CHAPTER 3

INTEGRATED FORCE METHOD AND ITS FORMULATION

3.1 PREAMBLE

Navier’s table problem (1785) perhaps was the initiator of the analysis of indeterminate structural problems. Navier wanted to calculate the four reactions induced at the foot of the table, but he had only three standard equilibrium conditions. The problem was one degree statically indeterminate. He developed one additional condition i.e. deformational compatibility condition simply by indirect approach, as the available compatibility formulation was just insufficient, incomplete or adhoc in nature to deal with the structural mechanics and Theory of Elasticity.

Despite of immature and insufficient compatibility condition, the development in structural mechanics continued but only through indirect approach and methods were categorized as either displacement method or force method. In the displacement method, the possible nodal displacements are constrained and by applying unit displacement in the direction of constrained displacement stiffness coefficients are worked out. Thus, by considering primary unknowns as displacements, first of all unknown joint displacements are calculated and then the secondary unknowns such as reaction and internal forces are calculated. In the force method, on the other hand, the complete structure is made statically determinate and displacements are calculated in the direction of redundants due to loading. By applying unit force in the direction of primary unknowns, the flexibility coefficients are calculated. Using governing compatibility equations, primary unknowns i.e, redundants, reactions and then other internal actions are calculated.

Infact, the names stiffness method and flexibility method are more diffuse name for the displacement and force methods, respectively. Generally
speaking these apply when stiffness and flexibility matrices, respectively, are important part of the modeling and solution process.

Now, the whole pretext regarding immature development of compatibility condition methods for structural mechanics and theory of elasticity can be understood with the help of a Pie diagram which is shown in Fig. 3.1.

![Pie Chart](image)

**Fig. 3.1 Equilibrium Equations and Compatibility Conditions**

It is a fact that all the indirect methods discussed above were just developed by using the three quarter of the pie chart Fig. 3.1 i.e. field and boundary part of equilibrium equations and field part of compatibility conditions. The remaining component was developed by Patnaik [7], by considering one additional condition which is known as force compatibility condition and Beltrami-Michell’s formulation was converted to Complete Beltrami Mitchell’s formulation. Thus, Structural Mechanics and Theory of Elasticity become full fledge computational tool for solving the various framed and continuum structure problems.
Using the additional condition in terms of forces with boundary compatibility condition, a new force method was evolved named as Integrated Force Method (IFM), which can complement all the available indirect methods. The IFM is a relatively young approach for the analysis of indeterminate structures, which makes use of both the fundamental equations i.e. equilibrium equations (EEs) and compatibility conditions (CCs). Formerly there was certain degree of asymmetry in the development and utilization of these two concepts. The underlying principle behind the EEs is force balancing likewise the compliance of forces and initial deformations are achieved through the CCs. Augmentation of these CCs with system EEs leads to IFM. It is possible by taking into account of these relations to obtain a complete system of equations which must be satisfied by stress components and thus the way is open for direct determination of forces without solving for components of displacements. In IFM all the internal forces and reactions are directly treated as primary unknowns unlike the displacements and redundants in well-known stiffness and flexibility methods respectively.

3.2 VARIATIONAL FUNCTIONALS FOR IFM

The IFM is one of the five formulations of structural mechanics, where others are Flexibility Method (FM), Displacement Method (DM), Mixed Method (MM) and Total Method (TM). The Pie chart depicted in Fig. 3.1 shows the role of three parts i.e. Field, Boundary equilibrium conditions and Field compatibility condition in the formulation. Using the stationary condition of variational formulations for the IFM yields the equilibrium equations and compatibility conditions as well as force and displacement boundary conditions. Also, it yields a new set of boundary conditions which was identified as the boundary compatibility conditions before few years. It opened a new thrust for theory of elasticity based problems by considering force boundary condition rather than displacement boundary condition. In
literature, it has been named as Completed Beltrami-Mitchell Formulation (CBMF), or known as IFM for Theory of Elasticity.

For a two dimensional elasticity problem, the variational functional \( \pi_s(\sigma, u, \sigma^e) \) of the IFM is obtained by adding the three expressions as follows.

\[
\pi_s = A + B - W
\]  \( \ldots (3.1) \)

In which the variables of the functional \( \pi \), for variational purpose are the displacements \( u \) and redundant stresses \( \sigma^e \). The BMF has two EEs and one CC in the field expressed in term of stresses. The variational variables of functional \( \pi_s^\text{BMF} \) are two displacements \( u \) and \( v \) and one redundant stress function which can be considered as Airy Stress Function \( \phi \) that gives,

\[
\sigma_x^e = \frac{\partial^2 \phi}{\partial y^2} V; \quad \sigma_y^e = \frac{\partial^2 \phi}{\partial x^2} V; \quad \tau_{xy}^e = -\frac{\partial^2 \phi}{\partial x \partial y} \]  \( \ldots (3.2) \)

The internal strain energy functional represents stresses and strains which are treated independently in explicit form. It is expressed as

\[
A(\sigma, u) = h\int_s \left[ \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] dxdy \]  \( \ldots (3.3) \)

The functional \( B(\beta, R) \) gives the compatibility conditions of the structural deformation in two dimensional elasticity. Thus the functional \( B(\epsilon, \sigma^e) \) can be written in the form as

\[
B(\epsilon, \sigma^e) = h\int_s \left[ \epsilon_x \sigma_x^e + \epsilon_y \sigma_y^e + \gamma_{xy} \tau_{xy}^e \right] dxdy \]  \( \ldots (3.4) \)

This \( B(\epsilon, \sigma^e) \) is the complementary strain energy function in which the strains \( \epsilon \) and redundant stresses \( \sigma^e \) are treated independently. For IFM, as per stress strain law, the strain is converted into stresses. So, the functional for a two dimensional elasticity problem can be written as

\[
B(\sigma, \sigma^e) = h\int_s \left[ \frac{\sigma_x - \mu \sigma_y}{E} \sigma_x^e + \frac{\sigma_y - \mu \sigma_x}{E} \sigma_y^e + (\frac{1 + \mu}{E}) \gamma_{xy} \tau_{xy}^e \right] dxdy \]  \( \ldots (3.5) \)

The potential of the external force \( W(P, u) \) has following three components;
\[
W(P,u) = h\int_S [B_xu + B_yv]dxdy + \int_{L_1} [\dot{P}_xu + \dot{P}_yv]dL_1 + \int_{L_2} [P_x\dot{u} + P_y\dot{v}]dL_2
= I_1 + I_2 + I_3 \quad \ldots \quad (3.6)
\]

The first integral (I_1) is for body forces B_x and B_y in the field S, second integral (I_2) is for the portion of boundary L_1 on which the external loading \( \dot{P}_x \) and \( \dot{P}_y \) is acting and remaining integral (I_3) is for the boundary L_2 on which displacements \( \dot{u} \) and \( \dot{v} \) are prescribed as shown in Fig. 3.2.

\[\text{Fig. 3.2 Boundary Compatibility Conditions in BVP}\]

Substituting Eqs. (3.3) to (3.6) in Eq.(3.1) after substituting Eq. (3.2) in Eq. (3.3) gives

\[
\begin{align*}
\pi_s^{BMF} &= h\int_S \left[\sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]dxdy + h\int_S \left[\frac{\sigma_x - \mu\sigma_y}{E} \sigma_x^e + \frac{\sigma_y - \mu\sigma_x}{E} \sigma_y^e + (\frac{1+\mu}{E})\gamma_{xy}\tau_{xy}^e\right]dxdy - h\int_S [B_xu + B_yv]dxdy - \int_{L_1} [\dot{P}_xu + \dot{P}_yv]dL_1 - \int_{L_2} [P_x\dot{u} + P_y\dot{v}]dL_2
\end{align*}
\]

\[\ldots \quad (3.7)\]

For the stationary conditions of the VF, in which u, v and \( \phi \) are non zero quantities, making their brackets including coefficients equals to zero leads to the following field equations and boundary conditions:

(1) The field equilibrium equation are

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0 \quad \text{and} \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + B_y = 0 \quad \ldots \quad (3.8)
\]

(2) The field compatibility conditions are
\[
\frac{\partial^2}{\partial x^2} (\sigma_y - \mu \sigma_x) + \frac{\partial^2}{\partial y^2} (\sigma_x - \mu \sigma_y) - 2(1+\mu) \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \quad \ldots (3.9)
\]

The Eq. (3.9) is CC which can be simplified by using equilibrium equation (3.8) and relation between body forces and its potential \( B_x = \frac{\partial u}{\partial x}, \quad B_y = \frac{\partial v}{\partial y} \) as

\[
\nabla^2 (\sigma_x + \sigma_y) + (1+\mu) \nabla^2 \psi = 0 \quad \ldots (3.10)
\]

Along the boundary, where forces are prescribed as \( \dot{P}_x \) and \( \dot{P}_y \), stress boundary conditions are considered which gives

\[
\sigma_x n_x + \tau_{xy} n_y = \dot{P}_x \quad \text{and} \quad \sigma_y n_y + \tau_{xy} n_x = \dot{P}_y \quad \ldots (3.11)
\]

\[
\frac{\partial}{\partial x} (\sigma_y - \mu \sigma_x) n_x + \frac{\partial}{\partial y} (\sigma_x - \mu \sigma_y) n_y - (1+\mu) \left( \frac{\partial \tau_{xy}}{\partial x} n_y + \frac{\partial \tau_{xy}}{\partial y} n_x \right) = 0 \quad \ldots (3.12)
\]

Here Eq. (3.11) is known as classical stress boundary condition and the boundary condition given by Eq. (3.12) is identified as the novel boundary compatibility condition. Due to which Beltarmi-Michell’s Formulation is known as Completed Beltarmi-Michell’s Formulation (CBMF) which is IFM procedure for Theory of Elasticity.

### 3.3 IMPORTANCE OF STRAIN FORMULATION

The strain formulation in theory of elasticity and development of compatibility condition in structural mechanics have neither understood properly and neither utilized in past. Due to this shortcoming, the important development in the direction of methods of analysis for framed and continuum structures got stuck up and diverted to indirect solution techniques. Also, because of this direct stress formulation, which calculates stresses and strains in the structures and continuum could not be developed properly. Using indirect methods, it has been calculated using mathematical differentiation of displacement function, which was developed through an approximate interpolation function. After understanding the importance of
strain formulation in terms of Boundary Compatibility Conditions (BCCs) and it has been derived using variational formulation and also verified using mathematical form of integral theorem. Navier’s method in elasticity and displacement method in structural mechanics have to take care of this additional condition as an extra step typically at inter-elemental boundaries in discrete element models.

Thus the completion of strain formulation led to revival of direct force calculation method in addition to availability of indirect method, known as IFM and IFMD which are applicable for structural mechanics based framed types problem and Completed Beltarmi-Michells Formulation (CBMF) in theory of elasticity which is based on strain formulation is used to solve 2D plane elasticity problems. Through CBMF now one can attempt problems of direct stress calculation, displacement calculation, mixed boundary value problem with the other major limitations and loop holes of classical force method also being removed; which was in past applicable to only stress boundary condition. The researchers found highest fidelity response by using both the new methods even with coarser FE models for different types of problems. Table 3.1[83] shows the different methods of structural mechanics and theory of elasticity with and without use of compatibility condition.

Table 3.1 Various Methods and Associated Variational Formulations

<table>
<thead>
<tr>
<th>Various Methods</th>
<th>Primary Variables</th>
<th>Variational Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elasticity Structures</td>
<td>Elasticity Structures</td>
<td>IFM Variational Formulation</td>
</tr>
<tr>
<td>Completed Beltrami-Mitchell Formulation (CBMF)</td>
<td>Stresses Forces</td>
<td></td>
</tr>
<tr>
<td><strong>Intgrated Force Method (IFM)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>EEs &amp; CCs Enforced</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Airy Formulation (AF)</td>
<td>Stress Function Redundants</td>
<td>Complementary Energy</td>
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<tr>
<td><strong>Redundant Force Method</strong></td>
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<tr>
<td><strong>Field CCs Enforced</strong></td>
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</table>
3.4 BASIC RELATIONS OF INTEGRATED FORCE METHOD

The concept of equilibrium of forces and compatibility of deformations are fundamental to analysis of framed structures. The equilibrium equations are written in terms of forces, which can be axial forces, shear forces, bending moments and twisting moments. The compatibility conditions are expressed in terms of deformations, which can be elongations, deflections and curvatures. Hence, it is utmost compulsion to express these developed CCs in terms of forces, so that it can be coupled and concatenated from bottom side in a matrix form with Equilibrium Equations, which are already available in terms of forces. So, to convert the deformations to forces, two additional sets of equations are required. These are the deformation displacement relations (DDRs) and force displacement relations (FDRs). Thus, the four sets of structural mechanics equations required in IFM analysis are:

1. Equilibrium equations (EEs),
2. Deformation displacement relations (DDRs),
3. Compatibility conditions (CCs), and
4. Force deformation relations (FDRs).
3.4.1 Equilibrium Equations (EEs)

A member as shown in Fig. 3.3, subjected to a system of external forces, is said to be in equilibrium, when it remains in a position of rest i.e. when the force resultants in the directions of the reference axes are equal to zero. This way three equations for forces and three for moments are available. Force balancing is the central concept behind the equilibrium equations in structural mechanics.

Fig. 3.3 Forces and Moments in Three Directions

Generally equilibrium equations are written by considering either reactions or applicable internal forces as the unknowns. The generation of equilibrium equation is illustrated here with reference to a fixed beam example having length L and flexural rigidity EI, which is subjected to transverse load P as shown in Fig. 3.4.

Case 1- External reactions as unknowns

In this case all the applicable external support reactions are treated as unknown forces. Two equilibrium equations for the beam case can be formulated as algebraic summation of forces along y-y axis and moments about z-z axis equals to zero as follows
Fig. 3.4 Fixed Beam Example

\[ \sum F_y = 0, \text{ gives } V_A + V_B - P = 0 \quad \ldots \ (3.13) \]

\[ \sum M_{zz} = 0, \text{ gives } M_A + V_B \times L - \frac{PL}{2} - M_B = 0 \quad \ldots \ (3.14) \]

**Case 2- Internal Moments as unknowns**

In this case beam is discretized into two segments i.e. AC and CB. Both the end moments, for each element, are considered as unknowns. Equilibrium equation in this case can be formulated at joint C by considering equilibrium between: (i) External force P and shearing forces of both the members, (ii) Internal moment induced on either side as shown in Fig. 3.5.

Equilibrium Equations are formulated as follows;

\[ \sum F_y = 0 \text{ at C gives } 2(M_1 - M_2)/L + 2(M_4 - M_3)/L - P = 0 \]

\[ \sum M_z = 0 \text{ at C gives } -M_2 + M_3 = 0 \quad \ldots \ (3.15) \]
3.4.2 Deformation Displacement Relations (DDRs)

Deformation is a change in geometrical shape of the structure due to applied loading. Deformation is often described in terms of strains in mechanics of structures. Displacement on the other hand specifies the position of a point with reference to its original or previous position. The deformation displacement relation (DDR) is an important relation in IFM and is a central component behind both equilibrium and compatibility conditions. The DDR can directly be derived from the equilibrium equations. It depends on the type of material, size and geometry of the member and the forces applied. The deformation is always associated with each type of force variable, i.e. extension in the rod is the deformation due to axial force, bending curvature is the deformation in beam bending, shear deformation is due to shearing forces and twisting angle is the deformation due to twisting action. In the derivation, deformation displacement relation in IFM can be written using the transpose of the equilibrium matrix derived for a particular system. In DDR the system displacements are mapped into the elemental deformations. Symbolically it can be represented as

\[
\{\beta\} = [B]^T \{d\} 
\]

where, \(\{\beta\}\) are the ‘n’ elemental deformations in given structure, \([B]\) is equilibrium matrix of size (m x n) and \(\{d\}\) are the nodal displacements.

3.4.3 Compatibility Conditions (CCs)

While studying the geometry of the system, the deformation of the small segment of complete structure must be consistent as well as continuous with the overall deformation pattern (Fig. 3.6). It is known as condition of continuity or compatibility.
Whenever structural member deforms due to external loading, at any point of the deflected shape, the displacement function must be continuous, differentiable, and single-valued which takes care of non-distortional and non-overlapping mechanism. The deformational compatibility conditions are necessary for the solution of indeterminate structural problems [76]. Also, in IFM sinking of supports and temperature variation can be considered as the initial strain which can be directly accounted through compatibility relations for solution purpose.

In IFM, first the DDR is derived and then the elimination of the displacements from the deformation displacement relation is carried out to obtain the compatibility conditions.

### 3.4.4 Force Deformation Relations (FDRs)

In order to analyse any problem using IFM, force deformation relations are necessary to convert all the compatibility conditions, which are in terms of displacements to independent variables which represent the internal forces for given structural members. The bottom most part of the equilibrium matrix [B] i.e. which represents the equilibrium equations needs this paradigm because upper few rows of [B] are already in terms of forces. Thus, final equation is [B] \{F\} = \{P\}. Likewise, the compatibility conditions (CCs) are written in terms of deformations \{\beta\} and thus [C] \{\beta\} = \{0\}.

The force deformation relation (FDR) can be obtained from the Hooke’s law by simply relating the internal stress developed to the force applied and then deformation to strain induced.
(i) **Derivation of FDR for an Axial Rod**

For a cylindrical rod having cross sectional area as A and length as L, subjected to an axial force F (**Fig. 3.7**), the FDR can be written as follows:

![Fig. 3.7 Bar subjected to an Axial Force](image)

As per strength of material relations, \( \varepsilon = \frac{\delta l}{L} \)

In IFM \( \varepsilon = \frac{\beta}{L}, \sigma = \frac{F}{A} \)

Now according to Hooke’s law stress and strain can be related as

\[
\varepsilon = \frac{\sigma}{E} \quad \text{or} \quad \frac{\beta}{L} = \frac{\sigma}{E} \quad \beta = \frac{\sigma L}{E} \quad \text{or} \quad \beta = \left( \frac{L}{AE} \right) F \quad \ldots (3.17)
\]

Where \( \left( \frac{L}{AE} \right) \) is the flexibility coefficient corresponding to an axial force F.

According to Castigliano’s theorem, the first derivative of strain energy U with respect to force P is equal to the deformation (\( \beta \) in IFM) corresponding to that force P.

\[
\beta = \frac{\partial W}{\partial P} = \frac{\partial U}{\partial P} \quad \ldots (3.18)
\]

Where strain energy \( U = \int_{0}^{L} \frac{\sigma \varepsilon}{2} A \, dx \)

Replacing \( \sigma \) by \( F/A \) and \( \varepsilon = \sigma/E \) by \( F/AE \), the strain energy for a uniform bar of area A corresponding to force F can be written as
\[ U = \int_{0}^{l} \frac{F^2}{2AE} \, dx = \frac{F^2L}{2AE} \] \hspace{1cm} \ldots (3.19)

\[ : \text{Deformation} \quad \beta = \frac{\partial U}{\partial P} = \left( \frac{L}{AE} \right) F \] \hspace{1cm} \ldots (3.20)

In indeterminate analysis the energy based derivation for FDR is preferred.

**ii) Derivation of FDR for a Beam Member**

Generally shear force and bending moment are the two governing internal forces for the bending deformations in the beams, which are related to each other. However, the strain energy due to shear force as it violates the basic assumption of stress-strain is neglected, only the strain energy due to the moments is taken into account.

From the Castigliano’s second theorem,

\[ \beta = \frac{\partial W}{\partial P} = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \int_{0}^{l} \frac{M^2}{2EI} \, dx \] \hspace{1cm} \ldots (3.21)

Where, \( U = \int \frac{M^2}{2EI} \, dx \)

To simplify the procedure, one may consider

\[ \beta = \int_{0}^{l} M \left( \frac{\partial M}{\partial P} \right) \frac{dx}{EI} \] \hspace{1cm} \ldots (3.22)

Beam response requires two internal unknown forces i.e. either two end moments or a pair of bending moment and shear force.

The FDR for these two cases is worked out as follows.

**Case 1:** Consider a beam as shown in Fig. 3.8 with two end moments \( M_1 \) and \( M_2 \) as unknowns with reactions developed as \( V_A \) and \( V_B \).
Taking moment at any section distance $x$ from $A$

$$M(x) = V_A x + M_1$$

Substituting the value of $V_A$ from the diagram and differentiating with respect to $M_1$ gives

$$\frac{\partial M(x)}{\partial M_1} = 1 - \frac{x}{L}$$

So, $\beta_A = \frac{\partial U}{\partial M_1} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial M_1} \, dx = \frac{L}{EI} \left( \frac{M_1}{3} + \frac{M_2}{6} \right)$

Similarly, $\beta_B = \frac{\partial U}{\partial M_2} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial M_2} \, dx = \frac{L}{EI} \left( \frac{M_2}{3} + \frac{M_1}{6} \right)$

It can be seen from the above equations that the coefficients associated with the two moment unknowns are the generalized flexibility coefficients. Thus, the two deformations for the beam can be represented in matrix form as:

$$\begin{bmatrix} \beta_A \\ \beta_B \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$\{\beta\} = [G] \{F\}$$

where $[G]$ is the flexibility matrix associated with $M_1$ and $M_2$ for the beam.

**Case 2:** Consider a beam as shown in Fig. 3.9 with shear force $V$ and bending moment $M$ at a distance $L$ from $A$ as unknowns. As seen above, the FDR for this case can be directly expressed through the flexibility coefficients related to the type of force unknowns.
Following the same procedure, the matrix form for deformation in terms of flexibility coefficients can be written as

\[
\begin{bmatrix}
\beta_A \\
\beta_B
\end{bmatrix} = \frac{1}{EI} \begin{bmatrix}
L & \frac{L^2}{2} \\
\frac{L^2}{2} & \frac{L^3}{3}
\end{bmatrix} \begin{bmatrix}
M \\
V
\end{bmatrix}
\]

... (3.25)

\[
\begin{bmatrix}
\beta
\end{bmatrix} = [G] \{F\}
\]

where [G] is the flexibility matrix associated with M and V for the beam.

### 3.5 NULL PROPERTY OF EEs AND CCs

Once the equilibrium between external loading and internal forces or external reactions is satisfied, a rectangular matrix [B] of size m x n is developed with m being the number of equilibrium equations considered along displacement directions and n being the total number of unknowns considered. Thus, r = n – m compatibility conditions are needed. After developing the equilibrium matrix [B], displacement deformation matrix is written as follows.

\[
\{\beta\}_{(n \times 1)} = [B]^T_{(n \times m)} \{\delta\}_{(m \times 1)}
\]

... (3.26)

Once \{\beta\} is developed, in terms of nodal displacements (\theta, \delta), one has to develop r number of compatibility conditions by eliminating the nodal displacements from the equations.
Let \( r = 2 \), then the problem is \( 2^0 \) degree of statically indeterminate. Converting the compatibility conditions in a matrix form gives

\[
[C]_{(r \times n)} \{\beta\}_{(n \times 1)} = \{0\} \quad \ldots (3.27)
\]

Substituting \( \{\beta\} \) from Eq.(3.26) gives,

\[
[C]_{(r \times n)} [B]^{T}_{(n \times m)} \{\delta\}_{(m \times 1)} = \{0\} \quad \ldots (3.28)
\]

As displacement vector \( \{\delta\} \neq \{0\} \), its coefficients matrix \([C]_{(r \times n)} [B]^{T}_{(n \times m)}\) must be zero. Thus,

\[
[C]_{(r \times n)} [B]^{T}_{(n \times m)} = \{0\} \quad \ldots (3.29)
\]

or

\[
[B]_{(m \times n)} [C]^{T}_{(n \times r)} = \{0\} \quad \ldots (3.30)
\]

Which is a must for the correctness of the solution for all the problems and it also confirms that the developed equilibrium matrix \([B]\) and the compatibility matrix \([C]\) are correct.

### 3.6 IFM SOLUTION PROCEDURE

The complete procedure is explained here with the help of a fixed beam example shown in **Fig. 3.4**.

**Step 1:** Develop Equilibrium Matrix \([B]\)

By referring **Fig. 3.4** and Eq. (3.15), one can write

\[
\begin{bmatrix}
\frac{2}{L} & -\frac{2}{L} & -\frac{2}{L} & \frac{2}{L} \\
0 & 1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix} = \begin{bmatrix}
P \\
0 \\
0 \\
0
\end{bmatrix}
\]

or

\([B]\{F\} = \{P\} \quad \ldots (3.31)\]

**Step 2:** Develop Displacement Deformation Relation \(\{\beta\}\)
\[
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix} =
\begin{bmatrix}
\frac{2}{L} & 0 \\
-\frac{2}{L} & 1 \\
-\frac{2}{L} & -1 \\
\frac{2}{L} & 0
\end{bmatrix}
\begin{bmatrix}
\delta \\
\Theta
\end{bmatrix}
\] ... (3.32)

Which can be written as

\[\{\beta\} = [B]^T \{d\}\]

which can be written in expanded form as follows:

\[
\beta_1 = \frac{2}{L} \delta, \quad \beta_2 = -\frac{2}{L} \delta + \Theta, \quad \beta_3 = -\frac{2}{L} \delta - \Theta\quad \text{and} \quad \beta_4 = \frac{2}{L} \delta
\] ... (3.33)

**Step 3**: Develop Compatibility Matrix \([C]\)

Let nodal displacements \(\delta\) and \(\Theta\), just below the central point load \(P\). To make \([B]\) matrix square, one needs to develop two additional conditions using displacement deformation relations \(\{\beta\}\), as \(r = n - m = 4 - 2 = 2\).

So, \(\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0\) and \(\beta_1 - \beta_4 = 0\)

Arranging in matrix form, one can write

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\] ... (3.34)

or \([C]\{\beta\} = \{0\}\), where \([C]\) is known as the compatibility matrix which is developed by eliminating the two nodal displacements \((\delta, \Theta)\) from the displacement deformation relation. Developing coefficients for all the \(\{\beta\}\) is always a mathematical hurdle in the further development.

Using Eq. (3.17), the null property of all the matrices is checked, which validate the above formulation.
Step 4: Develop Force Deformation Relation

As the problem is having four internal moments as unknowns, considering relative flexibility coefficients and using Eq. (3.4), one can write for AC and CB segments of fixed beam,

\[
\begin{bmatrix}
\beta_A \\
\beta_C
\end{bmatrix} = \frac{L}{12EI} \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
M_1 \\
M_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\beta_C \\
\beta_B
\end{bmatrix} = \frac{L}{12EI} \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
M_3 \\
M_4
\end{bmatrix}
\]

Substituting Eq. (3.24) into Eq. (3.22), all the DDRs can be converted to FDRs (Force deformation relations).

Thus, one can write

\[
\frac{L}{12EI} [3M_1 + 3M_2 + 3M_3 + 3M_4] = 0
\]

\[
\frac{L}{12EI} [2M_1 + M_2 - M_3 + 2M_4] = 0
\]

Arranging in matrix form,

\[
\frac{L}{12EI} \begin{bmatrix}
3 & 3 & 3 & 3 \\
2 & 1 & -1 & 2
\end{bmatrix} \begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Step 5: Develop Global Equilibrium Matrix

Concatenating the Eq. (3.25) by substituting related values of parameters in Eq. (3.19), one can have the global equilibrium matrix as
\[
\begin{bmatrix}
2 & -2 & 2 & 2 \\
0 & 1 & -1 & 0 \\
1 & 1 & 1 & 1 \\
4 & 4 & 4 & 4 \\
6 & 12 & -1 & 12 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Which can be written in the form

\[ [S][F] = [P] \] \hspace{1cm} \text{... (3.38)}

Solving the above matrix gives the value of internal moments as

\[
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
0.125 \\
-0.125 \\
-0.125 \\
0.125 \\
\end{bmatrix} \text{ kN-m} \] \hspace{1cm} \text{... (3.39)}

Which are matching with the exact solution of moments in a fixed beam at supports and at a mid span under a central point load.

**Step 6: Calculate Nodal Displacements**

The equation for nodal displacements is given by

\[ \{\delta\} = [J][G][M] \]

where \([J]\) = m rows from top of \([S]^{-1}\) matrix and is of size 2 x 4, \([G]\) consists of flexibility coefficients calculated from Eq. (3.24) of \([\beta]\) matrix as

\[
\begin{bmatrix}
\frac{1}{6} & \frac{1}{12} & 0 & 0 \\
\frac{1}{12} & \frac{1}{6} & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{12} \\
0 & 0 & \frac{1}{12} & \frac{1}{6} \\
\end{bmatrix}
\]

The solution obtained from Eq. (3.39) is as follows:

\[ w = 0.0052 \text{mm} \text{ and } \theta = 0.0 \text{ radians}, \text{ which is matching with the exact solution of deflection of a fixed beam under the central point load as } \frac{PL^3}{192EI} = 0.005208\text{mm}. \]
### 3.7 IFM FOR FRAMED STRUCTURES

In the IFM any structure is designated as “structure (n, m)” where n and m are the force and displacement degrees of freedom respectively. The n component of force vector \{F\} must satisfy m equilibrium equations along displacement directions, with \( r = (n - m) \) being the number of compatibility conditions. For framed structures, the equilibrium equations can be symbolized as

\[
[B] \{F\} = \{P\} 
\]

... (3.40)

The EEs represent a relation between the internal forces \{F\} and external loads \{P\}. The internal forces \{F\} are the prime variables of equilibrium. The equilibrium matrix \[B\] is always rectangular for statically indeterminate structures, where \( n > m \), and is a sparse matrix for large scale problems. The development of \[B\] matrix is very simple and straight forward.

The work done by external loading \{P\} of the structure by considering the nodal displacements \{X\} of the structure can be written as

\[
W = \frac{1}{2} \{P\}^T\{X\} = \frac{1}{2} \left[p_1X_1 + p_2X_2 + p_3X_3 + \ldots + p_mX_m \right] 
\]

... (3.41)

The internal strain energy of the structure can also be written considering the deformation of the elements as follows.

\[
IE = \frac{1}{2} \{F\}^T\{\beta\} = \frac{1}{2} \left[f_1\beta_1 \beta_1 + f_2\beta_2 \beta_2 + f_3\beta_3 \beta_3 + \ldots + f_n\beta_n \right] 
\]

... (3.42)

In which, \{\beta\} represents the vector of generalized internal deformations of the elements developed due to the straining.

According to the energy conservation theorem, the internal energy (IE) stored in the structure is equal to the work done (W) by the external loads \{P\}.

\[
IE = W \quad \text{or} \quad \frac{1}{2} \{F\}^T\{\beta\} = \frac{1}{2} \{P\}^T\{\delta\} 
\]

... (3.43)
Substituting the value of \{P\} from Eq. (3.40) in Eq. (3.43), one can eliminate \{F\}^T from the above equations. Thus, the equation becomes

\[
\{\beta\} = \{B\}^T\{\delta\} \quad \ldots \quad (3.44)
\]

The above deformation displacement relation represents \(n\) deformations expressed in terms of \(m\) displacements, which leads to \((n - m)\) constraints on the deformations of the elements. The constraint on deformations are called compatibility conditions and are expressed through a compatibility matrix \([C]\) and a generalized internal deformation vector \{\beta\} which can be written as

\[
[C]\{\beta\} = 0 \quad \ldots \quad (3.45)
\]

Now as per IFM these CCs are required to be augmented along with the system equilibrium equations that are already in terms of primal variables \{F\} i.e. internal forces. Therefore, it is required to express this compatibility matrix in terms of primal variables \{F\}. Noting that \{\beta\} = \{G\}\{F\}, one can write

\[
[C]\{\beta\} = [C][G]\{F\} \quad \ldots \quad (3.46)
\]

where \([G]\) is concatenated flexibility matrix.

Combining Eqs. (3.40) and (3.46), the coupled equations of IFM can be written as

\[
\begin{bmatrix} [B] \\ [C][G] \end{bmatrix} \{F\} = \begin{bmatrix} \{P\} \\ \{\partial R\} \end{bmatrix} \quad \text{or} \quad [S]\{F\} = \{P\} \quad \ldots \quad (3.47)
\]

Equation (3.47) is known as the basic equation of IFM. In which, \([S]\) is the global equilibrium equation matrix of size \((n \times n)\) which consists of two components The upper part \([S]\) is known as \([B]\) matrix, which is a sparse matrix of size \((m \times n)\) and is developed through basic equilibrium equations. Bottom part \([C][G]\) is known as the compatibility matrix which is of size \((r \times n)\), where \(r = n - m\). \{F\} is unknown vector of internal force of size \((n \times 1)\), which depends upon the type of problem. \{P\} is the vector of external loading of size \((n \times 1)\), in which \{\partial R\} is the vector related to the secondary effect to be
concatenated with respect to nodal displacement in \( \{P\} \) matrix. The solution of Eq. (3.47) can be obtained by inverting the \([S]\) matrix. However, before inverting normalization of major elements with respect to upper components has to be carried out. This is required, because \([C][G]\) always gives components that are of much less value compared to the components of \([B]\) matrix.

The nodal displacements \( \{\delta\} \) can be worked out using the following relation:

\[
\{\delta\} = [J][G]\{F\} \quad \ldots (3.48)
\]

where \([J]\) = m rows from top of \([S^{-1}]^T\) of size \((m \times n)\), \([G]\) = flexibility coefficients calculated from Eqs. (3.24) or (3.25) and \(\{F\}\) is the vector calculated from using Eq. (3.47).

### 3.8 IFM FOR CONTINUUM STRUCTURES

In reality, a physical system has three dimensional domain. Practical situation, however, may have geometry and loading condition such that a three dimensional problem may be idealized as one or two dimensional problem. Two dimensional simplifications implies that one may disregard one of the coordinate axes in these problems, for instance z axis and consider that the whole phenomena takes place in xy plane. Four common situation of 2D simplification are; (i) Plane strain problems, (ii) Plane stress problems (iii) Plate bending problems and (iv) Axisymmetric problems.

Problems involving long body whose geometry and loading do not vary significantly in the longitudinal direction are referred to as plane strain problems (Fig. 3.10). Examples of this type are long strip footing, retaining wall, dam and long underground tunnel. In a plane strain problem, the strain normal to the plane and loading is assumed to be zero. Thus, only in-plane strains \(\varepsilon_x, \varepsilon_y\) and \(\gamma_{xy}\) are nonzero whereas \(\varepsilon_z, \nu_{yz}\) and \(\gamma_{zx}\) are zero. In contrast to the plane strain condition, in which longitudinal dimension in
the z-direction is large compared to the z and y dimensions, the plane stress condition is characterized by very small dimension in the z-direction.

A thin plate loaded in its own plane is the well known example of plane stress approximation. In a plane stress situation, in-plane stresses i.e. \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) are non-zero and other stresses are zero. In plate bending problem, however, a thin plate is subjected to lateral load instead of inplane loading and in such cases moment-curvature relationship is considered for the analysis for the analysis of plate instead of stress-strain relationship. When the geometry, boundary condition, material properties and loading are identical with respect to axis of symmetry, the three dimensional problem can be reduced to an analogous two dimensional problem which is characterized as an axi-symmetric problem (Fig. 3.10). Typical examples, where the axisymmetric analysis may be sufficient are pressure vessels, water tanks etc.

![Fig. 3.10 Different Types of 2D Problems](image)
The basic equations of IFM for solving 2D continuum type of problems remain the same. Here also primarily internal forces \( \{F\} \) are worked out first and then utilizing the secondary equations the nodal displacements are calculated. The complete formulation consists of derivation of different types of element properties based on the discretization of different strain energies.

### 3.9 FORMULATION FOR PLANE STRESS/STRAIN PROBLEMS

The continuum problem is discretized into finite number of elements with ‘n’ and ‘m’ force and displacement degrees of freedom respectively. The governing equations are obtained by coupling the m equilibrium equations and \( r = n - m \) compatibility conditions. The equilibrium equations are expressed as

\[
[B]\{F\} = \{P\}
\]

with ‘r’ compatibility conditions as

\[
([C][G]\{F\} = \{\delta R\})
\]

The basic governing IFM equation for analysis is expressed as

\[
\begin{bmatrix}
[B] \\
[C] \\
[G]
\end{bmatrix}
\{F\} = \begin{bmatrix}
\{P\} \\
\{\partial R\}
\end{bmatrix} \quad \text{or} \quad [S] \{F\} = \{P\}
\]

From forces \( \{F\} \), the nodal displacements can be calculated using the following formula;

\[
\{\delta\} = [J] \{[G] \{F\} + \{\beta^0\}\}
\]

where \([J] = m \) top rows of matrix \([S]^{-1}\)\( ^T \).

The complete formulation requires the following 3 matrices:

1. The equilibrium matrix \([B]\), which works as a link between internal forces and external loads.
2. The compatibility matrix \([C]\), which governs the deformations by \([C]\{\beta\}\).
3. The global flexibility matrix \([G]\), which relates deformations to forces.
3.9.1 Formulation of Equilibrium Matrix \([B]\)

The ‘EE’ written in terms of forces at grid points of finite-element model represents the vectorial summation of ‘n’ internal forces \(\{F\}\) and ‘m’ external loads \(\{P\}\). The nodal EE in matrix notation gives a banded rectangular matrix \([B]\) of size \(m \times n\). The variational functional is evaluated as a portion of IFM functional which yields the basic elemental equilibrium matrix \([B^e]\) in explicit form.

\[
U^e = \int_S \left\{ N_x \left( \frac{\partial u}{\partial x} \right) + N_y \left( \frac{\partial v}{\partial y} \right) + N_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right\} dx dy = \int [N]^T \{\varepsilon\} ds \quad \text{... (3.50)}
\]

Where \(N_x, N_y, N_{xy}\) are the in-plane internal forces and \(\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}\) and \(\varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\) represents the respective strains.

Consider four-noded, 8 d.dof rectangular in-plane element of thickness \(t\) with dimensions \(2a \times 2b\) along \(x\) - and \(y\) - axes as shown in Fig. 3.11.

Fig. 3.11 Rectangular Element with Nodal Displacements

For the rectangular membrane element, the force field is chosen in terms of five independent forces as;

\[
[F]^T = \{ F_1 \ F_2 \ F_3 \ F_4 \ F_5 \}^T \quad \text{... (3.51)}
\]

Here the distribution of internal forces in terms of unknowns is considered as follows:
\[ N_x = F_1 + F_2 \frac{y}{b}, \quad N_y = F_3 + F_4 \frac{x}{a} \text{ and } N_{xy} = F_5 \quad \ldots (3.52) \]

Although the variation of normal forces is linear, the shear is constant. The displacement field should satisfy the continuity condition and the selected forces should satisfy the mandatory requirement. Displacement interpolation functions for generalized element are as follows:

\[
u(x, y) = \frac{1}{4} \left\{ \left( 1 - \frac{x}{a} \right) \left( 1 - \frac{y}{b} \right) u_1 + \left( 1 + \frac{x}{a} \right) \left( 1 - \frac{y}{b} \right) u_2 \right\} \quad \ldots (3.53)\]

\[
u(x, y) = \frac{1}{4} \left\{ \left( 1 - \frac{x}{a} \right) \left( 1 - \frac{y}{b} \right) v_1 + \left( 1 + \frac{x}{a} \right) \left( 1 - \frac{y}{b} \right) v_2 \right\} \quad \ldots (3.54)\]

Where \( u_1, v_1 \ldots u_4, v_4 \) are eight nodal displacements as shown in Fig. 3.11.

Substituting Eqs. (3.53) and (3.54) in the Eq. (3.50) and rearranging all force and displacement functions properly, one can obtain the elemental equilibrium matrix as follows

\[ U^e = [\delta]^T [B^e] [F] \]

where \([B^e] = \int_s [Z]^T [Y] \, ds \quad \ldots (3.55)\]

Here \([Z] = [L][N], [L] \) is the differential operator matrix, \([N] \) is the displacement interpolation function matrix and \([Y] \) is the force interpolation function matrix. Substituting and integrating yields the following non-symmetrical equilibrium matrix \([B^e] \) for the element, which represents the displacements of \( m \) rows from \( u_1 \) to \( v_4 \) in increasing order from top to bottom and \( n \) columns represents forces from \( F_1 \) to \( F_5 \) in increasing order from left to right.
\[ [B^e] = \begin{bmatrix}
-b & b/3 & 0 & 0 & -a \\
0 & 0 & -a/3 & a/3 & -b \\
b/3 & 0 & 0 & -a \\
0 & 0 & -a/3 & a/3 & b \\
b/3 & 0 & 0 & a \\
0 & 0 & a/3 & b \\
-b & -b/3 & 0 & 0 & a \\
0 & 0 & a/3 & -a/3 & -b
\end{bmatrix} \quad \ldots (3.56) \]

### 3.9.2 Element Flexibility Matrix \([G^e]\)

The basic elemental flexibility matrix is obtained by discretizing the complementary strain energy which gives

\[ [G^e] = \int_{s} [Y]^T [D][Y] \, dx \, dy \quad \ldots (3.57) \]

where \([Y]\) is the force interpolation function matrix, which is developed from Eq.(3.40) and \([D]\) is material property matrix. Substituting values in Eq. (3.44) and integrating within the domain \((2a \times 2b)\) with the origin at the center of the element yields the symmetrical flexibility matrix \([G^e]\) as follows;

\[ [G^e] = \frac{4ab}{Et} \begin{bmatrix}
1 & 0 & -\nu & 0 & 0 \\
0 & 1/3 & 0 & 0 & 0 \\
-\nu & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 0 & 2(1 + \nu)
\end{bmatrix} \quad \ldots (3.58) \]

### 3.9.3 Global Compatibility Matrix

The compatibility matrix is obtained from the deformation displacement relation \((\beta = [B]^T[X])\). In DDR all the deformations are expressed in terms of all the possible nodal displacements and the \(r\) compatibility conditions are
developed in terms of internal forces $F_1, \ldots, F_{2n}$, where ‘2n’ indicates the total number of internal forces in a given problem. So, global compatibility matrix $[C]$ can be evaluated by multiplying the global coefficients of $\{\beta\}$ of complete matrix $(r \times n)$ under consideration by global flexibility matrix, which is developed by putting all elemental flexibility matrix at diagonal position as per the numbering pattern of each element. One can check the null property of the matrix as per Eq. (3.30) for its mathematical validity.

### 3.10 FORMULATION FOR PLATE BENDING PROBLEMS

The procedure discussed above remains the same except the change in formulation of element matrices required for the solution of bending problem.

#### 3.10.1 Formulation of Equilibrium Matrix $[B^e]$

The ‘EE’ written in terms of forces at grid points of discrete model represents the vectorial summation of ‘n’ internal forces $\{F\}$ and ‘m’ external loads $\{P\}$. The nodal EE gives banded rectangular matrix $[B]$ of size $m \times n$. The variational functional is evaluated as a portion of IFM functional which yields the basic elemental equilibrium matrix $[B^e]$ in explicit form.

$$
U^e = \int \left\{ M_x \left( \frac{\partial^2 w}{\partial x^2} \right) + M_y \left( \frac{\partial^2 w}{\partial y^2} \right) + 2M_{xy} \left( \frac{\partial^2 w}{\partial x \partial y} \right) \right\} \, dx \, dy
= \int [M]^T \{\epsilon\} \, dx \, dy \tag{3.59}
$$

Where $M_x$, $M_y$ and $M_{xy}$ are the internal moments and $\{\epsilon\}^T = \left( \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2} \text{ and } 2 \frac{\partial^2 w}{\partial x \partial y} \right)$ represent the corresponding curvature terms.

Consider a four-noded, 12 ddof ($w_1$ to $\theta_{y4}$ with three degrees of freedom at each node) rectangular element of thickness $t$ with dimensions as $2a \times 2b$ along the $x$ and $y$ axes. The force field is chosen in terms of nine independent forces as;
\{F\} = [F_1, F_2 \ldots \ldots F_9]^T \quad \text{...(3.60)}

Relations between internal moments and independent forces are written as

\[ M_x = F_1 + F_2x + F_3y + F_4xy \]

\[ M_y = F_5 + F_6x + F_7y + F_8xy \]

\[ M_{xy} = F_9 \]

Arranging above equations in matrix form,

\[
\begin{pmatrix}
M_x \\
M_y \\
M_{xy}
\end{pmatrix} =
\begin{bmatrix}
1 & x & y & xy & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x & y & xy & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
F_e
\end{bmatrix}
\quad \text{...(3.61)}
\]

or \[ \{M\} = \{y\} \{F_e\} \], where \( \{F_e\} = [F_1, F_2, F_3\ldots\ldots\ldots\ldots F_9]^T \)

The variation of above forces is considered bilinear along both the directions. The displacement fields satisfy the continuity condition and the selected forces satisfy the mandatory requirement.

The Hermitian Interpolation function for the lateral displacement for rectangular plate bending element is as follows:

\[ w(x,y) = N_1(x,y)w_1 + N_1(x',y)\theta_{x1} + N_1(x,y')\theta_{y1} \ldots \ldots\ldots\ldots N_4(x,y')\theta_{y4} \quad \text{...(3.62)} \]

Where, \( N_1(x,y) = N_1(x)N_1(y) \), \( N_1(x',y) = N_1(x)N_1(y) \) and \( N_1(x,y') = N_1(x)N_1(y) \) and so on. Also,

\[ N_2(x) = \frac{y^3 + 3b^2y + 2b^3}{4b^3} \]

\[ N_2(y) = \frac{y^3 - 3a^2x + 2a^3}{4a^3} \]

\[ N_1'(x) = \frac{x^3 - ax^2 - a^2x + a^3}{4a^2} \]

\[ N_1'(y) = \frac{y^3 - by^2 - b^2y + b^3}{4b^2} \]

etc are associated with the nodal displacements \( w_1, \theta_{x1} \ldots \ldots \ldots \theta_{y4} \) as shown in Fig. 3.11
By arranging all force and displacement functions properly, one can discretize the Eq. (3.46) to obtain the elemental equilibrium matrix as follows.

$$U_e = \{\delta\}^T[B_e][F]$$

Where $$[B_e] = \int_s [Z]^T[Y] \, ds$$ and $$[Z] = [L][N]$$

where $$[L]$$ is the differential operator matrix, $$[N]$$ is the displacement interpolation function matrix and $$[Y]$$ is the matrix of force interpolation function. Substituting and integrating yields the following equilibrium matrix $$[B_e]$$ of size 12 x 9. Here the row from top to bottom represents the displacement components ($$X_1$$ to $$X_{12}$$), while column from left to right represents the independent unknowns ($$F_1$$ to $$F_9$$).
3.10.2 Formulation of Element Flexibility Matrix \([G_e]\)

The element flexibility matrix is obtained by discretizing the complementary strain energy; which gives

\[
[G_e] = \int_{s} [Y]^T[D][Y] \, dx\,dy
\]

where \([Y]\) is the force interpolation function matrix and \([D]\) is the material property matrix. Integrating within the domain \((2a \times 2b)\), with origin at center of the element, yields the symmetrical flexibility matrix \([G_e]\) as follows;

\[
[B_e] =
\begin{bmatrix}
0 & b & 0 & -\frac{2b^2}{5} & 0 & 0 & a & -\frac{2a^2}{5} & -2 \\
0 & \frac{b^2}{3} & 0 & -\frac{b^3}{15} & -a & \frac{2a^2}{5} & ab & -\frac{2a^2b}{5} & 0 \\
b & -ab & \frac{-2b^2}{5} & -\frac{2a^2b}{5} & 0 & 0 & -\frac{a^2}{3} & \frac{a^3}{15} & 0 \\
0 & b & 0 & -\frac{2b^2}{5} & 0 & 0 & -a & -\frac{2a^2}{5} & 0 \\
0 & -\frac{b^2}{3} & 0 & -\frac{b^3}{15} & a & -\frac{2a^2}{5} & ab & -\frac{2a^2b}{5} & 0 \\
-b & -ab & \frac{-2b^2}{5} & -\frac{2a^2b}{5} & 0 & 0 & a^2 & \frac{a^3}{3} & 0 \\
0 & -b & 0 & -\frac{2b^2}{5} & 0 & 0 & -a & -\frac{2a^2}{5} & -2 \\
0 & \frac{b^2}{3} & 0 & \frac{b^3}{5} & a & \frac{2a^2}{5} & ab & -\frac{2a^2b}{5} & 0 \\
b & -ab & \frac{-2b^2}{5} & -\frac{2a^2b}{5} & 0 & 0 & -\frac{a^2}{3} & -\frac{a^3}{15} & 0 \\
0 & -b & 0 & -\frac{2b^2}{5} & 0 & 0 & 0 & \frac{2a^2}{5} & 2 \\
0 & -\frac{b^2}{3} & 0 & -\frac{b^3}{15} & -a & -\frac{2a^2}{5} & ab & -\frac{2a^2b}{5} & 0 \\
0 & -ab & \frac{-2b^2}{5} & -\frac{2a^2b}{5} & 0 & 0 & \frac{a^2}{3} & \frac{a^3}{15} & 0 
\end{bmatrix} \quad \text{... (3.63)}
\]
\[ [C^c] = \begin{bmatrix}
1 & 0 & 0 & 0 & -\nu & 0 & 0 & 0 & 0 \\
0 & \frac{a^2}{3} & 0 & 0 & 0 & -\nu \frac{a^2}{3} & 0 & 0 & 0 \\
0 & 0 & \frac{b^2}{3} & 0 & 0 & 0 & -\nu \frac{b^2}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{a^2 b^2}{9} & 0 & 0 & 0 & -\nu \frac{a^2 b^2}{9} & 0 \\
-\nu & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -\nu \frac{a^2}{3} & 0 & 0 & 0 & \frac{a^2}{3} & 0 & 0 & 0 \\
0 & 0 & -\nu \frac{b^2}{3} & 0 & 0 & 0 & \frac{b^2}{3} & 0 & 0 \\
0 & 0 & 0 & -\nu \frac{a^2 b^2}{9} & 0 & 0 & 0 & \frac{a^2 b^2}{9} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2(1 + \nu) & 0
\end{bmatrix} \]

... (3.64)

3.10.3 Global Compatibility Matrix

The compatibility matrix is obtained from the deformation displacement relation (\(\{\beta\} = [B]^T\{\delta}\)). In DDR the deformations are expressed in terms of all the possible nodal displacements and the ‘r’ compatibility conditions are developed in terms of internal forces i.e., \(F_1, \ldots, F_{2n}\), where ‘2n’ are the total number of internal forces in a given problem. The global compatibility matrix \([C]\) can be evaluated by multiplying the global coefficients of \(\{\beta\}\) of complete matrix \((r \times n)\) by the global flexibility matrix; which is generated by putting all the elemental flexibility matrices at diagonal position as per the numbering of each element. Before using the global compatibility matrix for further calculations, one has to check the null property of the matrix to ensure the mathematical validity of the developed matrices.