CHAPTER 3

SEMI GENERALIZED $b$-CLOSED SETS IN
TOPOLOGICAL SPACES

3.1 INTRODUCTION

Levin (1970) introduced a new version of generalized closed sets in topology. Andrijevic (1996) introduced a class of generalized open sets in a topological space called $b$-open sets. This type of set was investigated by Ekici and Caldas (2004) under the name of $\gamma$-open sets. Dunham (1980) discussed the results on generalized closed sets in topology. Ganster (2007) analyzed some questions about $b$-open sets. Since the advent of these notions, several researches with interesting variety of conclusions have been recorded.

The present chapter introduces a new class of semi generalized $b$-closed sets, semi generalized $b$-open sets and $T_{sgb}$ -space and studies the relations with some other closed sets, open sets and spaces. Also, the properties of the closed sets and open sets have been discussed. Further, a new operator called semi generalized $b$-closure operator has been introduced in the present chapter and some of its properties studied.

3.2 SEMI GENERALIZED $b$-CLOSED SET

In this section, the definition of semi-generalized $b$-closed set and some of its characterizations are discussed.
**Definition 3.2.1:** A subset $A$ of a topological space $(X, \tau)$ is said to be a semi generalized $b$-closed set denoted by $sgb$ – closed set, if $b\text{Cl} (A) \subseteq G$, whenever $A \subseteq G$ and $G$ is semi open in $(X, \tau)$.

**Definition 3.2.2:** The set of all $sgb$-closed sets in a topological space $(X, \tau)$ are denoted by $sgbc(X)$.

**Theorem 3.2.3 :** Let $A$ be a $sgb$-closed subset of $(X, \tau)$, then $b\text{Cl} (A) – A$ does not contain any non-empty closed sets.

**Proof:** Let $F \in \text{Cl}(X)$ such that $F \subseteq b\text{Cl} (A) – A$ since $X – F$ is semi open.

$A \subseteq X - F$ and $A$ is $sgb$-closed.

It follows that $b\text{Cl} (A) \subseteq X – F$

Thus, $F \subseteq X – b\text{Cl} (A)$.

This implies that $F \subseteq (X-b\text{Cl} (A)) \cap (b\text{Cl} (A) – A) = \emptyset$.

hence $F = \emptyset$

**Corollary 3.2.4 :** Let $A$ be a $sgb$-closed set. Then $A$ is $b$-closed iff $b\text{Cl} (A) – A$ is closed.

**Proof:** Necessary part: Let $A$ be a $sgb$-closed set. If $A$ is $b$-closed, then $b\text{Cl} (A) – A = \emptyset$ which is closed set.

Converse part: Let $b\text{Cl} (A) – A$ be closed, then by Theorem 3.2.3 $b\text{Cl} (A) – A$ does not contain any non-empty closed subset and since $b\text{Cl} (A) – A$ is closed subset of itself. Then,

$b\text{Cl} (A) – A = \emptyset$

$b\text{Cl} (A) = A$

and $A$ is $b$-closed set.
**Theorem 3.2.5:** Let $B \subseteq A \subseteq X$ where $A$ is a sgb-closed set and semi-open set. $B$ is then sgb-closed relative to $A$ iff $B$ is sgb-closed in $X$.

**Proof:** Necessary part: It is first considered that since $B \subseteq A$ and $A$ are both sgb-closed and semi-open set, then $b\text{Cl} (A) \subseteq A$ and thus $b\text{Cl} (B) \subseteq b\text{Cl} (A) \subseteq A$

Now from the fact that,

$$A \cap b\text{Cl} (B) = b\text{Cl}_A (B)$$

$$b\text{Cl} (B) = b\text{Cl}_A (B) \subseteq A$$

If $B$ is sgb-closed relative to $A$ and $G$ is semi-open subset of $X$ such that $B \subseteq G$, then $B = B \cap A \subseteq G \cap A$ where $G \cap A$ is semi-open in $A$.

Hence, $B$ is sgb-closed relative to $A$,

$$b\text{Cl} (B) = b\text{Cl}_A (B) \subseteq G \cap A$$

$$\Rightarrow b\text{Cl} (B) \subseteq G$$

$\therefore$ $B$ is sgb-closed in $X$.

**Converse part:** If $B$ is sgb-closed set in $X$ and $G$ is semi-open subset of $A$ such that $B \subseteq G$, then $G = V \cap A$ for some semi-open subset $V$ of $X$.

As $B \subseteq V$ and $B$ is sgb-closed in $X$, $b\text{Cl} (B) \subseteq V$, thus,

$$b\text{Cl}_A (B) = b\text{Cl} (B) \cap A \subseteq V \cap A \subseteq G.$$ 

$$\Rightarrow b\text{Cl}_A (B) \subseteq G$$

$\therefore$ $B$ is sgb-closed relative to $A$

**Remark 3.2.6:** Let subset $A$ be semi-open and sgb-closed, $A \cap F$ is then sgb-closed whenever $F \in b\text{Cl} (X)$
Proof: Since A is sgb-closed and semi open set, then bCl (A) ⊆ A and thus A is b-closed. Hence, A ∩ F is b-closed in X which implies that A ∩ F is sgb-closed in X.

Theorem 3.2.7: If A is a sgb-closed set and B is any set such that A ⊆ B ⊆ bCl (A), B is then a sgb-closed set.

Proof: Let B ⊆ G where G is semi open set. Since A is sgb-closed set and A ⊆ G then bCl (A) ⊆ G and also bCl (A) = bCl (B).

Therefore, bCl (B) ⊆ G and hence B is sgb-closed set.

Theorem 3.2.8: Intersection of any two sgb-closed set is sgb-closed.

Proof: Let A and B be two sgb-closed set. ie, bCl (A) ⊆ G whenever A ⊆ G and G is semi open and bCl (B) ⊆ G wherever B ⊆ G and G is semi open.

Now, bCl (A ∩ B) = bCl (A) ∩ bCl (B) ⊆ G

Where A ∩ B ⊆ G and G is semi-open. Thus, intersection of any two sgb-closed set in sgb-closed set.

Remark 3.2.9: Union of any two sgb-closed set need not be a sgb-closed set as seen from the following example.

Example 3.2.10: Let X = {a, b, c}, \( \tau = \{X, \emptyset, \{a, b\}\} \) in this topology space (X, \( \tau \)), the subset \{a\} and \{b\} is sgb-closed but their union \{a, b\} is not sgb-closed set.

Theorem 3.2.11: Every b-closed set is sgb-closed set.

Proof: Let A be a b-closed set in X and let G be a semi-open set contains A in X. Now \( G \supseteq A = bCl A \). Hence, every b-closed set is sgb-closed set.
Remark 3.2.12: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.13: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$, this topological space $(X, \tau)$, the subset $\{a, c\}$ is sgb-closed which is not $b$-closed set.

Theorem 3.2.14: Every swg-closed set is sgb-closed set.

Proof: Let $A$ be a swg-closed set, then $\text{Cl} (\text{Int} A) \subseteq G$ where $A \subseteq G$ and $G$ are semi-open. Since every semi closed set is $b$-closed sets, $b\text{Cl } A \subseteq \text{Cl} (\text{Int} A) \subseteq G$ and $G$ is semi-open.

$\therefore A$ is sgb-closed set

Remark 3.2.15: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.16: If $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. In this topological space $(X, \tau)$, the subset $\{b\}$ is sgb-closed set which is not swg-closed set.

Theorem 3.2.17: Every $g\alpha$-closed set is sgb-closed set.

Proof: Let $A$ be a $g\alpha$-closed set then, $\alpha \text{Cl } A \subseteq G$ whenever $A \subseteq G$ and $G$ is $\alpha$-open. Since, every $\alpha$-closed sets are $b$-closed sets, $b\text{Cl} (A) \subseteq \alpha \text{Cl} A \subseteq G$ and $G$ is semi-open.

$\therefore A$ is sgb-closed.

Hence, every $g\alpha$-closed set is sgb-closed set.

Remark 3.2.18: The converse of the above theorem need not be true as seen from the following example.
Example 3.2.19: Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). In this topological space \((X, \tau)\), the subset \(\{b\}\) is sgb-closed set which is not \(g\alpha\)-closed set.

Theorem 3.2.20: Every sg-closed set is sgb-closed set.

Proof: Let \(A\) be a sg-closed set, then \(s\text{Cl} A \subseteq G\) whenever \(A \subseteq G\) and \(G\) is semiopen. Since every semi-closed set is b-closed sets, \(b\text{Cl} A \subseteq s\text{Cl} A \subseteq G\) and \(G\) is semi open. Therefore, \(A\) is sgb-closed set.

Hence, every sg-closed set is sgb-closed set.

Remark 3.2.21: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.22: Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a, b\}\} \). In this topological space \((X, \tau)\), the subsets \(\{a\}\) is sgb-closed set which is not sg-closed set.

Theorem 3.2.23: Every \(\hat{g}\)-closed set is sgb-closed set.

Proof: Let \(A\) be a \(\hat{g}\)-closed set, then \(\text{Cl} A \subseteq G\) whenever \(A \subseteq G\) and \(G\) is semi open. Since every closed set is b-closed sets, \(b\text{Cl} A \subseteq \text{Cl} A \subseteq G\) and \(G\) is semi open. Therefore, \(A\) is sgb-closed set.

Remark 3.2.24: The converse of the above theorem need not be true as seen from the following example.

Example 3.2.25: Let \( X = \{a, b, c\}, \tau = \{X, \emptyset, \{a, b\}\} \). In this topological space \((X, \tau)\), the subset \(\{a\}\) is sgb-closed set which is not \(\hat{g}\)-closed set.

Remark 3.2.26: The following examples show that the wg-closed sets and sgb-closed sets are independent.
Example 3.2.27: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is sgb-closed set which is not wg-closed set.

Example 3.2.28: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, c\}$ is wg-closed set which is not sgb-closed set.

Remark 3.2.29: The following examples show that gsp-closed sets and sgb-closed sets are independent.

Example 3.2.30: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is gsp closed set which is not sgb-closed set.

Example 3.2.31: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is sgb-closed set which is not gsp-closed set.

Remark 3.2.32: The following examples show that the $\alpha$ g-closed sets and sgb-closed sets are independent.

Example 3.2.33: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is $\alpha$ g-closed set which is not sgb-closed set.

Example 3.2.34: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is sgb-closed set which is not $\alpha$ g-closed set.

Remark 3.2.35: The following examples show that the gs-closed sets and sgb-closed sets are independent.

Example 3.2.36: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is gs-closed set which is not sgb-closed set.

Example 3.2.37: Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}\}$. In this topological space $(X, \tau)$, the subset $\{a\}$ is sgb-closed set which is not gs-closed set.
**Remark 3.2.38:** The following examples show that the #gs-closed sets and sgb-closed sets are independent.

**Example 3.2.39:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}, \{a, b\}\} \). In this topological space \((X, \tau)\), the set \( \{a, c\} \) is \#g-semi closed set which is not sgb-closed set.

**Example 3.2.40:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a, b\}\} \). In this topological space \((X, \tau)\), the subset \( \{a\} \) is sgb-closed set which is not \#g semi closed set.

**Remark 3.2.41:** The following examples show that the gp-closed sets and sgb-closed sets are independent.

**Example 3.2.42:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}, \{a, b\}\} \). In this topological space \((X, \tau)\), the set \( \{a, c\} \) is gp-closed set which is not sgb-closed set.

**Example 3.2.43:** Let \( X = \{a,b,c\}, \quad \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\) this topological space \((X, \tau)\), the subset \( \{a\} \) is sgb-closed set which is not gp-closed set.

**Remark 3.2.44:** The following examples show that the pg-closed sets and sgb-closed sets are independent.

**Example 3.2.45:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{b\}\} \). In this topological space \((X, \tau)\), the set \( \{a, b\} \) is pg-closed set which is not sgb-closed set.

**Example 3.2.46:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\). In this topological space \((X, \tau)\), the subset \( \{b, c\} \) is sgb-closed set which is not pg-closed set.

**Remark 3.2.47:** The following examples show that the *g-closed sets and sgb-closed sets are independent.
**Example 3.2.48:** Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{a, b\}$ is $^*g$-closed set which is not sgb-closed set.

**Example 3.2.49:** Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$. In this topological space $(X, \tau)$, the subset $\{b\}$ is sgb-closed set which is not $^*g$-closed set.

**Remark 3.2.50:** The following examples show that the $\widetilde{g}$-closed sets and sgb-closed sets are independent.

**Example 3.2.51:** Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. In this topological space $(X, \tau)$, the subset $\{b, c\}$ is $\widetilde{g}$-closed set which is not sgb-closed set.

**Example 3.2.52:** Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$. In this topological space $(X, \tau)$, the subset $\{b\}$ is sgb-closed set which is not $\widetilde{g}$-closed set.

**Remark 3.2.53:** The following examples show that the rg-closed sets and sgb-closed sets are independent.

**Example 3.2.54:** Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. In this topological space $(X, \tau)$, the subset $\{b, c\}$ is rg-closed set which is not sgb-closed set.

**Example 3.2.55:** Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. In this topological space $(X, \tau)$, the subset $\{b\}$ is sgb-closed set which is not rg-closed set.
**Remark 3.2.56**: From the above discussion, the Figures 3.1 and 3.2 are obtained.

Figure 3.1 Implied relationship of sgb – closed set

where $A \rightarrow B$ represent $A$ implies $B$

$A \not\rightarrow B$ represent $A$ does not imply $B$
This section introduces the new class of semi generalized b-open set in topological spaces and discusses their properties.

**Definition 3.3.1:** A subset $A$ of a topological space $(X, \tau)$ is called semi generalised b-open (denoted by sgb-open) set, if its complement that is $A^c$ is semi generalized b-closed.

The collection of all sgb-open sets in $X$ is denoted by $sgbO(X)$

**Theorem 3.3.2:** A subset $A \subseteq X$ is sgb-closed set if $F \subseteq b \text{Int} (A)$ whenever $F$ is closed set and $F \subseteq A$. 

**Figure 3.2 Independence of sgb-closed set**

where $A \nrightarrow B$ represent A does not implies B

$A \nleftarrow B$ represent B does not implies A

**3.3 SEMI GENERALIZED b-OPEN SETS**

This section introduces the new class of semi generalized b-open set in topological spaces and discusses their properties.
Proof: Let A be a sgb-open set and suppose \( F \subseteq A \) where \( F \) is closed, \( X-A \) is then a sgb-closed set contained in the semi open set \( X-F \). Hence, \( b\text{Cl} (X-A) \subseteq X-F \) and \( X-b\text{Int} (A) \subseteq X-F \). Thus \( F \subseteq b\text{Int} (A) \). Conversely, if \( F \) is a closed set with \( F \subseteq b\text{Int} (A) \) and \( F \subseteq A \) then \( X-b\text{Int} (A) \subseteq X-F \), then \( b\text{Cl} (X-A) \subseteq X-F \). Hence, \( X-A \) is sgb-closed set and \( A \) is a sgb-closed set.

3.4 SEPARATION AXIOMS OF \( T_{sgb} \)–SPACES

This section introduces a new class of topological space called \( T_{sgb} \)–space and studies the separation of axioms. Also, the relationship with some other spaces is also discussed.

Definition 3.4.1: A topological space \((X, \tau)\) is said to be \( T_{sgb} \)–space if every sgb-closed set is semi-closed set.

Theorem 3.4.2: Every \( T_{swg} \)–space is \( T_{sgb} \)–space.

Proof: Let \( X \) be \( T_{swg} \)–space and \( A \) be a swg-closed set in \( X \) then \( A \) is sgb-closed set by Theorem 3.2.14. As \( X \) is a \( T_{swg} \)–space, \( A \) is closed and hence it is semi-closed. Therefore, \( X \) is a \( T_{sgb} \)–space.

Remark 3.4.3: The converse of the above theorem need not be true as seen from the following example.

Example 3.4.4: Let \( X = \{a, b, c,\} \), \( \tau = \{X, \emptyset, \{a\}\} \). In this topological space \((X, \tau)\) is \( T_{sgb} \)–space and not \( T_{swg} \)–space, since the subset \( \{b\} \) is swg-closed which is not closed set.

Remark 3.4.5: The following examples show that the \( T_{sgb} \)–space and pre \( T_{\frac{1}{2}} \)–spaces are independent.
**Example 3.4.6:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}\}. \) In this topological space \((X, \tau)\) is T\(_{\text{sgb}}\)-space and not pre T\(_{\frac{1}{2}}\) -spaces, since the subset \(\{a, b\}\) is gp-closed set which is not pre-closed set.

**Example 3.4.7:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a, b\}\}. \) In this topological space \((X, \tau)\) is pre T\(_{\frac{1}{2}}\) -space and not T\(_{\text{sgb}}\) -spaces, since the subset \(\{a\}\) is sgb-closed set which is not semi-closed set.

**Remark 3.4.8:** The following examples show that the T\(_{\text{sgb}}\)-spaces and T\(_d\) -spaces are independent.

**Example 3.4.9:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}, \{a, b\}\}. \) In this topological space \((X, \tau)\) is T\(_{\text{sgb}}\)-space and not T\(_d\) -spaces, since the subset \(\{b\}\) is gs-closed set which is not g-closed set.

**Example 3.4.10:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{c\}, \{a, b\}\} \) this topological space \((X, \tau)\) is T\(_d\)-space and not T\(_{\text{sgb}}\) -spaces, since the subset \(\{b\}\) is sgb-closed set which is not semi-closed set.

**Remark 3.4.11:** The following examples show that the T\(_{\text{sgb}}\)-spaces and T\(_b\) -spaces are independent.

**Example 3.4.12:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}, \{a, b\}\}. \) In this topological space \((X, \tau)\) is T\(_{\text{sgb}}\)-space and not T\(_b\) -spaces, since the subset \(\{a, c\}\) is gs-closed set which is not closed set.

**Example 3.4.13:** Let \( X = \{a, b, c\}, \quad \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}. \) In this topological space \((X, \tau)\) is T\(_b\)-space and not T\(_{\text{sgb}}\) -spaces, since the subset \(\{c\}\) is sgb-closed set which is not semi-closed set.
**Remark 3.4.14:** The following examples show that the $T_{sgb}$-spaces and $T_{wg}$-spaces are independent.

**Example 3.4.15:** Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$. In this topological space $(X, \tau)$ is $T_{sgb}$-space and not $T_{wg}$-spaces, since the subset $\{a, b\}$ is wg-closed set which is not closed set.

**Example 3.4.16:** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$ is $T_{wg}$-space and not $T_{sgb}$-spaces, since the subset $\{c\}$ is sgb-closed set which is not semi-closed set.

**Remark 3.4.17:** The following examples show that the $T_{sgb}$-spaces and $\alpha T_d$-spaces are independent.

**Example 3.4.18:** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. In this topological space $(X, \tau)$ is $T_{sgb}$-space and not $\alpha T_d$-spaces, since the subset $\{b\}$ is $\alpha g$-closed set which is not g-closed set.

**Example 3.4.19:** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$ is $\alpha T_d$-space and not $T_{sgb}$-spaces, since the subset $\{a, b\}$ is sgb-closed set which is not semi-closed set.

**Remark 3.4.20:** The following examples show that the $T_{sgb}$-spaces and $\alpha T_b$-spaces are independent.

**Example 3.4.21:** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. In this topological space $(X, \tau)$ is $\alpha T_b$-space and not $T_{sgb}$-spaces, since the subset $\{c\}$ is sgb-closed set which is not semi-closed set.
Example 3.4.22: Let \( X = \{a, b, c\}, \quad \tau = \{X, \ \emptyset, \ {c}, \ {a, c}\} \). In this topological space \((X, \tau)\) is \(T_{s\text{gb}}\)-space and not \(\alpha T_b\)-spaces, since the subset \(\{b, c\}\) is \(\alpha g\)-closed set which is not closed set.

Remark 3.4.23: The following examples show that the \(T_{s\text{gb}}\)-spaces and \(T_{\alpha g}\)-spaces are independent.

Example 3.4.24: Let \( X = \{a, b, c\}, \quad \tau = \{X, \ \emptyset, \ {c}, \ {a, c}\} \). In this topological space \((X, \tau)\) is \(T_{s\text{gb}}\)-space and not \(T_{\alpha g}\)-space, since the subset \(\{b, c\}\) is \(\alpha g\)-closed set which is not \(g\alpha\)-closed set.

Example 3.4.25: Let \( X = \{a, b, c\}, \quad \tau = \{X, \ \emptyset, \ {c}, \ {a, b}\} \). In this topological space \((X, \tau)\) is \(T_{\alpha g}\)-space and not \(T_{s\text{gb}}\)-space, since the subset \(\{a\}\) is \(s\text{gb}\)-closed set which is not semi-closed set.

Remark 3.4.26: The following examples show that the \(T_{s\text{gb}}\)-spaces and \(T_{g s}\)-spaces are independent.

Example 3.4.27: Let \( X = \{a, b, c\}, \quad \tau = \{X, \ \emptyset, \ {a, b}\} \). In this topological space \((X, \tau)\) is \(T_{g s}\)-space and not \(T_{s\text{gb}}\)-space, since the subset \(\{b\}\) is \(s\text{gb}\)-closed set which is not semi-closed set.

Example 3.4.28: Let \( X = \{a, b, c\}, \quad \tau = \{X, \ \emptyset, \ {c}\} \). In this topological space \((X, \tau)\) is \(T_{s\text{gb}}\)-space and not \(T_{g s}\)-space, since the subset \(\{a, c\}\) is \(g s\)-closed set which is not \(s g\)-closed set.

Remark 3.4.29: The implication of the \(s\text{gb}\)-closed set and the independenrness of the \(s\text{gb}\)-closed set are not the same in the case of their corresponding spaces. Thus, a notable relationship under the analysis of separation axioms that is obtained is given in the Figure 3.3.
3.5 SEMI GENERALIZED b-CLOSURE OPERATOR

This section introduces a new class of topological operator called semi generalized b-closure operator and is discussed along with the $T_{sgb}$-space.
**Definition 3.5.1:** For any subset E of \((X, \tau)\), the following is defined,

\[
bCl^*(E) = \bigcap \{ A : E \subseteq A \in bD(X, \tau) \}
\]

where

\[
bD(X, \tau) = \{ A : A \subseteq X \text{ and } A \text{ is sgb-closed in } (X, \tau) \}
\]

**Theorem 3.5.2:** Let \(E\) and \(F\) be the two subsets of a space \((X, \tau)\). Then,

(i) \(E \subseteq bcl^*(E) \subseteq bcl(E) \subseteq cl(E)\)

(ii) \(bCl^*(\emptyset) = \emptyset\) and \(bCl^*(X) = X\)

(iii) \(bCl^*(E \cup F) \supseteq bCl^*(E) \cup bCl^*(F)\)

(iv) \(bCl^*(bCl^*E) = bCl^*(E)\) and \(bCl^*(bCl^*E) = bCl^*(E)\)

(v) if \(E\) is sgb-closed then \(bCl^*(E) = E\).

The proof follows immediately from the definitions and properties of sgb-closed sets.

**Theorem 3.5.3:** For each \(x \in X\), \(\{x\}\) is semi-closed or its compliment \(\{x\}^c\) is sgb-closed in a space \((X, \tau)\).

**Proof:** Suppose that \(\{x\}\) is not semi-closed in \((X, \tau)\). Since \(\{x\}^c\) is not semi-open. The space \(X\) itself is only semi-open set containing \(\{x\}^c\). Therefore, \(bCl(\{x\}^c)\) holds and \(\{x\}^c\) is sgb-closed.

**Theorem 3.5.4:** For a space \((X, \tau)\) if \(x \neq y\) then \(bCl^*(x) \neq bCl^*(y)\).

**Proof:** By the above Theorem, it is sufficient to prove the following, that is \(\{x\}^c\) is sgb-closed. Since \(\{y\} \subseteq \{x\}^c\), \(y \in bCl^*(\{y\}) \subseteq \{x\}^c\), \(bCl^*(\{y\}) \neq bCl^*(\{x\})\).
Definition 3.5.5: \( S.O.(X, \tau)^* = \{B: \ bcl^*(B^c) = B^c\} \)

Remarks 3.5.6: If \( E \in bD(X, \tau) \) (Def. 3.5.1) then \( E^c \in S.O(X, \tau)^* \)

Theorem 3.5.7: (i) \( S.O.(\tau) \subseteq S.O.(\tau)^* \) holds

(ii) A space \((X, \tau)\) is \( T_{sgb} \) if and only if \( S.O.(\tau) \subseteq S.O.(\tau)^* \) holds.

Proof: (i) Let \( E \in S.O.(\tau) \), its complement \( E^c \) then is semi-closed, if and only if \( E^c = bCl(E^c) \), which follows from Theorem 3.5.2 (i) that \( bCl^*(E^c) = E^c \) holds, that is \( E \in S.O.(\tau)^* \). (ii) Necessity: Since the semi-closed sets and the sgb-closed sets coincide by the assumption, \( bCl(E) = bCl^*(e) \) holds for every subset of \((X, \tau)\).

Therefore, \( S.O.(\tau) = S.O.(\tau) \)

Sufficiency: Let \( A \) be a sgb-closed set of \((X, \tau)\). By the Theorem 3.5.2(V), then \( A = bCl^*(A) \) and hence \( A^c \in S.O.(\tau) \)

Then \( A \) is semi-closed. Therefore \((X, \tau)\) is \( T_{sgb} \)-space.