MOMENTS OF GENERALIZED ORDER STATISTICS FROM ERLANG-TRUNCATED EXPONENTIAL DISTRIBUTION AND CHARACTERIZATION

1. Introduction

The concept of generalized order statistics as discussed in Chapter I describes random variables arranged in ascending order of magnitude and includes well-known concepts of ordered statistics that have been treated separately in statistical literature. The use of such an unifying concept has been steadily increasing over the year. Several authors utilized the generalized order statistics in their work, such as Kamps (1995), Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Kamps and Cramer (2001), Pawlas and Szynal (2001a), Ahmed and Fawzy (2003), Khan and Alzaid (2004), Khan et al. (2006), Ahmed (2007), Khan et al. (2007) among others. In this chapter explicit expressions and some recurrence relations for single and product moments of generalized order statistics from Erlang-truncated exponential distribution are studied. Further the results are deduced for moments of ordinary order statistics and record values and a characterization of this distribution has been considered on using the conditional moment of the generalized order statistics.

A random variable $X$ is said to have Erlang-truncated exponential distribution if its pdf is of the form

$$f(x) = \beta(1-e^{-\lambda})e^{-\beta x(1-e^{-\lambda})}, \quad x \geq 0, \quad \beta, \lambda > 0$$

and the corresponding df is

The part of the results of this chapter appeared in Khan et al. (2010b)
\[
\bar{F}(x) = e^{-\beta x(1-e^{-\lambda})}, \quad x \geq 0, \quad \beta, \lambda > 0,
\]  
(2.1.2)

where

\[
\bar{F}(x) = 1 - F(x).
\]

We note that \( f(x) \) and \( F(x) \) satisfy the relation

\[
f(x) = \beta(1 - e^{-\lambda})\bar{F}(x).
\]

(2.1.3)

The relation in (2.1.3) will be used to derive some recurrence relations for the moments of generalized order statistics from the Erlang-truncated exponential distribution.

2. Relations for single moments

As given in chapter I, the pdf of the \( r \)-th gos, \( X(r,n,m,k) \), \( 1 \leq r \leq n \), is

\[
f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma r-1} f(x) g_m^{r-1}(F(x)).
\]

(2.2.1)

First we prove some results, which may be needed subsequently.

**Lemma 2.1:** For the Erlang-truncated exponential distribution as given in (2.1.1) and any non-negative finite integers \( a \) and \( b \) with \( m \neq -1 \)

\[
I_j(a,0) = \frac{\Gamma(j+1)}{[\beta(1-e^{-\lambda})]^j} a^{j+1},
\]

(2.2.2)

where

\[
I_j(a,b) = \int_0^\infty x^j [\bar{F}(x)]^a g_m^b(F(x))dx.
\]

(2.2.3)

**Proof:** From (2.2.3), we have

\[
I_j(a,0) = \int_0^\infty x^j [\bar{F}(x)]^a dx.
\]

(2.2.4)

On substituting for \( \bar{F}(x) \) from (2.1.2) in equation (2.2.4), we establish the result given in (2.2.2).

**Lemma 2.2:** For the Erlang-truncated exponential distribution as given in (2.1.1) and any non-negative finite integers \( a \) and \( b \)
\[ I_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} I_j(a+u(m+1),0) \]  
\[ = \frac{\Gamma(j+1)}{[\beta(1-e^{-\lambda})]^{j+1}(m+1)^b} \sum_{u=0}^{b} (-1)^u \]  
\[ \times \binom{b}{u} \frac{1}{[a+u(m+1)]^{j+1}}, \quad m \neq -1 \]  
\[ = \frac{\Gamma(j+b+1)}{[\beta(1-e^{-\lambda})]^{j+1}a^{j+b+1}}, \quad m = -1, \]  
(2.2.5)  
(2.2.6)  
(2.2.7)

where \( I_j(a,b) \) is as given in (2.2.3).

**Proof:** On expanding \( g_m^b(F(x)) = \left[ \frac{1}{m+1} \{1 - (\bar{F}(x))^{m+1}\} \right]^b \) binomially in (2.2.3), we get when \( m \neq -1 \)

\[ I_j(a,b) = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \int_0^\infty x^j [\bar{F}(x)]^a [a+u(m+1)] \]  
\[ = \frac{1}{(m+1)^b} \sum_{u=0}^{b} (-1)^u \binom{b}{u} I_j(a+u(m+1),0). \]  

Making use of Lemma 2.1, we derive the result given in (2.2.6)

and when \( m = -1 \) that

\[ I_j(a,b) = 0, \quad \text{as} \quad \sum_{u=0}^{b} (-1)^u \binom{b}{u} = 0. \]

Since (2.2.6) is of the form \( \frac{0}{0} \) at \( m = -1 \), therefore we have

\[ I_j(a,b) = \frac{\Gamma(j+1)}{[\beta(1-e^{-\lambda})]^{j+1}} \sum_{u=0}^{b} (-1)^u \binom{b}{u} \frac{[a+u(m+1)]^{-j-1}}{(m+1)^b}. \]  
(2.2.8)

Differentiating numerator and denominator of (2.2.8) \( b \) times with respect to \( m \), we get
\[ I_j(a,b) = \frac{\Gamma(j+1)}{[\beta(1-e^{-\lambda})]^{j+1}} \sum_{u=0}^{b} (-1)^{u+b} \binom{b}{u} \frac{(j+1)(j+2)\cdots(j+b)u^b}{b! [a+u(m+1)]^{j+b+1}}, \quad b > 0. \]

On applying L’Hospital rule, we have

\[
\lim_{m \to -1} I_j(a,b) = \frac{\Gamma(j+b+1)}{b! [\beta(1-e^{-\lambda})]^{j+1}} \sum_{u=0}^{b} (-1)^{u+b} \binom{b}{u} u^b. \tag{2.2.9}
\]

But for all integers \( n \geq 0 \) and for all real numbers \( x \), we have Ruiz (1996)

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n = n!. \tag{2.2.10}
\]

Therefore,

\[
\sum_{u=0}^{b} (-1)^{u+b} \binom{b}{u} u^b = b!. \tag{2.2.11}
\]

Now on substituting (2.2.11) in (2.2.9), we have the result given in (2.2.7).

**Theorem 2.1:** For the Erlang-truncated exponential distribution as given in (2.1.1) and \( 1 \leq r \leq n, \ k = 1,2,\ldots, \ m \neq -1 \)

\[
E[X^j(r,n,m,k)] = \frac{[\beta(1-e^{-\lambda})] C_{r-1}}{(r-1)!} I_j(\gamma_r,r-1)
\]

\[
= \frac{C_{r-1}}{(r-1)! (m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}
\]

\[
\times \frac{\Gamma(j+1)}{[\beta(1-e^{-\lambda})]^{j} (\gamma_{r-u})^{j+1}}, \tag{2.2.13}
\]

where \( I_j(\gamma_r,r-1) \) is as defined in (2.2.3).
Proof: From (2.2.1), we have

\[ E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))dx \]

\[ = \frac{[\beta(1-e^{-\lambda})]C_{r-1}}{(r-1)!} I_j(\gamma_r, r-1) \]

upon using the relation in (2.1.3). Now making use of Lemma 2.2, we establish the result given in (2.2.13).

Special cases
i) Putting \( m=0, k=1 \) in (2.2.13), the explicit moments of the \( r \)-th order statistic of the Erlang-truncated exponential distribution can be obtained as

\[ E[X^j_{r:n}] = \frac{\Gamma(j+1)C_{r:n}}{[\beta(1-e^{-\lambda})]^j} \sum_{u=0}^{r-1} (-1)^u \frac{1}{u}(n-r+u+1)^j, \]

where

\[ C_{r:n} = \frac{n!}{(r-1)!(n-r)!} \]

which is the result obtained by Lieblein (1955) for exponential distribution at \( \beta(1-e^{-\lambda}) = c \).

ii) Putting \( m=-1 \) in (2.2.13), the explicit expression for the single moments of upper \( k \) record values for the Erlang-truncated exponential distribution may be obtained in view of (2.2.12) and (2.2.7) in the form

\[ E[X^j(r,n,-1,k)] = E[(Y_r^{(k)})^j] \] (as defined in Chapter I)

\[ = \frac{\Gamma(j+r)}{(r-1)!}[k\beta(1-e^{-\lambda})]^j \]

and hence

\[ E[Y_r^{(k)}] = \frac{\Gamma(r+1)}{(r-1)! k\beta(1-e^{-\lambda})} \]
as obtained by Grudzień and Syznal (1983) for exponential distribution at 
\( \beta(1 - e^{-\lambda}) = c \)
and for \( k = 1 \) (record values)
\[
E[(Y_r^{(1)})^j] = E[X^j_{U(r)}] \quad \text{(as defined in Chapter I)}
\]
\[= \frac{\Gamma(j + r)}{(r - 1)!\beta(1 - e^{-\lambda})^j} \]
as obtained by Ahsanullah (1987) for exponential distribution at \( j = 1, \beta(1 - e^{-\lambda}) = c. \)

Recurrence relations for single moments of \( gos \) from \( df \) (2.1.2) is obtained in the following theorem.

**Theorem 2.2:** For the distribution as given in (2.1.2) and \( n \in N, m \in \mathbb{R}, r = 1 \)
\[
E[X^{j+1}(1,n,m,k)] = \frac{j + 1}{\beta(1 - e^{-\lambda})} E[X^j(1,n,m,k)]
\]
(2.2.14)
and for \( 2 \leq r \leq n \)
\[
E[X^{j+1}(r,n,m,k)] = \frac{j + 1}{\beta(1 - e^{-\lambda})} E[X^j(r,n,m,k)]
\]
\[+ E[X^{j+1}(r-1,n,m,k)]. \quad (2.2.15)\]

**Proof:** From (2.2.1) and (2.1.3), we have
\[
E[X^j(r,n,m,k)] = \frac{\beta(1 - e^{-\lambda})C_{r-1}I_r(x)}{(r - 1)!}
\]
(2.2.16)
where
\[
I_t(x) = \int_0^\infty x^j [F(x)]^{\gamma_t} g_{t-1}^{-1}(F(x)) \, dx.
\]
Integrating by parts treating \( x^j \) for integration and the rest of the integrand for differentiation, we get
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\[I_t(x) = \frac{\gamma_t}{j+1} \int_0^\infty x^j+1 [F(x)]^{\gamma_t-1} f(x) g_m^{r-1}(F(x)) \, dx\]
\[- \frac{t-1}{j+1} \int_0^\infty x^j+1 [F(x)]^{\gamma_t+m} f(x) g_m^{t-2}(F(x)) \, dx.\]

Now substituting for \(I_1(x)\) and \(I_r(x)\) in equation (2.2.16), we establish the relations given in (2.2.14) and (2.2.15).

**Remark 2.1:** Putting \(j = 0, \beta(1 - e^{-\lambda}) = c\) in (2.2.15), the result for single moments of \(gos\), obtained by Kamps (1995) for exponential distribution is deduced.

**Remark 2.2:** Setting \(m = 0, k = 1\) in (2.2.15), we obtain recurrence relations for single moments of order statistics of the Erlang-truncated exponential distribution in the form

\[E[X_{r:n}^{j+1}] = \frac{j+1}{(n-r+1)\beta(1 - e^{-\lambda})} E[X_{r:n}^{j}] + E[X_{r-1:n}^{j+1}].\]

**Remark 2.3:** Setting \(m = -1, k \geq 1\) in (2.2.15), we get the recurrence relations for single moments of upper \(k\) records of the Erlang-truncated exponential distribution in the form

\[E[(Y_r^{(k)})^{j+1}] = \frac{j+1}{k\beta(1 - e^{-\lambda})} E[(Y_r^{(k)})^{j}] + E[(Y_{r-1}^{(k)})^{j+1}]\]

as obtained by Kamps (1995) for exponential distribution at \(j+1 = j, \beta(1 - e^{-\lambda}) = c\).

3. **Relations for product moments**

The joint pdf of \(X(r,n,m,k)\) and \(X(s,n,m,k)\), \(1 \leq r < s \leq n\) is given as

\[f_{X(r,n,m,k), X(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(x)]^m g_m^{r-1}(F(x))\]
\[\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_{s-1}} f(x) f(y). \quad (2.3.1)\]
Before coming to the main results we shall prove the following Lemmas.

**Lemma 3.1:** For the Erlang-truncated exponential distribution as given in (2.1.1) and non-negative integers $a, b, c$ with $m \neq -1$

$$I_{i,j}(a,0,c) = \sum_{w=0}^{j} \frac{\Gamma(j+1)\Gamma(i+w+1)}{w!\beta(1-e^{-\lambda})^{i+j-2}c^{j-w+1}[a+c]^{i+w+1}},$$

(2.3.2)

where

$$I_{i,j}(a,b,c) = \int_{0}^{\infty} \int_{x}^{\infty} x^iy^j[F(x)]^a [h_m(F(y)) - h_m(F(x))]^b [F(y)]^c dydx.$$

(2.3.3)

**Proof:** From (2.3.3), we have

$$I_{i,j}(a,0,c) = \int_{0}^{\infty} x^i[F(x)]^a G(x)dx,$$

(2.3.4)

where

$$G(x) = \int_{x}^{\infty} y^j[F(y)]^c dy.$$

(2.3.5)

On substituting for $F(y)$ from (2.1.2) in equation (2.3.5), we get

$$G(x) = \int_{x}^{\infty} y^j e^{-\beta y(1-e^{-\lambda})}c dy$$

$$= \sum_{w=0}^{j} x^w e^{-\beta x(1-e^{-\lambda})c} \Gamma(j+1)\frac{\Gamma(j+1)}{w!\beta(1-e^{-\lambda})c}^{j-w+1}$$

(Gradshteyn and Ryzhik, 2007, p-346).

Upon substituting this expression for $G(x)$ in (2.3.4) and then integrating the resulting expression, we establish the result given in (2.3.2).

**Lemma 3.2:** For the conditions as stated in Lemma 3.1

$$I_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v}$$

$$\times I_{i,j}(a+(b-v)(m+1),0, c + v(m+1)).$$

(2.3.6)
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\[\sum_{w=0}^{b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} \frac{\Gamma(j+1)}{w!} \beta(1-e^{-\lambda})^{i+j+2} \]

\[\times \frac{\Gamma(i+w+1)}{[c+v(m+1)]^{j-w+1}[a+c+b(m+1)]^{i+w+1}},\]

(2.3.7)

where \(I_{i,j}(a,b,c)\) is as given in (2.3.3).

**Proof:** Expanding \([h_m(F(y))-h_m(F(x))]^b\) binomially in (2.3.3) after noting that \(h_m(F(y))-h_m(F(x)) = g_m(F(y)) - g_m(F(x))\), we get

\[I_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} \]

\[\times \int_0^\infty \int_0^\infty x^i y^j [\overline{F}(x)]^{a+(b-v)(m+1)} [\overline{F}(y)]^{c+v(m+1)} dydx.\]

\[= \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} I_{i,j}(a+(b-v)(m+1),0,c+v(m+1)).\]

Making use of Lemma 3.1, we establish the result given in (2.3.7).

**Theorem 3.1:** For Erlang-truncated exponential distribution as given in (2.1.1) and for \(1 \leq r < s \leq n\), \(k = 1,2,\ldots\), \(m \neq -1\)

\[E[X^i (r,n,m,k) X^j (s,n,m,k)]\]

\[= \frac{[\beta(1-e^{-\lambda})]^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \]

\[\times I_{i,j}((u+1)(m+1), s-r-1, \gamma_s)\]

(2.3.8)

\[= \frac{\Gamma(j+1)C_{s-1}}{(r-1)!(s-r-1)!\beta(1-e^{-\lambda})^{i+j+2} (m+1)^{s-2}} \sum_{w=0}^{j} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} \]

\[\times (-1)^u+v \binom{r-1}{u} \binom{s-r-1}{v} \frac{\Gamma(i+w+1)}{w!(\gamma_{s-v})^{j-w+1}(\gamma_{r-u})^{i+w+1}}\]

(2.3.9)

and subsequently for \(s = r + 1\)
\[ E[X^i(r,n,m,k)X^j(r+1,n,m,k)] = \frac{\Gamma(j+1)C_r}{(r-1)!(\beta(1-e^{-\lambda}))^{i+j}(m+1)^{r-1}} \times \sum_{u=0}^{j} \sum_{r=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{\Gamma(i+w+1)}{w!(\gamma_{r+1})^{j-w+1}(\gamma_{r+1})^{i+w+1}}. \]  

**Proof:** From (2.3.1), we have

\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{[\beta(1-e^{-\lambda})]^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \int_0^{\infty} \int_{x}^{y} x^i y^j [F(x)]^{m+1} g_m^{-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{ys} dydx. \]  

(2.3.11)

Upon using the relation (2.1.3). Now expanding \( g_m^{-1}(F(x)) \) binomially in (2.3.11), we get

\[ E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{[\beta(1-e^{-\lambda})]^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \times \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} I_{i,j}((u+1)(m+1), s-r-1, \gamma_s). \]

Making use of Lemma 3.2, we establish the relation given in (2.3.9).

**Special cases**

i) Putting \( m = 0, k = 1 \) in (2.3.9), the explicit formula for the product moments of order statistics of the Erlang-truncated exponential distribution can be obtained as

\[ E[X^i_{r:n} X^j_{s:n}] = \frac{\Gamma(j+1)C_{r,s:n}}{[\beta(1-e^{-\lambda})]^{i+j}} \sum_{w=0}^{j} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u} \binom{s-r-1}{v} \frac{\Gamma(i+w+1)}{w!(n-s+v+1)^{j-w+1}(n-r+u+1)^{i+w+1}}, \]

where

\[ C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}. \]
ii) Putting $m = -1$ in (2.3.10), we deduce the explicit expression for the product moments of upper $k$ record values of the Erlang-truncated exponential distribution as

$$E [(Y_{r}^{(k)})^i (Y_{r+1}^{(k)})^j] = \frac{\Gamma(j+1)}{(r-1)!} [k \beta (1-e^{-\lambda})]^i j \sum_{w=0}^{j} \frac{\Gamma(i+w+r)}{w!}$$

and for $k = 1$ (record values)

$$E [X_{U(r)}^i X_{U(r+1)}^j] = \frac{\Gamma(j+1)}{(r-1)!} [\beta (1-e^{-\lambda})]^i j \sum_{w=0}^{j} \frac{\Gamma(i+w+r)}{w!}.$$

Making use of (2.1.3), we can derive recurrence relations for product moments of $gos$ from (2.1.2).

**Theorem 3.2:** For the given Erlang-truncated exponential distribution in (2.1.2) and $n \geq 2$, $m \in \mathbb{R}, 1 \leq r < r+1 \leq n$

$$E [X^i (r,n,m,k)X^{j+1} (r+1,n,m,k)]$$

$$= \frac{j+1}{\beta (1-e^{-\lambda}) \gamma_{r+1}} E [X^i (r,n,m,k)X^{j} (r+1,n,m,k)]$$

$$+ E [X^{i+j+1} (r,n,m,k)]$$

(2.3.12)

and for $1 \leq r < s \leq n$, $s - r \geq 2$ and $i, j \geq 0$

$$E [X^i (r,n,m,k)X^{j+1} (s,n,m,k)] = \frac{j+1}{\beta (1-e^{-\lambda}) \gamma_s}$$

$$\times E [X^i (r,n,m,k)X^{j} (s,n,m,k)] + E [X^i (r,n,m,k)X^{j+1} (s-1,n,m,k)].$$

(2.3.13)

**Proof:** From (2.3.1), we have

$$E [X^i (r,n,m,k)X^{j} (s,n,m,k)] = \frac{C_{s-1}}{(r-1)! (s-r-1)!}$$

$$\times \int_{0}^{\infty} x^i [\bar{F}(x)]^m f(x) g_m^{r-1} (F(x)) J(x) dx,$$

(2.3.14)
where
\[
J(x) = \int_x^\infty y^j [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^\gamma_y f(y) \, dy
\]
\[
= \beta (1 - e^{-\lambda}) \int_x^\infty y^j [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^\gamma_y dy
\]
upon using the relation in (2.1.3). Integrating now by parts treating \( y^j \) for integration and the rest of the integrand for differentiation, we obtain when \( s = r + 1 \) that
\[
J(x) = \frac{\beta (1 - e^{-\lambda})}{j + 1} \left\{ -x^{j+1}[\bar{F}(x)]^\gamma_y + \gamma_y [\bar{F}(x)]^\gamma_{y+1} \int_x^\infty y^{j+1}[\bar{F}(y)]^\gamma_y f(y) \, dy \right\}
\]
(2.3.15)
and when \( s > r + 1 \) that
\[
J(x) = \beta (1 - e^{-\lambda}) \gamma_s \int_x^\infty y^{j+1} [h_m(F(y)) - h_m(F(x))]^{s-r-1}
\times [\bar{F}(y)]^\gamma_y f(y) \, dy - \frac{(s-r-1)}{j + 1}
\]
\times \int_x^\infty y^{j+1} [h_m(F(y)) - h_m(F(x))]^{s-r-2} [\bar{F}(y)]^\gamma_y f(y) \, dy \right\}
\]
(2.3.16)

Upon substituting the above expressions for \( J(x) \) in (2.3.14), we have after simplifications the recurrence relations given in (2.3.12) and (2.3.13).

**Remark 3.1:** Setting \( m = 0, \, k = 1 \) in (2.3.12) and (2.3.13), we obtain recurrence relations for product moments of order statistics of the Erlang-truncated exponential distribution in the form
\[
E[X_{r;n}^i X_{r+1;n}^{j+1}] = \frac{j + 1}{\beta (1 - e^{-\lambda}) (n - r)} E[X_{r;n}^i X_{r+1;n}^j] + E[X_{r;n}^i X_{r+1;n}^{j+1}]
\]
and

\[
E[X_{r:n}^i X_{s:n}^{j+1}] = \frac{j + 1}{\beta(1 - e^{-\lambda})(n - s + 1)} E[X_{r:n}^i X_{s:n}^j] + E[X_{r:n}^i X_{s-1:n}^{j+1}].
\]

**Remark 3.2:** Putting \(m = -1, \ k \geq 1\) in (2.3.13), we get the recurrence relations for product moments of upper \(k\) records of the Erlang-truncated exponential distribution in the form

\[
E[(Y_r^{(k)})^i (Y_s^{(k)})^{j+1}] = \frac{j + 1}{\beta(1 - e^{-\lambda})k} E[(Y_r^{(k)})^i (Y_s^{(k)})^{j+1}] + E[(Y_r^{(k)})^i (Y_s^{(k)})^{j+1}].
\]

**Remark 3.3:** At \(j = 0\) in (2.3.9), we have

\[
E[X^i (r,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \times \binom{r-1}{u} \binom{s-r-1}{v} \frac{\Gamma(i+1)}{[\beta(1 - e^{-\lambda})]^i (\gamma_{s-v})(\gamma_{r-u})^{i+1}}.
\]

Making use of the result that (Athar et al., 2010)

\[
\sum_{v=0}^{s-r-1} (-1)^v \binom{s-r-1}{v} \frac{1}{\gamma_{s-v}} = \frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{t=r+1}^{s} \gamma_t}
\]

and simplifying the resulting expression, we obtain the relation given in (2.2.13).

**Remark 3.4:** At \(i = 0\) in (2.3.13), recurrence relations for product moments reduces to relations for single moments as obtained in (2.2.15).

4. **Characterization**

Let \(X(r,n,m,k), \ r = 1,2,\ldots,n\) be gos, then the conditional pdf of \(X(s,n,m,k)\) given \(X(r,n,m,k) = x, \ 1 \leq r < s \leq n\), in view of (2.2.1) and (2.3.1), is
\[
\begin{align*}
&f_{X(s,n,m,k)\mid X(r,n,m,k)}(y \mid x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \\
&\quad \times \left[ (h_m(F(y)) - h_m(F(x))]^{s-r-1} \left[ F(y) \right]^{\gamma_s-1} \right] f(y), \quad x < y. 
\end{align*}
\] (2.4.1)

**Theorem 4.1:** Suppose \( F(x) < 1, \) for all \( x \in (0, \infty) \) be a distribution function of the random variable \( X \) and \( F(0) = 0, \) \( F(\infty) = 1, \) then
\[
\overline{F}(x) = e^{-\beta x(1-e^{-\lambda})}, \quad x \geq 0, \quad \beta, \lambda > 0
\] (2.4.2)
if and only if for \( 1 \leq r < s \leq n \)
\[
E[X(s,n,m,k) \mid X(r,n,m,k) = x] = x + \frac{1}{\beta(1-e^{-\lambda})} \sum_{j=1}^{s-r} \frac{1}{\gamma_{r+j}}. 
\] (2.4.3)

**Proof:** We have from (2.4.1)
\[
\begin{align*}
&\quad \quad E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \\
&\quad \quad \times \int_{x}^{\infty} \left[ 1 - \left( \frac{\overline{F}(y)}{F(x)} \right)^{m+1} \right]^{s-r-1} \left( \frac{\overline{F}(y)}{F(x)} \right)^{\gamma_s-1} \frac{f(y)}{F(x)} dy. 
\end{align*}
\] (2.4.4)
By setting \( z = \frac{\overline{F}(y)}{F(x)} = e^{-\beta(y-x)}(1-e^{-\lambda}) \) from (2.1.2) in (2.4.4), we obtain
\[
\begin{align*}
&\quad \quad E[X(s,n,m,k) \mid X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \\
&\quad \quad \times \int_{0}^{1} \left[ x - \frac{\ln z}{\beta(1-e^{-\lambda})} \right] z^{\gamma_s-1}(1-z^{m+1})^{s-r-1} dz \\
&\quad \quad = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} (xA_1 - A_2), 
\end{align*}
\] (2.4.5)
where
\[
A_1 = \int_{0}^{1} z^{\gamma_s-1}(1-z^{m+1})^{s-r-1} dz 
\] (2.4.6)
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and

\[ A_2 = \frac{1}{\beta(1-e^{-\lambda})} \int_0^1 \ln z \gamma_{s-1} z^{s-1} (1-z^{m+1})^{s-r-1} \, dz. \tag{2.4.7} \]

Again by setting \( t = z^{m+1} \) in (2.4.6) and (2.4.7), we get

\[
A_1 = \frac{1}{m+1} \int_0^{t^{m+1}} \frac{k+n-s-1}{t^{n-s-1}} (1-t)^{s-r-1} \, dt
= \frac{(m+1)^{s-r-1} \Gamma(s-r)}{\prod_{j=1}^{s-r} \gamma_{r+j}}
\]

and

\[
A_2 = \frac{1}{\beta(1-e^{-\lambda})(m+1)^2} \int_0^1 \ln t \frac{k+n-s-1}{t^{n-s-1}} (1-t)^{s-r-1} \, dt
= \frac{\Gamma \left( \frac{k}{m+1} + n-s \right) \Gamma(s-r)}{\beta(1-e^{-\lambda})(m+1)^2 \Gamma \left( \frac{k}{m+1} + n-r \right)}
\times \left[ \psi \left( \frac{k}{m+1} + n-s \right) - \psi \left( \frac{k}{m+1} + n-r \right) \right],
\]

where

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) \quad \text{[Gradshtein and Ryzhik, 2007, p-540]}. \]

Since [Gradshtein and Ryzhik, 2007, p-905]

\[
\psi(x-n) - \psi(x) = -\sum_{k=1}^{n} \frac{1}{x-k}.
\]

Therefore,

\[
A_2 = -\frac{(m+1)^{s-r-1} \Gamma(s-r) s-r}{\beta(1-e^{-\lambda}) \prod_{j=1}^{s-r} \gamma_{r+j}} \sum_{j=1}^{s-r} \frac{1}{\gamma_{r+j}}.
\]

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Substituting these expressions for $A_1$ and $A_2$ in equation (2.4.5) and simplifying the resulting expression, we derive the relation in (2.4.3).

To prove sufficient part, we have from (2.4.3) and (2.4.1)

$$
\frac{C_{s-1} \gamma_s f(x)}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\infty \gamma y[(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}]^{s-r-1} \times [\overline{F}(y)]^{\gamma_s-1} f(y)dy = [\overline{F}(x)]^{\gamma_{r+1}} H_r(x),
$$

(2.4.8)

where

$$H_r(x) = x + \frac{1}{\beta(1-e^{-\lambda})} \sum_{j=1}^{s-r} \frac{1}{\gamma_{r+j}}.
$$

Differentiating (2.4.8) both sides with respect to $x$ and rearranging the terms, we get

$$
- \frac{C_{s-1}[\overline{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_x^\infty \gamma y[(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}]^{s-r-2} \times [\overline{F}(y)]^{\gamma_s-1} f(y)dy = H'_r(x)[\overline{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} H_r(x)[\overline{F}(x)]^{\gamma_{r+1}-1} f(x)
$$
or

$$
- \gamma_{r+1} H_{r+1}(x)[\overline{F}(x)]^{\gamma_{r+2}+m} f(x) = H'_r(x)[\overline{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} H_r(x)[\overline{F}(x)]^{\gamma_{r+1}-1} f(x).
$$

Therefore,

$$
\frac{f(x)}{\overline{F}(x)} = -\frac{H'_r(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]}
$$

$$
= \beta(1-e^{-\lambda})
$$

which gives

$$\overline{F}(x) = e^{-\beta x(1-e^{-\lambda})}, \ x \geq 0, \ \beta, \ \lambda > 0.
$$

**Remark 4.1:** Setting $\beta(1-e^{-\lambda}) = \alpha$ in (2.4.3), the result for conditional moment of gos, obtained by Khan and Alzaid (2004) and Khan *et al.* (2006) for exponential distribution is deduced.