2.1 **INTRODUCTION**

Redundancy is one of the important and effective way to increase the probability of successful operation of the system and hence useful to increase the net profit earned by the system. Two-unit repairable redundant systems have been widely studied in the literature of reliability. Various researchers including [57, 66, 91] have analyzed two unit repairable redundant system under the common assumption that the repair of failed unit is completed without any time limitation. There may be situations where after the patience limit (prefixed) of repair time it is better (economically as well as technically) to replace the failed unit by new one.

In practice there may be situations where the cold standby unit with perfect switching device is not instantaneously available to operate it takes random activation time to operate.

Taking above facts in view, in the present study, a two non-identical unit cold standby system is analyzed under the concept of patience time (prefixed) of repair with activation time of standby unit. By using regenerative point technique, the following measures of system effectiveness are obtained—

(i) State transition probabilities and mean sojourn times in different states.

(ii) The reliability of the system and MTSF.
(iii) Expected up time of the system in (0, t) and in steady state.

(iv) Expected busy period of the repairman in (0, t) and in steady state.

(v) Expected down time of the system in (0, t) and in steady state.

(vi) Net expected profit earned by the system in (0, t) and in steady state.

2.2 SYSTEM DESCRIPTION AND ASSUMPTIONS

(i) The system consists of two non identical units in standby configuration. Initially p-unit is operative and o-unit is kept as cold standby.

(ii) Each unit of the system has two modes—normal (N) and total failure (F).

(iii) Upon failure of p-unit, standby unit (o-unit) goes for activation. During activation the repair of failed p-unit is not permissible.

(iv) A single repair facility (group of skilled persons) is always available with the system to repair as well as replacement of the failed unit and p-unit gets priority in repair over o-unit.

(v) When p-unit fails it goes for repair immediately and if the repair of failed p-unit is not completed before the patience limit (prefixed) of repair time then it replaced by now one, But the repair time of o-unit is not bounded with a patience limit.

(vi) Each repaired unit is as good as new.

(vii) The failure time distribution of each unit is negative exponential while repair time as well as activation time distribution is taken as general.

(viii) The distribution of lead time and replace time of p-unit are exponential with different parameters.

2.3 NOTATIONS AND SYMBOLS FOR THE SYSTEM STATES

\[ E \] : Set of regenerative states i.e. \( E = \{S_0 - S_6\} \).

\( \lambda_1, \lambda_2 \) : failure rate of p-unit/o-unit.

\( G_1(\cdot), G_2(\cdot) \) : c.d.f of repair time of the failed p-unit/o-unit.
\( Q_i(u) \) : p.d.f of \( U_i \) (\( i = 1 \) for lead time and \( i = 2 \) replacement time) i.e. \( \phi_i = \beta_i e^{-\lambda_i u} \ u \geq 0 \).

\( H(\cdot) \) : c.d.f of activation time of o-unit.

**Symbols for the States of the System**

\( N_{1o}/N_{2o} \) : p-unit/o-unit is operative and in \( N \)-mode.

\( N_{2s}/N_{2A} \) : o-unit in \( N \)-mode under cold standby activation.

\( F_{1r}/F_{1wr} \) : p-unit in \( F \)-mode and under repair/waiting for repair.

\( F_{2r}/F_{2wr} \) : o-unit in \( F \)-mode and under repair/waiting for repair.

\( F_{1Rp} \) : p-unit in \( F \)-mode and under replacement.

Using the above notations and symbols the possible states and the transitions among the states are shown in Figure-1.

### 2.4 TRANSITION PROBABILITIES AND SOJOURN TIMES

In for I we observe that the epochs of transition from \( S_2 \) to \( S_3 \) is non-regenerative as the future probabilistic behavior on this epoch depends upon the previous state. The all other entrance epochs are regenerative.

By simple probability arguments, the non-zero elements of \( Q = [Q_{ij}(t)] \) can be obtained as follows—

**a**  **Transition probabilities**

\[
Q_{01}(t) = \Pr[\text{System transits from state } S_0 \text{ to } S_1 \text{ during } u, u + du; u \leq t]
\]

\[
= \int_0^t \alpha_1 e^{-\alpha_1 u} du = 1 - e^{-\alpha_1 t}
\]

Similarly,

\[
Q_{12}(t) = \int_0^t dH(u)
\]
Fig. 2.1

Transition Diagram

Symbols:
- $S_i$: State i
- $F_i$: Event i
- $N_i$: Node i
- $\alpha_i$, $\beta_i$: Transition rates
- $\beta_1$, $\beta_2$, $\alpha_1$, $\alpha_2$: Specific rates

States and Events:
1. $S_1$: $N_{10}$, $N_{2A}$
2. $S_2$: $F_{lr}$, $N_{2O}$
3. $S_3$: $F_{lr}$, $F_{2wr}$
4. $S_4$: $N_{1O}$, $F_{2T}$
5. $S_5$: $F_{rp}$, $N_{2O}$
6. $S_6$: $F_{rp}$, $F_{5wr}$

Transitions:
- $\gamma_1$, $\gamma_2$: Transition functions
- $b_1$, $b_2$: Rates associated with specific events

Legend:
- •: Regenerative Point
- ×: Non-regenerative

States Indicators:
- Up State
- Failed State
- Tentative
\[ Q_{20}(t) = \int_0^t dG_1(u) e^{(\alpha_2 + \beta_1)u} \]

\[ Q_{23}(t) = \alpha_2 \int_0^t e^{- (\alpha_2 + \beta_1)u} \tilde{G}_1(u) \, du \]

\[ Q_{25}(t) = \beta_1 \int_0^t e^{-(\alpha_2 + \beta_1)u} \tilde{G}_1(u) \, du \]

\[ Q_{34}(t) = \int_0^t dG_1(u) e^{-\beta_1 u} \]

\[ Q_{36}(t) = \beta_1 \int_0^t e^{-\beta_1 u} \tilde{G}_1(u) \, du \]

\[ Q_{40}(t) = \int_0^t dG_2(u) e^{-\alpha_1 u} \]

\[ Q_{43}(t) = \alpha_1 \int_0^t e^{-\alpha_1 u} \tilde{G}_2(u) \, du \]

\[ Q_{50}(t) = \beta_2 \int_0^t e^{- (\beta_2 + \alpha_2)u} \, du = \frac{\beta_2}{\beta_2 + \alpha_2} [1 - e^{-(\beta_2 + \alpha_2)t}] \]

\[ Q_{56}(t) = \alpha_2 \int_0^t e^{- (\beta_2 + \alpha_2)u} \, du = \frac{\alpha_2}{\beta_2 + \alpha_2} [1 - e^{-(\beta_2 + \alpha_2)t}] \]

\[ Q_{64}(t) = \beta_2 \int_0^t e^{-\beta_2 u} \, du = 1 - e^{-\beta_2 t} \]  \hspace{1cm} (1-12)

(b) **Steady State transition probabilities**

These can be obtained by using

\[ p_{ij} = \lim_{t \to \infty} Q_{ij}(t) \]

Therefore

\[ p_{01}(t) = \alpha_1 \int_0^\infty e^{\alpha_1 u} \, du = 1 \]
The two step steady state transition probabilities can be obtained by using the relationship

\[ p_{ik}^{(k)} = p_{ik}p_{kj} \quad \text{[Chapman – Kolmogrov Equation]} \]

Hence

\[ p_{24}^{(3)} = p_{23}p_{34} = \frac{\alpha_2}{\alpha_2 + \beta_1} [1 - \tilde{G}_1(\alpha_2 + \beta_1)] \tilde{G}_1(\beta_1) \]

\[ p_{26}^{(3)} = p_{23}p_{36} = \frac{\alpha_2}{\alpha_2 + \beta_1} [1 - \tilde{G}_1(\alpha_2 + \beta_1)][1 - \tilde{G}_1(\beta_1)] \quad (25–26) \]

It can be easily verified that

\[ p_{01} = p_{12} = p_{64} = 1 \]

\[ p_{20} + p_{23} + p_{25} = 1 \quad \text{or} \quad p_{20} + p_{24}^{(3)} + p_{26}^{(3)} + p_{25} = 1 \]

\[ p_{34} + p_{36} = 1 \quad , \quad p_{40} + p_{43} = 1 \]

\[ p_{50} + p_{56} = 1 \quad (27–32) \]

(c) **Mean Sojourn Times**—Mean sojourn time \( \psi_{i} \) in state \( S_{i} \) is defined as the expected time for which the system stays in state \( S_{i} \) before transiting to any
other state. Let $X_i$ denote the sojourn time in state $S_i$, then the mean sojourn time in state $S_i$ is given by

$$\psi_i = \int P(X_i > t) \, dt$$

So that,

$$\psi_0 = \int e^{-\alpha_1 t} \, dt = 1/\alpha_1$$

Similarly,

$$\psi_1 = \int H(t) \, dt, \quad \psi_2 = [1 - \tilde{G}_1(\beta_1 + \alpha_2)] / (\beta_1 + \alpha_2)$$

$$\psi_3 = [1 - \tilde{G}_1(\beta_1)] / (\beta_1), \quad \psi_4 = [1 - \tilde{G}_2(\alpha_1)] / (\alpha_1)$$

$$\psi_5 = \frac{1}{\alpha_2 + \beta_2}, \quad \psi_6 = \frac{1}{\beta_2}$$

(33–39)

### 2.5 RELIABILITY AND MEAN TIME TO SYSTEM FAILURE (MTSF)

Let the random variable ‘$T_i$’ denotes the time to system failure (TSF) when the system starts state $S_i \in E$, then reliability of the system is given by

$$R_1(t) = P[T_1 > t]$$

To determine it, we regard the failed states $S_3$, $S_4$, $S_5$ and $S_6$ as absorbing.

By simple probabilistic arguments, we observe that $R_0(t)$ is the sum of the following contingencies—

(i) The system remains up in state $S_0$ up to time $t$ without any transition from $S_0$. The probability of this contingency is $Z_0(t) = e^{-\alpha_1 t}$.

(ii) System transits from state $S_0$ to $S_1$ during time $(u, u + du); u \leq t$ and then starting from state $S_1$ it remains up during the remaining time $(t - u)$. The probability of this contingency is
\[ \int_0^1 q_{01}(u) du R_1(t - u) = q_{01}(t) \otimes R_1(t) \]

Therefore,
\[ R_0(t) = Z_0(t) + q_{01}(t) \otimes R_1(t) \]

Similarly,
\[ R_1(t) = q_{12}(t) \otimes R_2(t) \]
\[ R_2(t) = Z_2(t) + q_{20}(t) \otimes R_0(t) \]

(40 – 42)

where,
\[ Z_2(t) = e^{-(a_2 + |h|)n} G_2(t) \]

Taking Laplace transform (L.T.) of the relations (40–42), we have
\[ R_0^*(s) = Z_0^*(s) + q_{01}^*(s) R_1^*(s) \]
\[ R_1^*(s) = q_{12}^*(s) R_2^*(s) \]
\[ R_2^*(s) = Z_2^*(s) + q_{02}^*(s) R_0^*(s) \]

The solution of \( R_i^*(s) \) can be written in matrix form as follows—

\[
\begin{bmatrix}
R_0^* \\
R_1^* \\
R_2^*
\end{bmatrix} =
\begin{bmatrix}
1 & -q_{01}^* & 0 \\
0 & 1 & -q_{12}^* \\
-q_{20}^* & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
Z_0^* \\
Z_1^* \\
Z_2^*
\end{bmatrix}
\]

For brevity, the argument ‘s’ is omitted from \( q_0^*(s), Z_i^*(s) \) and \( R_i^*(s) \).

Computing the element of the first row of the inverse matrix, we have
\[ R_0^*(s) = \frac{N_1(s)}{D_1(s)} \] (43)

where,
\[ N_1(s) = \gamma^* + Z_1^* q_{01}^* q_{12}^* \]

and
Taking the inverse Laplace transform (ILT) of (43), one may get the reliability of the system for the known values of parameters.

The mean time to system failure (MTSF) is given by

$$E(T_0) = \lim_{s \to 0} R_0^*(s) = \frac{N_1}{D_1}$$

(44)

using the results

$$Z_1'(0) = \psi_1 \text{ and } q_1'(0) = p_{ij}$$

One can get,

$$N_1 = \psi_0 + \psi_2 p_{01} p_{12}$$

and

$$D_1(0) = 1 + p_{01} p_{12}$$

2.6 **Availability Analysis**

Let $A_i(t)$ be the probability that the system is up(operative) at epoch $t$, when initially it starts operation from $S_i \in E$.

By using the similar probabilistic arguments as in earlier section, one can easily obtain the following recurrence relations among $A_0(t)$—

$$A_0(t) = Z_0(t) + q_{01}(t) \odot A_1(t)$$

Similarly,

$$A_1(t) = q_{12}(t) \odot A_2(t)$$

$$A_2(t) = Z_2(t) + q_{20}(t) \odot A_0(t) + q_{24}(t) \odot A_4(t) + q_{25}(t) \odot A_5(t) + q_{26}(t) \odot A_6(t)$$

$$A_3(t) = q_{34}(t) \odot A_4(t) + q_{36}(t) \odot A_6(t)$$

$$A_4(t) = \ldots + q_{43}(t) \odot A_3(t)$$
\[ A_2(t) = q_{50}(t) \otimes A_0(t) + q_{56}(t) \otimes A_6(t) \]

\[ A_6(t) = q_{64}(t) \otimes A_4(t) \quad (45 - 51) \]

Taking the Laplace transform (LT) of the above set of equations.

The solution for \( A_i^*(s) \) can be written in the matrix form as follows—

\[
\begin{pmatrix}
  A_0^* \\
  A_1^* \\
  A_2^* \\
  A_3^* \\
  A_4^* \\
  A_5^* \\
  A_6^*
\end{pmatrix} =
\begin{pmatrix}
  1 & -q_{01}^* & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & -q_{12}^* & 0 & 0 & 0 & 0 \\
  -q_{20}^* & 0 & 1 & 0 & -q_{24}^{(3)*} & -q_{25}^* & -q_{26}^{(3)*} \\
  0 & 0 & 0 & 1 & -q_{34}^* & 0 & -q_{36}^* \\
  -q_{40}^* & 0 & 0 & -q_{43}^* & 1 & 0 & 0 \\
  -q_{50}^* & 0 & 0 & 0 & 0 & 1 & -q_{56}^* \\
  0 & 0 & 0 & 0 & -q_{64}^* & 0 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
  Z_0^* \\
  Z_2^* \\
  Z_2 \\
  Z_2 \\
  Z_2 \\
  Z_2 \\
  0
\end{pmatrix}
\]

Simplifying the above matrix system for \( A_0^*(s) \), one gets

\[ A_0^*(s) = \frac{N_2(s)}{D_2(s)} \quad (52) \]

where,

\[ N_2(s) = Z_0^*[1-q_{43}^*(q_{34}^* + q_{36}^* q_{64}^*)] + Z_2^* q_{01}^* q_{12}^*[1-q_{43}^*(q_{34}^* + q_{64}^* q_{36}^*)] \]

\[ + Z_2^* q_{01}^* q_{12}^*[q_{25}^* q_{64}^* q_{56}^* + q_{24}^* q_{64}^* q_{26}^*] \]

and

\[ D_2(s) = 1 - q_{43}^*(q_{34}^* + q_{36}^* q_{64}^*) - q_{01}^* q_{12}^*[q_{25}^* (q_{40}^* q_{64}^* q_{36}^* + q_{50}^*) - q_{43}^* q_{20}^*(q_{34}^* + q_{64}^* q_{36}^*) - q_{25}^* q_{43}^* q_{50}^* q_{64}^* q_{26}^* + q_{25}^* q_{34}^* q_{43}^* q_{50}^* q_{56}^* q_{26}^*] \]

The steady-state availability of the system is given by

\[ A_0 = \lim_{t \to \infty} A_0(t) = \lim_{s \to 0} s A_0^*(s) = \lim_{s \to 0} s \frac{N_2(s)}{D_2(s)} \]

we have

\[ D_2(s)|_{s=0} = -q_{43}^* - q_{34}^* - q_{36}^* - q_{64}^* - q_{01}^* - q_{12}^* - q_{25}^* - q_{24}^* - q_{40}^* - q_{43}^* - q_{20}^* - q_{43}^* q_{20}^* \]
\[(p_{34} + p_{64}p_{36}) - p_{25}p_{36}p_{43}p_{50}p_{64} - p_{25}p_{34}p_{43}p_{50} + p_{26}^{(3)}p_{40}p_{64} + p_{40}^{(3)}p_{24} - p_{20}\]

\[= p_{40} - p_{25}(p_{40}p_{56} + p_{50}) + p_{43}p_{25}p_{50} - p_{40}^{(3)}p_{26} - p_{40}^{(3)}p_{24}\]

\[= p_{40}p_{20} + p_{43}p_{20} - p_{20}\]

\[= 0\]

Therefore, the steady state availability by L. Hospital’s rule is given by

\[A_0 = \lim_{s \to 0} \frac{N_2(s)}{D_2(s)} = \frac{N_2}{D_2}\]  \(\text{(53)}\)

where,

\[N_2 = p_{40}(\psi_0 + \psi_1) + \psi_4(p_{25}p_{56} + p_{24} + p_{26})\]

To obtain \(D'_2\), we note that \(q_{ij}^{*}(0) = tq_{ij}(t) \, dt = -m_{ij}\) So, we collect the coefficient of \(m_{ij}\)'s in \(D'_2\) as follows—

(i) Coefficient of \(m_{01} = p_{25}(p_{40}p_{56} + p_{50}) - p_{43}p_{25}p_{50} - p_{40}(1 - p_{25}) = C_0\) (say)

(ii) Coefficient of \(m_{12} = p_{25}(p_{40} - p_{43}p_{50}) - p_{40}(1 - p_{25}p_{56}) + p_{25}p_{50} = C_1\) (say)

(iii) Coefficient of \(m_{20} = p_{01}p_{12} - p_{01}p_{12}p_{43}(p_{34} + p_{36}) - 1 - p_{43} = p_{40} = C_2\) (say)

(iv) Coefficient of \(m_{25} = p_{01}p_{12}(p_{40}p_{56}p_{64} + p_{50}) - p_{01}p_{12}p_{43}p_{50}(p_{34} + p_{36})\)

\[= p_{40}p_{56} + p_{50} - p_{43}p_{50} = p_{40} = C_2\) (say)

(v) Coefficient of \(m_{24}^{(3)} = p_{40} = C_2\) (say)

(vi) Coefficient of \(m_{25} = p_{01}p_{12}(p_{40}p_{64} + p_{50}) - p_{01}p_{12}p_{43}(p_{34}p_{50} + p_{36}p_{50})\)

\[= p_{40}p_{56} + p_{50} - p_{43}p_{50} = p_{40}p_{56} + p_{40}p_{50} = p_{40} = C_2\) (say)

(vii) Coefficient of \(m_{26}^{(3)} = p_{40} = C_2\) (say)

(viii) Coefficient of \(m_{34} = p_{43} - p_{01}p_{12}p_{43}p_{50} - p_{01}p_{12}p_{43}p_{20}\)

\[= p_{43} - p_{12}p_{43}p_{50} - p_{12}p_{43}p_{20} = C_3\) (say)
(ix) Coefficient of $m_{36} = p_{43}p_{64} - p_{01}p_{12}p_{43}p_{25}p_{50}p_{64} - p_{01}p_{12}p_{43}p_{20}p_{64}$

$$= p_{43} - p_{43}p_{25}p_{50} - p_{43}p_{20} = C_3$$

(x) Coefficient of $m_{40} = p_{01}p_{12}p_{25}p_{64}p_{56} + p_{01}p_{12}p_{26}^{(3)}p_{64} + p_{01}p_{12}p_{24}^{(3)}$

$$= p_{25}p_{56} + p_{26}^{(3)} + p_{24}^{(3)} = C_4 \text{(say)}$$

(xi) Coefficient of $m_{43} = (p_{34} + p_{36}p_{64}) - p_{01}p_{12}p_{25}(p_{64}p_{56} + p_{50})$

$$= 1 - p_{25}p_{50} - p_{20}$$

$$= 1 - p_{25} - p_{20} + p_{25}p_{56}$$

$$= p_{25}p_{56} + p_{26}^{(3)} + p_{24}^{(3)} = C_4$$

(xii) Coefficient of $m_{50} = p_{01}p_{12}p_{25} - p_{01}p_{12}p_{43}p_{25}[p_{34} + p_{36}p_{64}]$

$$= p_{25} - p_{43}p_{25} = p_{25}p_{40} = C_5 \text{(say)}$$

(xiii) Coefficient of $m_{56} = p_{01}p_{12}p_{25}p_{40}p_{64} = p_{25}p_{40} = C_5$

(xv) Coefficient of $m_{64} = p_{43}p_{36}(1 - p_{20}) + p_{26}^{(3)}p_{40} - p_{43}p_{25}p_{36}p_{50} = C_6 \text{(say)}$

Therefore,

$$D'_2 = m_{01}C_0 + m_{12}C_1 + (m_{20} + m_{24}^{(3)} + m_{25} + m_{26}^{(3)})C_2 + (m_{34} + m_{36})C_3$$

$$+ (m_{40} + m_{43})C_4 + m_{50} + m_{56}C_5 + m_{64}C_6$$

On applying the relations

$$\psi_i = \sum_j m_{ij} \text{ and } m_{ij}^{(k)} = m_{ik}p_{kj} + m_{kj}p_{ik}$$

One can get

$$D'_2 \sim \psi \sim + \psi \sim - \psi \sim - \psi \sim + \sim + \sim + \psi \sim + \psi \sim + \psi \sim + \psi \sim C_6$$
2.7 BUSY PERIOD ANALYSIS

Let $B_1(t)$ be the probability of the repair facility being busy either in repair of failed unit or replacement of the failed unit at time ‘t’ when system initially starts from state $S_1 \in E$. By using simple probabilistic arguments following set of recurrence relations can be easily developed—

\[
B_0(t) = q_{01}(t) \oplus B_1(t)
\]

\[
B_1(t) = q_{12}(t) \oplus B_2(t)
\]

\[
B_2(t) = q_{20}(t) \oplus B_0(t) + q_{24}^{(3)}(t) \oplus B_4(t) + q_{25}(t) \oplus B_5(t) + q_{26}^{(3)}(t) \oplus B_6(t) + q_{23}(t) \oplus Z_3(t)
\]

\[
B_3(t) = Z_3(t) + q_{34}(t) \oplus B_4(t) + q_{36}(t) \oplus B_6(t)
\]

\[
B_4(t) = q_{40}(t) \oplus B_0(t) + q_{43}(t) \oplus B_3(t)
\]

\[
B_5(t) = Z_5(t) + q_{50}(t) \oplus B_0(t) + q_{56}(t) \oplus B_6(t)
\]

\[
B_6(t) = Z_6(t) + q_{64}(t) \oplus B_4(t)
\]

(54 – 60)

where,

\[
Z_3(t) = e^{-(a_1 + \beta_1)t} ; \quad Z_5(t) = e^{-(a_5 + \beta_2)t} \quad \text{and} \quad Z_6(t) = e^{-\beta_2 t}
\]

Taking L.T. of above set of equations and solving for $B_0(t)$, one can get

\[
B_0(t) = \frac{N_3(s)}{D_2(s)}
\]

where,

\[
N_3(s) = q_{01}^{*} q_{12}^{*} q_{43}^{*} Z_3^{*} [q_{25}^{*} q_{56}^{*} q_{64}^{*} + q_{26}^{(3)*} q_{64}^{*} + q_{24}^{(3)*}] + Z_3^{*} q_{01}^{*} q_{12}^{*} q_{25}^{*} Z_6^{*} q_{01}^{*} q_{12}^{*} q_{43}^{*} q_{25}^{*} q_{34}^{*} q_{56}^{*} + q_{24}^{(3)*} q_{36}^{*} - q_{26}^{(3)*} q_{34}^{*} - q_{01}^{*} q_{12}^{*} Z_6^{*} [1 - q_{25}^{*} q_{56}^{*}]
\]

and $D_2(s)$ is already defined in availability analysis.
In steady state

$$B_0 = \frac{N_3}{D'_{2}}$$

(61)

where,

$$N_3 = \psi_3 p_{43} [p_{25} p_{56} + p_{26}^{(3)} + p_{24}^{(3)}] + \psi_5 p_{25} p_{40} + \psi_6 p_{43} [p_{25} p_{34} p_{56} + p_{24}^{(3)} p_{36} - p_{26}^{(3)} p_{34}] - \psi_6 [1 - p_{25} p_{56}]$$

and $D'_{2}$ is already defined in availability analysis.

2.8 **DOWN TIME ANALYSIS**

Let $D_i(t)$ be the probability of the system is down at epoch $t$ when system initially starts from state $S_i \in E$. By using simple probabilistic arguments following set of recurrence relations can be easily developed—

$$D_0(t) = q_{01}(t) \odot D_1(t)$$

$$D_1(t) = Z_1(t) + q_{12}(t) \odot D_2(t)$$

$$D_2(t) = q_{20}(t) \odot D_0(t) + q_{24}^{(3)}(t) \odot D_4(t) + q_{25}(t) \odot D_5(t) + q_{26}^{(3)}(t) \odot D_6(t)$$

$$D_3(t) = q_{34}(t) \odot D_4(t) + q_{36}(t) \odot D_6(t)$$

$$D_4(t) = \ldots \ldots \odot D_3(t)$$

$$D_5(t) = q_{50}(t) \odot D_0(t) + q_{56}(t) \odot D_6(t)$$

$$D_6(t) = q_{64}(t) \odot D_4(t)$$

(62 – 68)

where,

$$Z_1(t) = H(t)$$

Taking L.T. of above set of equations and solving for $D_0(t)$, one can get

$$D_0(t) = \frac{N_4(s)}{D_2(s)}$$

(69)

where,

$$N_4(s) = \ldots \ldots - q_{36} q_{14} + q_{36} q_{64})$$
and $D_2(s)$ is already defined in availability analysis.

In steady state

$$D_0 = \frac{N_3}{D'_2}$$

where,

$$N_3 = \Psi_4 p_{40}$$

and $D'_2(s)$ is already defined in availability analysis.

### 2.9 PROFIT FUNCTION ANALYSIS

The net expected cost/profit (gain) incurred during $(0, t)$ is given by

$$P(t) = \text{Expected total revenue during } (0, t) - \text{Expected total expenditure during } (0, t)$$

$$= K_0 \mu_{up}(t) - K_1 \mu_{b}(t) - K_2 \mu_{d}(t)$$

(70)

Where, $K_0$ is the revenue per-unit up time by the system, $K_1$ be the amount per unit of time paid to the repair facility and $K_2$ is the loss per unit of time when the system is in down state. Also,

$$\mu_{up}(t) = \text{Expected up time of the system during } (0, t)$$

$$= \int_0^t A_0(u)du$$

so that,

$$\mu_{up}^*(s) = \frac{A_0^*(s)}{s}$$

(71)

Similarly,

$\mu_{b}^*(s)$ = Expected busy period of the repair facility during $(0, t)$
\[ = \int_{0}^{t} B_0(u) \, du \]

so that,

\[ \mu_b^*(s) = \frac{B_0^*(s)}{s} \quad (72) \]

\[ \mu_d(t) = \text{Expected down time of the system during (0, t)} \]

\[ = \int_{0}^{t} D_0(u) \, du \]

so that,

\[ \mu_d^*(s) = \frac{D_0^*(s)}{s} \quad (73) \]

The expected profit per-unit time in steady-state is given by

\[ P = \lim_{t \to \infty} \frac{\mathcal{L}(P(t))}{t} = \lim_{s \to 0} s^2 \mathcal{F}^*(s) \]

\[ = K_0 \lim_{s \to 0} s \mathcal{F}_0(s) - K_1 \lim_{s \to 0} s \mathcal{F}_0(s) - K_2 \lim_{s \to 0} s \mathcal{F}_0(s) \]

\[ = K_0 A_0 - K_1 B_0 - K_2 D_0 \quad (74) \]

2.10 **PARTICULAR CASE**

When repair time distribution for p-unit and o-unit is taken as –ve exponential with parameter \( \lambda_1 \) and \( \lambda_2 \) respectively and the activation time distribution is also taken as –ve exponential with parameter \( \theta \), the changes are as follows—

\[ P_{20} = \frac{\lambda_1}{\lambda_1 + \alpha - \beta} \quad , \quad P_{\alpha 1} = \frac{\alpha_2}{-\alpha + \beta} \quad , \quad P_{\beta 1} = \frac{\beta_1}{+\alpha + \beta_1} \]
\[ p_{34} = \frac{\lambda_1}{\lambda_1 + \beta_1}, \quad p_{36} = \frac{\beta_1}{\lambda_1 + \beta_1}, \quad p_{40} = \frac{\lambda_2}{\lambda_2 + \alpha_1} \]
\[ p_{43} = \frac{\lambda_1}{\lambda_2 + \beta_1}, \quad p_{24}^{(3)} = \frac{\alpha \lambda_1}{(\lambda_1 + \beta_1)(\lambda_1 + \alpha_2 + \beta_1)} \]
\[ p_{26}^{(3)} = \frac{\alpha \lambda_1}{(\lambda_1 + \beta_1)(\lambda_1 + \alpha_2 + \beta_1)}, \quad \psi_1 = \frac{1}{\theta}, \quad \psi_2 = \frac{1}{\lambda_1 + \beta_1 + \alpha_2} \]
\[ \psi_3 = \frac{1}{\lambda_1 + \beta_1}, \quad \psi_4 = \frac{\lambda}{\lambda_2 + \alpha_1} \]  
(75 – 87)

2.11 GRAPHICAL ANALYSIS

For a more concrete study of the system’s behaviour we plot curves for MTSF and profit function w.r.t failure rate \( \alpha_1 \).

Fig. 2.2 shows the variation in MTSF w.r.t \( \alpha_1 \) for three different values of repair rate of p-unit \( \lambda_1 = 0.3, 0.2, 0.1 \), while other parameters are kept fixed as \( \alpha_2 = 0.02, \beta_1 = 0.001, \beta_2 = 0.002, \lambda_2 = 0.01, \theta = 0.001 \). From the graph it is observed that the MTSF uniformly decreases as the value of \( \alpha_1 \) increases and tends to vanish as \( \alpha_1 \) becomes large. Also the value of MTSF increases as the value of repair rate of p-unit \( \lambda_1 \) increases.

Fig. 2.3 shows the variation in profit w.r.t \( \alpha_1 \) for different values of \( \lambda_1 \) and \( \theta \) (other parameter kept fixed as above with \( K_0 = 3000; K_1 = 1300; K_2 = 550 \)). From the graph it is clear that the profit decreases as the failure rate \( \alpha_1 \), activation parameter \( \theta \) increases. It is also observed that the profit increases as the repair rate \( \lambda_1 \) increases.
BEHAVIOUR OF MTSF w.r.t. $\alpha_1$ FOR DIFFERENT VALUES OF $\lambda_1$