self mapping on \( E \) and \( \text{cl}T(E) \) is compact, then \( T \) has a fixed point.

**Corollary 2.3.7.** ([5]-Theorem 2.1) Let \( M \) be a nonempty closed and \( q \)-starshaped subset of a normed space \( X \) and \( T \) and \( f \) be self-maps such that \( T(M) \subseteq f(M) \). Suppose that \( T \) commutes with \( f \) and \( q \in F(f) \). If \( \text{cl}T(M) \) is compact, \( f \) is continuous and linear and \( T \) is \( f \)-nonexpansive on \( M \), then \( M \cap F(T) \cap F(f) \neq \emptyset \).

**Theorem 2.3.8.** Let \( M \) be a nonempty subset of a convex metric space \( X \) and \( y_1, y_2 \in X \). Suppose that \( T \) and \( I \) be self-maps of \( M \) such that \( T \) is \( I \)-nonexpansive and \( T \) and \( I \) satisfies (2.3.5). Suppose that the set \( F(I) \) is nonempty. Let the set \( D \), of best simultaneous approximants to \( y_1, y_2 \), is nonempty starshaped with respect to an element \( q \in F(I) \) and \( D \) is invariant under \( T \) and \( I \). Assume further that \( I \) is affine continuous on \( D \) with \( I(D) = \)
\( D \). Then \( D \cap F(T) \cap F(I) \neq \emptyset \) provided one of the following condition holds:

(i) \( clT(D) \) is compact;

(ii) \( T \) is compact map;

(iii) \( wclT(D) \) is weakly compact, \( I \) is weakly continuous, \( I - T \) is demiclosed at 0.

**Proof.** As \( I \) is continuous and \( D \) is closed, therefore \( F(f) \) is closed. Using the weak commutativity of \( T \) with \( I \), we obtain \( T(F(I)) \subseteq F(I) \). Thus \( clT(F(I)) \subseteq clF(I) = F(I) \). Since \( I \) is affine w.r.t. \( q \) and \( q \in F(I) \), so \( F(I) \) is \( q \)-starshaped. The desired conclusion follows from Theorem 2.3.4.

The following corollary follows from Theorem 2.3.8 as condition (i) implies that \( M \) is \( T \)-invariant.

**Corollary 2.3.9.** Let \( X, M, y_1, y_2, I \) and \( T \) be as in Theorem 2.3.8. Assume that \( T \) satisfies the following condition:

(i) \( d(Tx, y_i) \) for all \( x \in X \) and \( i = 1, 2 \). Suppose that the set \( M \), of best simultaneous \( M \)-approximants to \( y_1 \) and \( y_2 \), is nonempty compact and star-shaped with respect to an element \( q \in F(I) \). Then \( D \) contains a \( T \)- and \( I \)-invariant point.

### 2.4 Banach operator pairs and common fixed points

The study of a common fixed point of a pair of commuting mappings was initiated as soon as the first fixed point result was proved. This problem become more challenging and seems to be a vital interest in view of historically significant and negatively settled problem that a pair of commuting continuous self-mappings on the unit interval \([0,1]\) need not have a common fixed point [65]. Since then, many fixed point theorists have attempted to find
weaker forms of commutativity that may ensure the existence of a common fixed point for a pair of self-mappings on a metric space. In this context, the notion of Banach operator pair have been of significant interest for generalizing results in metric fixed point theory for single valued mappings.

In 1974, Subrahmanyam [150] obtained the fixed point of a continuous Banach operator of type $k$ in a complete metric space. A self map $T$ of a subset $D$ of $X$ is called Banach operator of type $k$, if

$$\|Tx - T^2y\| \leq k \|x - Tx\|$$

Recently, Chen and Li [26] introduce the notion of Banach operator pair as a new class of non commuting maps, defined as:

**Definition 2.4.1.** The order pair $(T, S)$ of two self map of a metric space $(X, d)$ is called a Banach operator pair, if the set $F(S)$ is $T-$invariant i.e. $T(F(S)) \subseteq F(S)$. Obviously the commuting pair $(T, S)$ is a Banach operator pair but the converse is not true in general. Also, if $(T, S)$ is a Banach operator pair, then $(S, T)$ need not be a Banach operator pair.

**Example 2.4.1.** ([26], Example-1) Let $T$ and $S$ be two self-mapping of $X = \mathbb{R}^2$ defined by

$$T(s, t) = (s^2 + t^2 + s - 1, s^2 + t^2 + t - 1)$$

$$S(s, t) = ((s - t)^2 + 2s - t, (3 - t)^2 + s)$$

for $(s, t) \in \mathbb{R}^2$. Computation shows that

$$F(T) = ((s, t) \in \mathbb{R}^2 : s^2 + t^2 - 1 = 0);$$

$$F(S) = ((s, t) \in \mathbb{R}^2 : s - t = 0 or s - t + 1 = 0);$$

$$C(T, S) = ((s, t) \in \mathbb{R}^2 : 2st - s + t - 1 = 0)$$

The following assertion can be verified easily.

(i) $T(F(S)) \subseteq F(S)$, and hence $(T, S)$ is a Banach operator pair.
(ii) \((S,T)\) is not a Banach operator pair since \((1,0) \in F(T), \text{ but } S(1,0) = (3, 2) \notin F(S)\).

(iii) \(T\) and \(S\) do not commute on the \(C(T,S)\) since, for example, taking \((1, \frac{2}{3}) \in C(T,S)\) then

\[
ST(1, \frac{2}{3}) = \left(\frac{17}{9}, \frac{14}{9}\right) \neq \left(\frac{305}{81}, \frac{278}{81}\right) = TS(1, \frac{2}{3}).
\]

**Definition 2.4.2.** Let \(D\) be a nonempty subset of a metric space \((X, d)\) and \(T : D \to D\). Then \(T\) is said to be demicompact if whenever \(\{x_n\}\) is a bounded sequence of points of \(D\) such that \(d(x_n, Tx_n) \to 0\), then \(\{x_n\}\) has a convergent subsequence.

**Definition 2.4.3.** Let \(T, S : D \to D\). Then \(T\) is said to be \(S\)-asymptotically nonexpansive if there exist a sequence \(\{k_n\}\) of real numbers in \([1, \infty)\) with \(k_n \geq k_{n+1}, k_n \to 1\) as \(n \to \infty\) such that

\[
d(T^n(x), T^n(y)) \leq k_n d(Sx, Sy), \quad \text{for all } x, y \in D
\]

**Definition 2.4.4.** Map \(T\) is said to be asymptotically regular if for all \(x \in D\),

\[
d(T^n(x), T^{n+1}(x)) \to 0.
\]

Next, we establish some result for new class of non commuting pair in convex metric space and then establish relevance in the context of invariant approximation.

We shall be using following lemma in proving common fixed point result in convex metric spaces.

**Lemma 2.4.2.** Let \(D\) be a closed subset of a metric space \(X\), and let \((T, S)\) be banach operator pair on \(D\). Assume that \(cl(T(D))\) is compact, and \(T, S\) satisfies

\[
d(Tx, Ty) \leq h \ d(Sx, Sy),
\]

for all \(x, y \in D\) and \(0 \leq h < 1\). If \(T\) and \(S\) are continuous, \(F(S)\) is nonempty, then there exist a common fixed point of \(T\) and \(S\).
Theorem 2.4.3. Let $D$ be a nonempty closed subset of a convex metric space $(X,d)$ with property (I), $S$ and $T$ be self mapping of $D$ and $q \in F(S)$. If $D$ is $q$--starshaped, $\text{cl}(T(D))$ is compact, $(T,S)$ is a Banach operator pair and $T$ is $S$--asymptotically nonexpansive and asymptotically regular, then $S$ and $T$ have a common fixed point in $D$.

Proof. for each $n$, define $T_n : D \to D$ as $T_n(x) = W(T^n(x), q, \lambda_n), x \in D,$ where $\lambda_n = \frac{n}{k_n}$ and $k_n$ is a sequence of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}, k_n \to 1$. Since $T(D) \subseteq D$ and $D$ is $q$--starshaped, it follows that $T$ maps $D$ into $D$. As $(T,S)$ is a Banach operator pair, hence $T(F(S)) \subseteq F(S)$. Which implies $T^n(F(S)) \subseteq F(S)$. for each $n \geq 1$. since $F(S)$ is $q$--starshaped, hence for each $x \in F(S)$

$$T_n(x) = W(T^n(x), q, \lambda_n) \in F(S).$$

Since $T_n(x) \in F(S)$ for each $x \in F(S)$. Thus $(T_n, S)$ is banach operator pair. Since $T$ is $S$--asymptotically nonexpansive , it follows that

$$d(T_n(x), T_n(y)) = d(W(T^n(x), q, \lambda_n), W(T^n(y), q, \lambda_n)) \leq \lambda_n d(T^n(x), d(T^n(y))$$

$$\leq \lambda_n k_n d(Sx, Sy) \leq \mu_n d(Sx, Sy)$$

i.e. $T_n$ is $S$ contraction. Since $\text{cl}(T(D))$ is compact and $T$ is continuous, $\text{cl}(T_n(D))$ is also compact. Hence by Lemma 2.4.2, there exist $x_n \in D$ such that $x_n \in F\mu(T_n, S)$ for each $n \in N$. Since $\{T^n x_n\}$ is a sequence in compact set $\text{cl}(T(D))$, there exists a subsequence $\{T^{n_i} x_{n_i}\}$ of $\{T^n x_n\}$ such that $\{T^{n_i} x_{n_i}\} \to z \in \text{cl}(T(D))$. Therefore,

$$x_{n_i} = T_{n_i} x_{n_i} = W(T^{n_i}(x_{n_i}), q, \lambda_{n_i}) = z.$$ 

Since $S$ is continuous and $S(x_{n_i}) = x_{n_i}$, it follows that $z \in F(S)$. Since $T$ is $S$--asymptotically nonexpansive and $S$ is continuous, it follows that

$$d(T^{n_i} x_{n_i}, T^{n_i} z) \leq k_{n_i} d(Sx_{n_i}, Sz) \to 0.$$
Therefore, \( \lim n x_n = \lim T^n z = z \). Since \( S \) is continuous \( \lim ST^n(z) = Sz \). Since \( T \) is asymptotically regular and \( S(z) = z \), it follows that

\[
d(z, Tz) \leq d(z, T^n z) + d(T^n z, T^{n+1} z) + d(T^{n+1} z, Tz)
\]

\[
\leq d(z, T^n z) + d(T^n z, T^{n+1} z) + k_1 d(S(T^n z), Sz) \to 0.
\]

Hence \( z \in F(T, S) \).

**Theorem 2.4.4.** Let \( D \) be a nonempty complete bounded subset of a convex metric space \( (X, d) \) with property (I), \( S \) and \( T \) be self mapping of \( D \) and \( q \in F(S) \). If \( D \) is \( q \)-starshaped, \( T \) is demicompact, \( S \) is continuous, \( (T, S) \) is a Banach operator pair and \( T \) is \( S \)-asymptotically nonexpansive and asymptotically regular, then \( S \) and \( T \) have a common fixed point in \( D \).

**Proof.** Defining \( T_n(x) = W(T^n x, q, \lambda_n) \) and proceeding as in theorem 2.4.3, we see that \( T_n \) is \( S \)-contraction and so by Lemma 2.4.2, there exists \( x_n \in D \) such that \( x_n \in F(T_n, S) \) for each \( n \in N \). Therefore,

\[
d(x_n, T^n x_n) = d(T_n x_n, T^n x_n) = d(w(T^n x_n, q, \lambda_n), T^n x_n)
\]

\[
\leq \lambda_n d(T^n x_n, T^n x_n) + (1 - \lambda_n) d(q, T^n x_n) \to 0
\]

Since \( (T, S) \) is a Banach operator pair and \( S x_n = x_n \), so \( ST^n x_n = T^n S x_n = T^n x_n \). Thus we have

\[
d(x_n, T x_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, T x_n)
\]

\[
\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(S(T^n x_n), S x_n)
\]

\[
\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n) \to 0.
\]

Since \( T \) is demicompact, \( \{x_n\} \) has a subsequence \( \{x_{n_i}\} \) such that \( x_{n_i} \to z \in D \). Since \( T \) is continuous, \( T(x_{n_i}) \to Tz \). Therefore,

\[
d(z, Tz) \leq d(z, x_{n_i}) + d(x_{n_i}, T x_{n_i}) + d(T x_{n_i}, Tz) \to 0.
\]
hence $Tz = z$. Since $S$ is continuous and $x_{n_i} = Sx_{n_i}$, it follows that $Sz = z$. Thus $z \in F(T, S)$.

Remark 2.4.1. A commuting pair $(T, S)$ is a Banach operator pair and affine mapping on set $S$ implies that $F(S)$ is $q-$starshaped, hence we get following corollary.

**Corollary 2.4.5.** ([155]- Theorem 2.6) Let $D$ be a nonempty closed subset of a normed linear space $X$, $S$ and $T$ be self mapping of $D$ with $T(D) \subseteq S(D)$ and $q \in F(S)$. If $D$ is $q-$starshaped, $\text{cl}(T(D))$ is compact, $S$ is continuous and affine with respect to $q$, $S$ and $T$ are commuting and $T$ is $S-$asymptotically nonexpansive and asymptotically regular, then $S$ and $T$ have a common fixed point in $D$.

**Corollary 2.4.6.** Let $D$ be a nonempty closed subset of a convex metric space $(X, d)$ with property (I), $S$ and $T$ be self mapping of $D$ and $T(D) \subseteq S(D)$ and $q \in F(S)$. If $D$ is $q-$starshaped, $\text{cl}(T(D))$ is compact, $S$ is continuous and affine with respect to $q$, $S$ and $T$ are commuting and $T$ is $S-$asymptotically nonexpansive and asymptotically regular, then $S$ and $T$ have a common fixed point in $D$.

**Corollary 2.4.7.** ([155]- Theorem 2.7) Let $D$ be a nonempty complete bounded subset of a normed linear space $X$, $S$ and $T$ be self mapping of $D$ with $T(D) \subseteq S(D)$ and $q \in F(S)$. If $D$ is $q-$starshaped, $T$ is demicompact, $S$ is continuous and affine with respect to $q$, $S$ and $T$ are commuting and $T$ is $S-$asymptotically nonexpansive and asymptotically regular, then $S$ and $T$ have a common fixed point in $D$.

**Corollary 2.4.8.** Let $D$ be a nonempty complete bounded subset of a convex metric space $(X, d)$ with property (I), $S$ and $T$ be self mapping of $D$ with $T(D) \subseteq S(D)$ and $q \in F(S)$. If $D$ is $q-$starshaped, $T$ is demicompact, $S$
is continuous and affine with respect to $q$, $T$ and $S$ are commuting and $T$ is $S$–asymptotically nonexpansive and asymptotically regular, then $S$ and $T$ have a common fixed point in $D$.

**Theorem 2.4.9.** Let $D$ be a nonempty subset of convex metric space $(X,d)$ and $y_1, y_2 \in X$. Suppose that $T$ and $S$ are self mapping of $D$. Assume that the set $K$, of best simultaneous $D$-approximants to $y_1$ and $y_2$, is nonempty and invariant under $T$ and $S$, $(T,S)$ is a Banach operator on $K$ and $T$ is $S$–asymptotically nonexpansive and asymptotically regular on $K$. $K_0 := F(S) \cap K$ is closed and starshaped w.r.t. an element $q \in D_0$. If $cl(T(D_0))$ is compact, then $D$ contains a $T$– and $S$– invariant point.

**Proof.** Proof is similar to that of Theorem 2.4.3.

### 2.5 Invariant best simultaneous approximation for weak asymptotic contractions

**Definition 2.5.1.** [158] Let $\Phi$ denote the collection of all function $\varphi$ from $R^+ := [0, \infty) \to R^+$ satisfying the properties (i) $\varphi$ is continuous (ii) $\varphi(t) < t$ for all $t > 0$. A continuous mapping $T$ from a complete metric space $(X,d)$ into itself is said to be weakly asymptotic contraction if for an arbitrary $\epsilon > 0$, there is an integer $n_\epsilon \geq 1$ such that as $\epsilon \to 0$, $n_\epsilon \to \infty$ and

$$d(T^{n_\epsilon}x, T^{n_\epsilon}y) \leq \varphi(d(x, y)) + \epsilon.$$ 

**Definition 2.5.2.** A convex metric space $X$ is said to satisfy condition $(P)$

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at \( p \in K \) (where \( K \) is starshaped and \( p \) is star centre) if for any \( x, y \in X, \lambda \in I \)
\[
d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda \varphi(d(x, y)) + \epsilon.
\]

**Definition 2.5.3.** Let \((X, d)\) be a metric space, \( W \) a convex structure on \( X \) and \( K \) a starshaped subset (with respect to star centre \( p \in K \)) of \( X \). A mapping \( T : X \to X \) is said to satisfy property \((W)\) in \( K \) if,

\[
(i) W(T^{(n+1)}x, p, k) = W(Tx, p, k(\frac{n}{n+1})) \text{ for each } n \in N \text{ and for all } x \in K
\]
and
\[
(ii) W(x, p, 1) = x
\]

To establish our next result we shall make use of the following result due to Xu([158], Theorem 3).

**Theorem 2.5.1.** [158] Let \((X, d)\) be a complete metric space and let \( T : X \to X \) is a weakly asymptotic contraction. Assume that \( T \) has a bounded orbit at some \( x \in X \). Then \( T \) has a unique fixed point \( z \).

**Theorem 2.5.2.** Suppose \( X \) is a complete convex metric space satisfying condition \((P)\). Let \( G \) be a nonempty subset and \( A \) be a bounded subset of \( X \). Also let \( T \) be a weakly asymptotic self contraction of \( G \). If the set \( \text{cent}_G(A) \) of best simultaneous \( G \)–approximates to \( A \) is nonempty, compact, starshaped, \( T \)-invariant and \( T \) is continuous on \( \text{cent}_G(A) \), then \( \text{cent}_G(A) \) contains a \( T \)-invariant point.

**Proof.** Let \( p \) be the star-centre of \( \text{cent}_G(A) \). Then \( W(x, p, \lambda) \in \text{cent}_G(A) \) for each \( x \in \text{cent}_G(A) \). Let for an arbitrary \( \epsilon > 0 \), there is an integer \( n_\epsilon \geq 1 \) such that as \( \epsilon \to 0 \), \( n_\epsilon \to \infty \) and \( \{k_{n_\epsilon}\}_{n_\epsilon=1}^\infty \) be a real sequence with \( 0 \leq k_{n_\epsilon} < 1 \) such that \( \lim_{n_\epsilon \to \infty} k_{n_\epsilon} = 1 \). Define \( T_{n_\epsilon} : \text{cent}_G(A) \to \text{cent}_G(A) \) by
\[
T_{n_\epsilon}x = W(T^{n_\epsilon}x, p, k_{n_\epsilon})
\]
for all \( x \in \text{cent}_G(A) \) and \( n_\epsilon \geq 1 \). Since \( p \) is star-center of \( \text{cent}_G(A) \) and \( T(\text{cent}_G(A)) \in \text{cent}_G(A) \) it follows that \( T_{n_\epsilon} \) maps \( \text{cent}_G(A) \) to itself for each \( n_\epsilon \). Now applying condition (P), we obtain

\[
d(T_{n_\epsilon}x, T_{n_\epsilon}y) = d(W(T^{n_\epsilon}x, p, k_{n_\epsilon}), W(T^{n_\epsilon}y, p, k_{n_\epsilon}))
\leq k_{n_\epsilon}d(T^{n_\epsilon}x, T^{n_\epsilon}y)
\leq k_{n_\epsilon}(\varphi(d(x, y)) + \epsilon)
\leq \lim_{n_\epsilon \to \infty} k_{n_\epsilon}(\varphi(d(x, y)) + \epsilon)
= \varphi(d(x, y)) + \epsilon
\]

for all \( \varphi \in \Phi \) and, thereby, implying that \( T_{n_\epsilon} \) is a weakly asymptotic contraction for each \( n_\epsilon \geq 1 \). It follows, by Theorem 2.5.1, that each \( T_{n_\epsilon} \) has a fixed point, say \( z_{n_\epsilon} \). Since \( \text{cent}_G(A) \) is compact, \( \{ z_{n_\epsilon} \} \) has a convergent subsequence \( \{ z(n_\epsilon) \}_i \) such that \( z(n_\epsilon)_i \to z \) (say) as \( i \to \infty \). Now

\[
z = \lim_{i \to \infty} z(n_\epsilon)_i = \lim_{i \to \infty} T(n_\epsilon)_i z(n_\epsilon)_i = \lim_{i \to \infty} W(Tz(n_\epsilon)_i, p, k(n_\epsilon)_i)
= \lim_{i \to \infty} W(Tz(n_\epsilon)_i, p, k(n_\epsilon)_i, \left(\frac{(n_\epsilon)_i - 1}{(n_\epsilon)_i}\right)) = W(Tz, p, 1) = Tz
\]

Hence, \( \text{cent}_G(A) \) contains a \( T \)-invariant point. This completes the proof.

Remark 2.5.1. Unification of the concept of weakly asymptotic contraction with property (W) in convex metric structure improves several known results in invariant best simultaneous approximation.