Chapter 5

Nambu-Goto String and Weyl Symmetry

5.1 Weyl Degree of Freedom in Nambu-Goto String through Field Transformaion

In this chapter we will study, how Weyl degree of freedom can be incorporated in Nambu-Goto (NG) string in light-cone gauge. Bosonic strings are formulated using two alternative actions namely the NG action [150] and the Polyakov action [151]. In the Polyakov formulation, one uses the metric of the string world-sheet, $g_{\alpha\beta}$, to manifest the Weyl and re-parametrization symmetries at the classical level. These symmetries are useful to eliminate the degrees of freedom of $g_{\alpha\beta}$. On the other hand, in the NG string only string coordinates $X_\mu (\mu = 0, ..., D - 1)$ are used as dynamical variables and hence no Weyl freedom is present from the beginning. We consider the NG string theory in BFV formulation at the sub-critical dimensions, where the BF fields appear as the conformal degrees of freedom [155, 156]. Using FFBRST transformations we will connect the action in conformal gauge to light cone gauge. In this way we will be able to incorporate Weyl degrees of freedom in NG string.
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5.1.1 BRST for Nambu-Goto action

NG action for the string coordinates \( X^\mu, \mu = 0, 1, \ldots, D - 1 \) on a two-dimensional world-sheet, parametrized by \( x^\alpha = (\tau, \sigma), \alpha = 0, 1 \) is written as [150]

\[
S_0 = \int d^2 x (-\det G_{\alpha\beta})^{\frac{1}{2}}
\]

where

\[
G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu
\]  

The momentum conjugate to \( X^\mu \) for this theory is then written as

\[
P_\mu = \sqrt{-G} \partial_\alpha X^\mu G^{\alpha 0}
\]

where \( G = \det G_{\alpha\beta} \). The Hamiltonian corresponding to this system vanishes. This system has two primary constraints which generate two re-parameterizations of string world-sheet and are written as,

\[
\phi_\pm = \frac{1}{4} (P_\mu^2 + (\partial_\sigma X^\mu)^2) \pm \frac{1}{2} \partial_\sigma X^\mu P_\mu
\]

These constraints are first class at the classical level but appears as second class at quantum level due to conformal anomaly. To convert them into first class constraints, we will introduce the new field \( \theta \) and its momentum conjugate \( \Pi_\theta \) in the action. The new effective constraints then take the form

\[
\tilde{\phi}_\pm = \phi_\pm + \frac{k}{\sqrt{2}} (\partial_\sigma \Pi_\theta \pm (\partial_\sigma^2 \theta)) + \frac{1}{4} (\Pi_\theta \pm \partial_\sigma \theta)^2
\]

where \( k \) is a constant which is fixed as [159]

\[
k = \frac{(25 - D)}{24\pi}
\]  

We further extend the phase space by introducing following pair of fields

\[
(C^\pm, \bar{P}_\pm), \quad (P^\pm, \bar{C}_\pm), \quad (N^\pm, B_\pm)
\]  

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Now, the action is written in the extended phase space as

$$ S = \int d^2\sigma [\dot{X}^\mu P_\mu + \theta \Pi_\theta + \dot{C}^a \bar{P}_a + \{\psi, Q\}] \quad (5.8) $$

where BRST charge $Q$ is given as

$$ Q = \int d\sigma [C^\pm (\tilde{\phi}_\pm + \bar{P}_\pm \partial_\sigma C^\pm) + B^\pm P_\pm] \quad (5.9) $$

One can easily find out that $Q^2$ is nilpotent in nature and gauge-fixing functional takes the form

$$ \Psi = \int d\sigma (i\dot{C}_a X^a + \bar{P}_a N^a) \quad (5.10) $$

where $X^a$ does not depend on ghost, anti-ghost, $B$ and $N$ fields.

After eliminating all the non-dynamical variables, BRST transformation for the dynamical variables is written as [155, 159]

$$ \delta X^\mu = -\frac{1}{2} (C^a \partial_a X^\mu) $$

$$ \delta \theta = -\frac{1}{2} (C^a \partial_a \theta) + \frac{k}{2\sqrt{2}} (\partial_+ C^+ - \partial_- C^-) $$

$$ \delta C^\pm = -\frac{1}{4} C^\pm \partial_\pm C^\pm $$

$$ \delta \tilde{C}_\pm = -\frac{1}{4} \partial_\pm X^\mu \partial_\pm X_\mu \pm \bar{C}_\pm \partial_\pm C^\pm \pm \frac{1}{2} \partial_\pm \tilde{C}_\pm C^\pm - \frac{1}{4} \partial_\pm \theta \partial_\pm \theta $$

$$ \mp \frac{k}{2\sqrt{2}} \partial_\pm \partial_\pm \theta \quad (5.11) $$

which leaves action in Eq.(5.8) invariant.

Now, total Lagrangian density has the form

$$ \mathcal{L} = \mathcal{L}_x + \mathcal{L}_{gf} + \mathcal{L}_{gh} \quad (5.12) $$

where $\mathcal{L}_x$ denote the string part of the Lagrangian density and
gauge-fixing and ghost terms are defined as

$$\lambda(\mathcal{L}_{gf} + \mathcal{L}_{gh}) = -i\delta(C_a\chi^a)$$  \hspace{1cm} (5.13)$$

where $\lambda$ is infinitesimal Grassmann parameter. In the next section we are going to discuss BRST symmetric Polyakov action in conformal as well as in light-cone gauges.

### 5.1.2 Polyakov Action

Following the technique in ref. [158] we convert NG action to a Polyakov action as

$$\mathcal{L}_x = -\frac{1}{2}\tilde{g}^{ab}\partial_a X^\mu \partial_b X^\mu - \frac{1}{2}\tilde{g}^{ab}\partial_a \theta \partial_b \theta$$  \hspace{1cm} (5.14)$$

Here $\theta$ dependent term in the above Lagrangian density brings extra degrees of freedom in the system. We need a re-parametrization invariant measure in the path integral formulation to construct the BRST symmetry of this theory. Using the methods described in [160, 161] we construct the BRST transformation as

$$\begin{align*}
\delta X^\mu &= -(C^a \partial_a X^\mu) \\
\delta \theta &= -(C^a \partial_a \theta) + \frac{k}{\sqrt{2}} (\partial_a C^a) \\
\delta C^a &= -C^b \partial_b C^a \\
\delta \bar{C}_a &= iB_a \\
\delta \tilde{g}^{ab} &= \partial_a C^a \tilde{g}^{cb} + \partial_b C^b \tilde{g}^{ac} - \partial_e (C^e \tilde{g}^{ab}) + 2\partial_c C^c \theta \tilde{g}^{ab}
\end{align*}$$  \hspace{1cm} (5.15)$$

We define the generating functional in path-integral formulation as

$$Z = \int D\phi \exp \left( i \int d^2x (\mathcal{L}_x + \mathcal{L}_{gf} + \mathcal{L}_{gh}) \right)$$  \hspace{1cm} (5.16)$$

where Lagrangian density is given by Eqs.(5.14, 5.13) and $D\phi$ is the
generic notation for path integral measure. Transformation in Eq.(5.15) leaves the effective action invariant. Now we fix the gauge more specifically and discuss BRST invariant effective theories in conformal as well as light cone gauges.

### 5.1.3 Conformal Gauge

Conformal gauge has been used extensively in the discussion of various problems. It has been used to study strings, gravity etc in path-integral and covariant operator formalism. This gauge is very useful to remove conformal anomaly, to introduce Weyl symmetry and in renormalizing the theory [157, 159, 160].

The conformal gauge condition is expressed as \( \tilde{g}^{ab} = \eta^{ab} \) [156, 160] and is incorporated into the following gauge-fixing and FP ghost term in a BRST invariant manner,

\[
L_{cf} = \lambda (L_{gf} + L_{gh}) = -i\delta^B (\bar{C}^0 \tilde{g}^{++} + \bar{C}^1 \tilde{g}^{--})
\]

(5.17)

Here \( \bar{C}^0 \) and \( \bar{C}^1 \) are anti-ghost fields.

### 5.1.4 Light-cone Gauge

On the other hand, light cone gauge is used to eliminate unphysical degrees of freedom and also in decoupling of ghost fields. Light-cone gauge has also been used in Kaku-Kikkawa string field theory, in showing the ultraviolet finiteness of \( N = 4 \) supersymmetric YM theory, in dimensional regularization, in gravity, supergravity, string and superstrings theories [162].

The light-cone gauge condition, \( (X^+ = f(\sigma), \tilde{g}^{++} = 0) \) [156, 161] is incorporated into the following gauge-fixing and ghost term in a BRST
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\[ L_{lc} = \lambda (L_{gf} + L_{gh}) = -i\delta B (\bar{C}^0 \tilde{g}^{++} + \bar{C}^1 (X^+ - f(\sigma))). \]  

(5.18)

Here \( f(\sigma) \) is an arbitrary function of \( \sigma^0 \) and \( \sigma^1 \).

Now we proceed to use FFBRST to address the Weyl degree of freedom in NG string formulation.

5.1.5 Connection between generating functionals in conformal and light-cone gauges

In this subsection, we construct the FFBRST transformation with an appropriate finite parameter to obtain the generating functional corresponding to \( L_{cf} \) from that of corresponding to \( L_{lc} \). We calculate the Jacobian corresponding to such a FFBRST transformation following the method outlined in chapter 2 and show that it is a local functional of fields and accounts for the difference of the two FP effective actions.

The generating functional corresponding to the FP effective action \( S_{cf} \) is written as

\[ Z_{cf} = \int D\phi \exp(iS_{cf}[\phi]) \]  

(5.19)

where \( S_{cf} \) is given by

\[ S_{cf} = \int d^2x (L_x + L_{cf}) \]  

(5.20)

Now, to obtain the generating functional corresponding to \( S_{lc} \), we apply the FFBRST transformation with a finite parameter \( \Theta[\phi] \), which is obtained from the infinitesimal but field dependent parameter, \( \Theta'[\phi(\kappa)] \); through \( \int_0^\kappa \Theta'[\phi(\kappa)] d\kappa \). We construct \( \Theta'[\phi(\kappa)] \) as,

\[ \Theta'[\phi] = i \int d^2x [\gamma \bar{C}^1 \{(X^+ - f(\sigma)) - \tilde{g}^{++}\}] \]  

(5.21)
Here $\gamma$ is arbitrary constant parameter and all the fields depend on the parameter $k$. The infinitesimal change in the Jacobian corresponding to this FFBRST transformation is calculated using Eq.(2.52)

$$\frac{1}{J(k)} \frac{dJ(k)}{dk} = -i \int d^2x \gamma [-iB^1 \{ (X^+ - f(\sigma)) - \tilde{g}^- - (C^a \partial_a X^+) \tilde{C}^1 \\
- \delta \tilde{g}^- - \tilde{C}^1]$$

(5.22)

To express the Jacobian contribution in terms of a local functional of fields, we make an ansatz for $S_1$ by considering all possible terms that could arise from such a transformation as

$$S_1[\phi(k), k] = \int d^2x [\xi_1 iB^1(X^+ - f(\sigma)) + \xi_2 iB^1 \tilde{g}^- + \xi_3 (C^a \partial_a X^+) \tilde{C}^1 + \xi_4 \delta \tilde{g}^- \tilde{C}^1 + \xi_5 iB^0 \tilde{g}^{++} + \xi_6 \tilde{C}^0 \delta \tilde{g}^{++}]$$

(5.23)

where all the fields are considered to be $k$ dependent and we have introduced arbitrary $k$ dependent parameters $\xi_n = \xi_n(k)(n = 1, 2, ..., 6)$ with initial condition $\xi_n(k = 0) = 0$. It is straightforward to calculate

$$\frac{dS_1}{dk} = \int d^2x [\xi_1' iB^1(X^+ - f(\sigma)) + \xi_2' iB^1 \tilde{g}^- + \xi_3' (C^a \partial_a X^+) \tilde{C}^1 + \xi_4' \delta \tilde{g}^- \tilde{C}^1 + \xi_5' iB^0 \tilde{g}^{++} + \xi_6' \tilde{C}^0 \delta \tilde{g}^{++} + \Theta' (-\xi_1 C^a \partial_a X^+ iB^1 + \xi_2 \delta \tilde{g}^- iB^1 + \xi_3 (iB^1) C^a \partial_a X^+ + \xi_3 (C^b \partial_b C^a) \partial_a X^+ \tilde{C}^1 + \xi_3 C^a \partial_a (-C^b \partial_b X^+) \tilde{C}^1 - \xi_4 iB^1 \delta \tilde{g}^- + \xi_5 \delta \tilde{g}^{++} iB^0
- \xi_6 iB^0 \delta \tilde{g}^{++} \} \right]$$

(5.24)

where $\xi'_n = \frac{d\xi_n}{dk}$. Now we will use the condition of Eq.(2.55).

$$\int D\phi \exp[i(S_{cf}[\phi(k)] + S_1[\phi(k), k])] \int d^2x [(-\gamma + \xi_1') iB^1(X^+ - f(\sigma)) + (\gamma + \xi_2') iB^1 \tilde{g}^- + (\gamma + \xi_3') (C^a \partial_a X^+) \tilde{C}^1 + (\gamma + \xi_4') \delta \tilde{g}^- \tilde{C}^1 + \xi_5' iB^0 \tilde{g}^{++} + \xi_6' \tilde{C}^0 \delta \tilde{g}^{++} + \Theta' \{ (-\xi_1 + \xi_3) C^a \partial_a X^+ iB^1 + (\xi_2 + \xi_4) \delta \tilde{g}^- iB^1 + (\xi_5 - \xi_6) \delta \tilde{g}^{++} iB^0 \} = 0$$

(5.25)
The terms proportional to $\Theta'$ which are regarded as nonlocal, vanishes independently. These will give relations between $\xi$. Making remaining local terms in Eq.(5.25) vanish will give relations between $\xi$ and $\gamma$. Solving these Eqs.(A.18, A.19), we will get following results.

The differential equations for $\xi_n(k)$ can be solved with the initial conditions $\xi_n(0) = 0$, to obtain the solutions

$$\xi_1 = \gamma k, \hspace{1em} \xi_2 = -\gamma k, \hspace{1em} \xi_3 = \gamma k, \hspace{1em} \xi_4 = \gamma k, \hspace{1em} \xi_5 = \xi_6 = 0 \quad (5.26)$$

Putting values of these parameters in expression of $S_1$, and choosing arbitrary parameter $\gamma = -1$, we obtain,

$$S_1[\phi(1), 1] = \int d^2x \left[ -iB^1(X^+ - f(\sigma)) + iB^1 \bar{g}^{--} - (C^a \partial_a X^+) \bar{C}^1 ight]$$

Thus the FFBRST transformation with the finite parameter $\Theta$ that is defined by Eq.(5.21) changes the generating functional $Z_{cf}$ as in Eq.(2.56)

$$Z_{cf} = \int D\phi \exp(iS_{cf}[\phi])$$

$$= \int D\phi' \exp[i(S_{cf}[\phi'] + S_1[\phi', 1])]$$

$$= \int D\phi \exp[i(S_{cf}[\phi] + S_1[\phi, 1])]$$

$$= \int D\phi \exp(iS_{lc}[\phi]) \equiv Z_{lc} \quad (5.28)$$

Here $S_{lc}$ is defined as

$$S_{lc} = \int d^2x (L_x + L_{lc}) \quad (5.29)$$

In this way FFBRST transformation with the finite field dependent parameter in Eq.(5.21) connects generating functional for the Polyakov action in conformal gauge to that of in the light-cone gauge.
5.1.6 Conclusion

In this present work we have demonstrated how Weyl degrees of freedom are incorporated in the formulation of NG string through certain field transformation. Weyl degrees of freedom are first identified in conformal gauge using BFV formulation. Then we have established the connection between conformal gauge to light-cone in Polyakov type action for NG string using the technique of FFBRST transformation, which connects various theories through the non-trivial Jacobian of path integral measure. The non-local BRST transformation by Igarashi et al in [158] is nothing but a particular type of FFBRST transformation. The parameter $\lambda$ in the non-local transformation in [158] is identified with FFBRST parameter $\Theta'$.

In the next chapter we will discuss about issue of ghost number current anomaly in bosonic string in conformal gauge and its removal using FFBRST transformation [196]. We consider Polyakov theory of Bosonic strings in conformal gauge which are used to study conformal anomaly. However it exhibits ghost number anomaly. We show how this anomaly can be avoided by connecting this theory to that of in background covariant harmonic gauge which is known to be free from conformal and ghost number current anomaly, by using suitably constructed finite field dependent BRST transformation.