Chapter 2

Second-Order Nonlinear Neutral Difference Equations with Mixed Arguments
2. Second-Order Nonlinear Neutral Difference Equations with Mixed Arguments

2.1 Introduction

This chapter is concerned with the second-order nonlinear neutral difference equations of mixed type of the form

\[ \Delta (\varphi_n \Delta (v_n + \delta v_{n-\tau_1} + \gamma v_{n+\tau_2})) + \xi_n v_{n+1-\sigma_1}^\alpha + \eta_n v_{n+1+\sigma_2}^\beta = 0, \quad n \geq n_0 > 0, \quad (2.1.1) \]

subject to the following conditions:

(A1) \( \{\varphi_n\} \) is a sequence of positive real numbers for all \( n \geq n_0 \);

(A2) \( \{\xi_n\} \), and \( \{\eta_n\} \) are sequences of real numbers;

(A3) \( \delta \) and \( \gamma \) are arbitrary constants;

(A4) \( \tau_1, \tau_2, \sigma_1, \) and \( \sigma_2 \) are nonnegative integers;

(A5) \( \alpha \) and \( \beta \) are ratios of odd positive integers.

By a solution of difference equation, we mean a nontrivial real sequence \( \{v_n\} \) defined for all \( n \geq n_0 \), and satisfying the given difference equation for all \( n \geq n_0 - \Phi \) where \( \Phi = \max(\tau_1, \sigma_1) \).

A nontrivial solution \( \{v_n\} \) of (2.1.1) is said to be weakly oscillatory if \( \{v_n\} \) is nonoscillatory, and \( \{\Delta v_n\} \) is oscillatory for all \( n \geq n_0 \).

From the review of literature, one can observe that compared to the study on oscillatory behavior of ordinary and delay difference equations, the study of neutral type difference equations of mixed type received less attention, see for example [5, 9, 56, 57, 60], and the references
contained therein. Therefore in this chapter, we study the oscillatory and asymptotic behavior of all solutions of (2.1.1).

Let $S$ denote the set of all nontrivial solutions of (2.1.1). In view of their asymptotic behavior, all solutions of (2.1.1) may be priori divided into the following four mutually disjoint classes:

\[
\mathcal{M}^+ = \{\{v_n\} \in S \text{ such that } v_n \Delta v_n \geq 0 \text{ for all } n \geq n_1 \geq n_0\},
\]

\[
\mathcal{M}^- = \{\{v_n\} \in S \text{ such that } \{v_n\} \text{ is nonoscillatory, and } v_n \Delta v_n \leq 0 \text{ for all } n \geq n_1 \geq n_0\},
\]

\[
\text{OS} = \{\{v_n\} \in S \text{ such that } \{v_n\} \text{ is oscillatory}\},
\]

\[
\text{WOS} = \{\{v_n\} \in S \text{ such that } \{v_n\} \text{ is weakly oscillatory}\}.
\]

In [16, 24, 25, 30, 54], the authors considered the equation (2.1.1) with $\gamma \equiv 0$ and $\eta_n \equiv 0$ for all $n \geq n_0$, and studied the oscillatory and asymptotic behavior of all solutions of (2.1.1).

In [1, 2], the authors considered the equation (2.1.1) with $\gamma \equiv 0$ and $\eta_n \equiv 0$ for all $n \geq n_0$, and classified all solutions of (2.1.1) into the above said four classes, and obtained conditions for the existence/nonexistence of solutions of (2.1.1) in these classes.

In [47–50], the authors considered the equation (2.1.1) with $\delta \equiv 0$, $\gamma \equiv 0$, and $\eta_n \equiv 0$ for all $n \geq n_0$, and discussed the existence/nonexistence of solutions of (2.1.1) in the above said classes.

Following this trend, in this chapter we consider the cases $\xi_n \geq 0$, $\eta_n \geq 0$, and $\{\xi_n\}$, $\{\eta_n\}$ changes sign for all large $n$, to obtain conditions in order that every solution of (2.1.1) is in these classes or not. Also we study the asymptotic behavior of nonoscillatory solutions of (2.1.1) in the classes $\mathcal{M}^+$ and $\mathcal{M}^-$. 8
In Section 2.2, we obtain some sufficient conditions for the existence/nonexistence of solutions of (2.1.1) in the above said classes. In Section 2.3, we study the asymptotic behavior of solutions of (2.1.1) in the classes $\mathcal{M}^+$ and $\mathcal{M}^-$. In Section 2.4, examples are provided to illustrate the main results. The results obtained in this chapter generalize, and extend some of the known results in [1, 2, 47–50].

### 2.2 Existence/Nonexistence of Solutions

We write $r_n = v_n + \delta v_{n-\tau_1} + \gamma v_{n+\tau_2}$ for all $n \geq n_0$. First, we examine the existence/nonexistence of solutions of (2.1.1) in the class $\mathcal{M}^+$.

**Theorem 2.2.1.** Assume that the conditions

1. (H1) $\delta \geq 0$, and $\gamma \geq 0$,
2. (H2) $\{\eta_n\}$ is any real sequence, and $\xi_n \leq 0$ for all $n \geq n_0$,
3. (H3) $\alpha \geq \beta$,

hold. If

$$\lim_{n \to \infty} \sup_{s=n_0}^{n-1} (L\xi_s + \eta_s) = \infty$$

(2.2.1)

for any $L > 0$, then $\mathcal{M}^+ = \phi$.

**Proof.** Let $\{v_n\}$ be a solution of (2.1.1) in the class $\mathcal{M}^+$. Without loss of generality, we may assume that $\{v_n\}$ is a positive solution of (2.1.1) in the class $\mathcal{M}^+$. Then there exists an integer $n_1 \geq n_0$ such that $v_{n_1-\Phi} > 0$, and $\Delta v_n \geq 0$ for all $n \geq n_1$ (The proof for the other case $v_n < 0$
and $\Delta v_n \leq 0$ for all $n \geq n_1$ is similar). Then by $(H_1)$ we have $r_n > 0$, and $\Delta r_n \geq 0$ for all $n \geq n_1$. Now

$$
\Delta \left( \frac{\varphi_n \Delta r_n}{v_n^{\beta}+\sigma_2} \right) = \Delta \left( \frac{\varphi_n \Delta r_n}{v_n^{\beta}+\sigma_2} \right) - \frac{\varphi_n \Delta r_n}{v_n^{\beta}+\sigma_2} \Delta \left( \frac{v_n^{\beta}}{v_n^{\beta}+\sigma_2} \right) \\
\leq -\xi_n v_n^{\alpha-\beta} - \eta_n
$$

(2.2.2)

for all $n \geq n_1$, where we have used $v_n > 0$, $\Delta v_n \geq 0$, and $\Delta r_n \geq 0$ for all $n \geq n_1$. Since $\{v_n\}$ is positive and nondecreasing and $\alpha \geq \beta$, there exists an integer $n_2 \geq n_1$ such that $v_n^{\alpha-\beta} \geq \mathcal{L} > 0$ for all $n \geq n_2$. Therefore (2.2.2) becomes

$$
\Delta \left( \frac{\varphi_n \Delta r_n}{v_n^{\beta}+\sigma_2} \right) \leq -(\mathcal{L} \xi_n + \eta_n)
$$

(2.2.3)

for all $n \geq n_2$. Summing the last inequality from $n_2$ to $n - 1$, we obtain

$$
\frac{\varphi_n \Delta r_n}{v_n^{\beta}+\sigma_2} \leq -\sum_{s=n_2}^{n-1} (\mathcal{L} \xi_s + \eta_s) + \frac{\varphi_{n_2} \Delta r_{n_2}}{v_{n_2}^{\beta}+\sigma_2}.
$$

(2.2.4)

Now taking limit inferior on both sides of the inequality (2.2.4), and using the condition (2.2.1), we see that

$$
\liminf_{n \to \infty} \frac{\varphi_n \Delta r_n}{v_n^{\beta}+\sigma_2} = -\infty,
$$

which means $\Delta r_n < 0$ for all $n \geq n_2$, which is a contradiction to $\Delta r_n \geq 0$ for all $n \geq n_2$. Now the proof is completed.

**Theorem 2.2.2.** Assume that the condition $(H_1)$ holds. Also assume that

$$(H_4) \ \{\xi_n\} \text{ is any real sequence, and } \eta_n \geq 0 \text{ for all } n \geq n_0,$$

$$(H_5) \ \beta \geq \alpha,$$

hold. If

$$
\limsup_{n \to \infty} \sum_{s=n_0}^{n-1} (\xi_s + \mathcal{L}_1 \eta_s) = \infty
$$

(2.2.5)

for any $\mathcal{L}_1 > 0$, then $\mathcal{M}^+ = \phi$. 

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Proof. Let \( \{v_n\} \) be a solution of (2.1.1) such that \( \{v_n\} \in M^+ \). Without loss of generality, we may assume that there exists an integer \( n_1 \geq n_0 \) such that \( v_{n-\Phi} > 0 \), and \( \Delta v_n \geq 0 \) for all \( n \geq n_1 \) (The proof for the opposite case \( v_n > 0 \), and \( \Delta v_n \leq 0 \) for all \( n \geq n_1 \) is similar). Then by \((H_1)\) we have \( r_n > 0 \), and \( \Delta r_n \geq 0 \) for all \( n \geq n_1 \). Now \( \Delta (\varphi_n \Delta r_n) \leq -\xi_n - \eta_n v^{\beta - \alpha}_{n+1+\sigma_2} \) for all \( n \geq n_1 \), where we have used the monotonicity of \( \{v_n\} \). Since \( \{v_n\} \) is nondecreasing and \( \beta \geq \alpha \), there exists an integer \( n_2 \geq n_1 \) such \( v^{\beta - \alpha}_{n+1+\sigma_2} \geq L_1 > 0 \) for all \( n \geq n_2 \). Now the inequality (2.2.6) becomes

\[
\Delta \left( \frac{\varphi_n \Delta r_n}{v^{\alpha}_{n-\sigma_1}} \right) \leq -\left( \xi_n + L_1 \eta_n \right) \tag{2.2.7}
\]

for all \( n \geq n_2 \). Summing the last inequality from \( n_2 \) to \( n - 1 \), we obtain

\[
\frac{\varphi_n \Delta r_n}{v^{\alpha}_{n-\sigma_1}} \leq -\sum_{s=n_2}^{n-1} \left( \xi_s + L_1 \eta_s \right) + \frac{\varphi_{n_2} \Delta r_{n_2}}{v^{\alpha}_{n_2-\sigma_1}} \tag{2.2.8}
\]

for all \( n \geq n_2 \). Now taking limit infimum on both sides of the inequality (2.2.8) and using the condition (2.2.5), we obtain

\[
\liminf_{n \to \infty} \frac{\varphi_n \Delta r_n}{v^{\alpha}_{n-\sigma_1}} = -\infty. \tag{2.2.9}
\]

But (2.2.9) implies that \( \Delta r_n < 0 \) for all \( n \geq n_2 \), which is a contradiction to \( \Delta r_n \geq 0 \) for all \( n \geq n_2 \). This completes the proof of the theorem. \( \Box \)

**Theorem 2.2.3.** Assume that the conditions \((H_1)\), \((H_4)\), and \((H_6)\) \( \alpha = \beta \),
hold. If
\[ \limsup_{n \to \infty} \sum_{s=n_0}^{n-1} (\xi_s + \eta_s) = \infty, \] (2.2.10)
then \( M^+ = \phi. \)

**Proof.** The proof is similar to that of Theorem 2.2.2, and hence the details are omitted. \( \square \)

**Theorem 2.2.4.** Assume that the conditions \((H_2), (H_3),\) and

\( (H_7) \ -1 < \delta + \gamma \leq 0 \) with \( \delta < 0, \) and \( \gamma > 0, \)

hold. If
\[ \sum_{s=n_0}^{\infty} \frac{1}{\varphi_s} = \infty, \] (2.2.11)
and
\[ \sum_{n=n_0}^{\infty} (L\xi_n + \eta_n) = \infty, \] (2.2.12)
for any \( L > 0, \) then \( M^+ = \phi. \)

**Proof.** Suppose equation (2.1.1) has a solution \( \{v_n\} \) in the class \( M^+. \) Then without loss of
genenity, we may assume that there exists an integer \( n_1 \geq n_0 \) such that \( v_{n-\phi} > 0, \) and
\( \Delta v_n \geq 0 \) for all \( n \geq n_1 \) (The proof for other case \( v_n < 0, \) and \( \Delta v_n \leq 0 \) for all \( n \geq n_1 \) is similar).

Since \( \{v_n\} \) is nondecreasing, and by \( (H_7) \) we have
\[ r_n = v_n + \delta v_{n-\tau_1} + \gamma v_{n+\tau_2} \geq (1 + \delta + \gamma)v_{n-\tau_1} > 0 \]
for all \( n \geq n_1. \) Now from (2.1.1) and proceeding as in Theorem 2.2.1, we have
\[
\Delta(v_n \Delta r_n) = -\left( \frac{\xi_n v_{n+1-\sigma_1}^\alpha + \eta_n v_{n+1+\sigma_2}^\beta}{v_{n+1+\sigma_2}^\beta} \right) v_{n+1+\sigma_2}^\beta \\
\leq -(L\xi_n + \eta_n) v_{n+1+\sigma_2}^\beta, \ n \geq n_1. \] (2.2.13)
Therefore by condition (2.2.12), there exists an integer $n_2 \geq n_1$ such that $\Delta(\varphi_n \Delta r_n) \leq 0$ for all $n \geq n_2$. Now we prove that $\varphi_n \Delta r_n \geq 0$ for all $n \geq n_2$. If not, then
\[
\Delta r_n \leq \frac{\varphi_{n_2} \Delta r_{n_2}}{\varphi_n} < 0 \quad \text{for all } n \geq n_2.
\] (2.2.14)

Summing the last inequality from $n_2$ to $n - 1$, we obtain
\[
r_n \leq \sum_{s=n_2}^{n-1} \frac{\varphi_{n_2} \Delta r_{n_2}}{\varphi_s} \leq \varphi_{n_2} \Delta r_{n_2} \sum_{s=n_2}^{n-1} \frac{1}{\varphi_s}.
\] (2.2.15)

Now combining (2.2.11), and (2.2.15), we see that $r_n \to -\infty$ as $n \to \infty$, which is a contradiction to the positivity of $r_n$. Thus $\varphi_n \Delta r_n \geq 0$ for all $n \geq n_2$. Now summing the inequality (2.2.13) from $n_2$ to $n - 1$, we obtain
\[
\varphi_n \Delta r_n \leq -\sum_{s=n_2}^{n-1} (L \xi_s + \eta_s) \nu_{n+1+\sigma_2}^{\beta} + \varphi_{n_2} \Delta r_{n_2}
\]
for all $n \geq n_2$. Now taking limit infimum as $n \to \infty$ on both sides of the last inequality and using the condition (2.2.12), we see that
\[
\liminf_{n \to \infty} \frac{\varphi_n \Delta r_n}{\nu_{n+\sigma_2}^{\beta}} = -\infty,
\]
which contradicts the fact that $\Delta r_n \geq 0$ for all $n \geq n_2$. Now the proof is completed. \(\square\)

**Theorem 2.2.5.** Assume that the conditions $(H_4)$, $(H_5)$, $(H_7)$, and (2.2.11) hold. If
\[
\sum_{n=n_0}^{\infty} (\xi_n + \nu_1 \eta_n) = \infty,
\] (2.2.16)
for any $\nu_1 > 0$, then $\mathcal{M}^+ = \phi$.

**Proof.** The proof is similar to that of Theorem 2.2.4, and hence it is omitted. \(\square\)

**Theorem 2.2.6.** Assume that the conditions $(H_4)$, $(H_6)$, $(H_7)$, and (2.2.11) hold. If
\[
\sum_{n=n_0}^{\infty} (\xi_n + \eta_n) = \infty,
\] (2.2.17)
then $\mathcal{M}^+ = \phi$. 

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Proof. The proof is similar to that of Theorem 2.2.4, and hence it is omitted.

Next, we discuss the existence/nonexistence of solutions of (2.1.1) in the class $\mathcal{M}^-$.

**Theorem 2.2.7.** Assume that the conditions $(H_1)$, and $(H_5)$ hold. Further assume that

$(H_8)$ \( \{\xi_n\} \) is any real sequence, and \( \eta_n \leq 0 \) for all \( n \geq n_0 \),

$(H_9)$ \( \tau_1 \leq \sigma_1 \),

$(H_{10})$ \( \int_0^p \frac{du}{u^{\alpha}} < \infty \), and \( \int_0^0 \frac{du}{u^{\alpha}} < \infty \) for some \( p > 0 \)

hold. If

$$\limsup_{n \to \infty} \frac{1}{s=n_0} \sum_{s=n_0}^{n-1} \sum_{u=n_0}^{s-1} (\xi_u + \mathcal{L}_2 \eta_u) = \infty$$

(2.2.18)

for any \( \mathcal{L}_2 > 0 \), then \( \mathcal{M}^- = \phi \).

**Proof.** Suppose \( \{v_n\} \) is a solution of (2.1.1) in the class \( \mathcal{M}^- \). Without loss of generality, we may assume that there exists an integer \( n_1 \geq n_0 \) such that \( v_{n-\Phi} > 0 \), and \( \Delta v_n \leq 0 \) for all \( n \geq n_1 \) (The proof is similar when \( v_n < 0 \), and \( \Delta v_n \geq 0 \) for all \( n \geq n_1 \)). Then by \( (H_1) \) \( r_n > 0 \), and \( \Delta r_n \leq 0 \) for all \( n \geq n_1 \). Now

$$\Delta \left( \frac{\varphi_n \Delta r_n}{v_{n-\sigma_1}^{\alpha}} \right) = \frac{\Delta(\varphi_n \Delta r_n)}{v_{n+1-\sigma_1}^{\alpha}} - \frac{\varphi_n \Delta r_n \Delta v_{n-\sigma_1}^{\alpha}}{v_{n+1-\sigma_1}^{\alpha} v_{n-\sigma_1}^{\alpha}} \leq -\xi_n - \eta_n v_{n+1+\sigma_2}^{\beta-\alpha}$$

(2.2.19)

for all \( n \geq n_1 \), where we have used \( \{v_n\} \) is nonincreasing, and \( \{\eta_n\} \) is nonpositive. Since \( \beta \geq \alpha \), and \( \{v_n\} \) is positive and nonincreasing, there exists a constant \( \mathcal{L}_2 > 0 \) such that \( v_{n+1+\sigma_1}^{\beta-\alpha} \leq \mathcal{L}_2 \).

Using the last inequality in (2.2.19), and then summing the resulting inequality from \( n_1 \) to \( n-1 \), we obtain

$$\frac{\Delta r_n}{v_{n-\sigma_1}^{\alpha}} \leq \frac{-1}{\varphi_n} \sum_{s=n_1}^{n-1} (\xi_s + \mathcal{L}_2 \eta_s)$$

(2.2.20)
for all \( n \geq n_1 \). Using the monotonicity of \( \{v_n\} \) and \( \tau_1 \leq \sigma_1 \), we have
\[
r_n = v_n + \delta v_{n-\tau_1} + \gamma v_{n-\sigma_2} \leq (1 + \delta + \gamma)v_{n-\tau_1} \leq (1 + \delta + \gamma)v_{n-\sigma_1}
\] (2.2.21)
for all \( n \geq n_1 \). Now combining (2.2.20) and (2.2.21), and using the fact \( \Delta r_n \leq 0 \) for all \( n \geq n_1 \), we obtain
\[
\int_{r_n}^{r_{n+1}} \frac{ds}{s^\alpha} \leq \frac{\Delta r_n}{r_n^\alpha} \leq \frac{-1}{(1 + \delta + \gamma)^\alpha} \sum_{s=n_1}^{n-1} (\xi_s + L_2 \eta_s)
\] (2.2.22)
for all \( n \geq n_1 \). Summing the last inequality from \( n_1 \) to \( n-1 \) yields
\[
\int_{r_n}^{r_{n+1}} \frac{ds}{s^\alpha} \leq \frac{-1}{(1 + \delta + \gamma)^\alpha} \sum_{s=n_1}^{n-1} \frac{1}{\phi_s} \sum_{u=n_1}^{s-1} (\xi_u + L_2 \eta_u)
\]
or
\[
\int_{r_n}^{r_{n+1}} \frac{ds}{s^\alpha} \geq \frac{1}{(1 + \delta + \gamma)^\alpha} \sum_{s=n_1}^{n-1} \frac{1}{\phi_s} \sum_{u=n_1}^{s-1} (\xi_u + L_2 \eta_u).
\]
Now by condition (2.2.18), we see that
\[
\limsup_{n \to \infty} \int_{r_n}^{r_{n+1}} \frac{ds}{s^\alpha} = \infty
\]
which contradicts \((H_{10})\). This completes the proof of the theorem. \( \square \)

**Theorem 2.2.8.** Assume that the conditions \((H_1), (H_6), (H_8), (H_9), \) and \((H_{10})\) hold. If
\[
\limsup_{n \to \infty} \sum_{s=n_0}^{n-1} \frac{1}{\phi_s} \sum_{u=n_0}^{s-1} (\xi_u + \eta_u) = \infty,
\] (2.2.23)
then \( \mathcal{M}^- = \phi \).

**Proof.** The proof is similar to that of Theorem 2.2.7, and hence the details are omitted. \( \square \)

In the following theorem, we examine the existence/nonexistence of solutions in the class \( \mathcal{WOS} \).
Theorem 2.2.9. Assume that \((H_1)\) holds. Further assume that

\[(H_{11}) \; \xi_n \geq 0, \text{ and } \eta_n \geq 0 \text{ for all } n \geq n_0,\]

\[(H_{12}) \; \left(1 - \delta - \gamma \frac{\mathcal{R}_{n+2}}{\mathcal{R}_n}\right) > 0 \text{ for all } n \geq n_0, \text{ where } \mathcal{R}_n = \sum_{s=n_0}^{n-1} \frac{1}{\varphi_s}\]

hold. Then \(\text{WOS} = \phi\).

Proof. Let \(\{v_n\}\) be a solution of (2.1.1) in the class \(\text{WOS}\). Then by definition \(\{v_n\}\) is nonoscillatory, and \(\{\Delta v_n\}\) is oscillatory. Therefore, without loss of generality there is an integer \(n_1 \geq n_0\) such that \(v_{n-\Phi} > 0\) for all \(n \geq n_1\). From (2.1.1), we have

\[\Delta (\varphi_n \Delta r_n) = -\xi_n v_{n+1-\sigma_1} - \eta_n v_{n+1+\sigma_2} \leq 0, \; n \geq n_1.\]

Hence \(\{\varphi_n \Delta r_n\}\) is nonincreasing, and of one sign for all \(n \geq n_1\). Since \(\varphi_n > 0\), we have either \(\Delta r_n > 0\) or \(\Delta r_n < 0\) for all \(n \geq n_1\). First assume that \(\Delta r_n > 0\) for all \(n \geq n_1\). Since

\[r_n = r_{n_1} + \sum_{s=n_1}^{n-1} \frac{\varphi_s \Delta r_s}{\varphi_s} \geq \varphi_n \Delta r_n \mathcal{R}_n\]

or

\[r_n - \varphi_n \Delta r_n \mathcal{R}_n \geq 0 \text{ for all } n \geq n_1. \tag{2.2.24}\]

Now

\[\Delta \left(\frac{r_n}{\mathcal{R}_n}\right) = \frac{\varphi_n \Delta r_n - r_n}{\mathcal{R}_n \mathcal{R}_{n+1} \varphi_n} \leq 0, \; n \geq n_1,\]

by using (2.2.24). Therefore \(\left\{\frac{r_n}{\mathcal{R}_n}\right\}\) is nonincreasing for all \(n \geq n_1\). Since \(r_n\) is increasing and \(\frac{r_n}{\mathcal{R}_n}\) is nonincreasing, we have

\[v_n \geq \left(1 - \delta - \gamma \frac{\mathcal{R}_{n+\sigma_2}}{\mathcal{R}_n}\right) r_n, \; n \geq n_1. \tag{2.2.25}\]

From the last inequality, and \((H_{12})\) we conclude that \(\{v_n\}\) is nondecreasing for all \(n \geq n_1\), which is a contradiction.
Next assume that $\Delta r_n < 0$ for all $n \geq n_1$. From the definition of $r_n$, we have $r_n \geq v_n$, and since $r_n$ is decreasing we have $\{v_n\}$ is decreasing for all $n \geq n_1$, which is again a contradiction. Now the proof is completed. 

\[ \square \]

At last, we examine the existence/nonexistence of solutions in the class $\mathcal{OS}$.

**Theorem 2.2.10.** Assume that the conditions $(H_1)$, $(H_5)$, $(H_{11})$, and (2.2.11) hold. If

\[ \sum_{n=n_0}^{\infty} (A_{n-\sigma_1}^\alpha \xi_n + A_{n+\sigma_2}^\beta \eta_n) = \infty, \]  

(2.2.26)

where $\varphi_n = \left(1 - \delta - \gamma \frac{R_{n+1} + \rho_2}{R_n}\right)$ such that $A_{n-\sigma_1} > 0$ and $A_{n+\sigma_2} > 0$ for all $n \geq n_0$, then the class $\mathcal{OS} \neq \emptyset$.

**Proof.** Let $\{v_n\}$ be a nonoscillatory solution of (2.1.1). Then without loss of generality, we may assume that $v_{n-\Phi} > 0$ for all $n \geq n_1 \geq n_0$. (The proof for the case $v_n < 0$ for all $n \geq n_1$ is similar). Proceeding as in Theorem 2.2.9, we see that $\Delta r_n > 0$ or $\Delta r_n < 0$ for all $n \geq n_1$. Also from the proof of Theorem 2.2.4, the condition (2.2.11) implies that the case $\Delta r_n < 0$ for all $n \geq n_1$ cannot happen. Next, we consider the case $\Delta r_n > 0$ for all $n \geq n_1$. From (2.1.1), $(H_{12})$, and (2.2.25) we have

\[ \Delta(\varphi_n \Delta r_n) + A_{n-\sigma_1}^\alpha \xi_n r_{n+1-\sigma_1}^\alpha + A_{n+\sigma_2}^\beta \eta_n r_{n+1+\sigma_2}^\beta \leq 0, \]  

(2.2.27)

for all $n \geq n_1$. Define

\[ w_n = \frac{\varphi_n \Delta r_n}{r_{n-\sigma_1}^\alpha}, \quad n \geq n_1. \]

Then $w_n > 0$ for all $n \geq n_1$, and

\[ \Delta w_n \leq -A_{n-\sigma_1}^\alpha \xi_n - A_{n+\sigma_2}^\beta \eta_n r_{n+1+\sigma_2}^\beta - \frac{\varphi_n \Delta r_n}{r_{n+1-\sigma_1}^\alpha} \Delta r_{n-\sigma_1}^\alpha \leq -A_{n-\sigma_1}^\alpha \xi_n - A_{n+\sigma_2}^\beta \eta_n r_{n+1+\sigma_2}^\beta, \quad n \geq n_1 \]  

(2.2.28)
where we have used \( r_n > 0 \), and nondecreasing for all \( n \geq n_1 \). Since \( A_{n+\sigma_2} > 0 \), \( \beta \geq \alpha \), and \( \{r_n\} \) is positive and nondecreasing, there exists an integer \( n_2 \geq n_1 \) such that \( r_{n+1+\sigma_2} \geq L_1 > 0 \) for all \( n \geq n_2 \). Using the last inequality in (2.2.28), and then summing the resulting inequality from \( n_2 \) to \( n-1 \), we obtain

\[
\sum_{s=n_2}^{n-1} (A_{s+\sigma_1}^\alpha \xi_s + L_1 A_{s+\sigma_2}^\beta \eta_s) \leq w_{n_2} - w_n \leq w_{n_2}.
\]

Letting \( n \to \infty \) in the above inequality, we obtain a contradiction to (2.2.26). Hence \( \{v_n\} \in OS \). This completes the proof.

\[\square\]

\section{2.3 Behavior of solutions in the classes \( \mathcal{M}^+ \) and \( \mathcal{M}^- \)}

Now we study the behavior of solutions of (2.1.1) in the classes \( \mathcal{M}^+ \) and \( \mathcal{M}^- \).

\textbf{Theorem 2.3.1.} Assume that conditions \((H_1),(H_5),(H_8)\), and \((H_9)\) hold. If condition (2.2.18) holds, then for every solution \( \{v_n\} \) of (2.1.1) in the class \( \mathcal{M}^- \), we have

\[
\lim_{n \to \infty} v_n = 0.
\]

\textit{Proof.} Proceeding as in Theorem 2.2.7, we have

\[
\limsup_{n \to \infty} \int_{r_n}^{r_{n+1}} \frac{ds}{s^\alpha} = \infty
\]

which means \( r_n \to 0 \) as \( n \to \infty \). Since \( 0 < v_n \leq r_n \) we obtain \( \lim_{n \to \infty} v_n = 0 \). The proof is now completed. \(\square\)

\textbf{Theorem 2.3.2.} Assume that conditions \((H_1),(H_6),(H_8)\), and \((H_9)\) hold. If condition (2.2.23) holds, then for every solution \( \{v_n\} \) of (2.1.1) in the class \( \mathcal{M}^- \), we have

\[
\lim_{n \to \infty} v_n = 0.
\]
Proof. The argument used in Theorem 2.2.8, leads to

\[ \limsup_{n \to \infty} \int_{r_n}^{r_{n+1}} \frac{ds}{s^{\alpha}} = \infty, \]

which implies that \( r_n \to 0 \) as \( n \to \infty \). Since \( 0 < v_n \leq r_n \), we have \( \lim_{n \to \infty} v_n = 0 \). The proof is now completed.

Finally, we examine the asymptotic behavior of solutions of (2.1.1) in the class \( M^+ \).

**Theorem 2.3.3.** Assume that the conditions \((H_1),(H_2)\) and \((H_3)\) hold. If

\[ \limsup_{n \to \infty} \sum_{s=n_0}^{n-1} (\mathcal{L} \xi_s + \eta_s) \sum_{k=n_0}^{s} \frac{1}{\varphi_k} = \infty \]  \hspace{1cm} \text{(2.3.1)}

for any \( \mathcal{L} > 0 \), then every solution of (2.1.1) in the class \( M^+ \) is unbounded.

**Proof.** Let \( \{v_n\} \) be a solution of (2.1.1) in the class \( M^+ \). Without loss of generality, we may assume that there exists an integer \( n_1 \geq n_0 \) such that \( v_n > 0 \), and \( \Delta v_n \geq 0 \) for all \( n \geq n_1 \) (Since the proof for the other case is similar). Then by \((H_1)\) we have \( r_n > 0 \), and \( \Delta r_n \geq 0 \) for all \( n \geq n_1 \). Define a sequence \( \{w_n\} \) by

\[ w_n = -\frac{\varphi_n \Delta r_n}{v_n^\beta + \sigma} \sum_{s=n_1}^{n-1} \frac{1}{\varphi_s} \]

for all \( n \geq n_2 \geq n_1 \). Then \( w_n < 0 \) for all \( n \geq n_2 \), and

\[ \Delta w_n = -\Delta r_n \frac{1}{v_n^\beta + \sigma} - \Delta (\varphi_n \Delta r_n) \sum_{s=n_1}^{n} \frac{1}{\varphi_s} + \Delta r_{n+1} \Delta v_n^\beta \sum_{s=n_1}^{n} \frac{1}{\varphi_s} \]

\[ \geq -\Delta r_n \frac{1}{v_n^\beta + \sigma} + (\mathcal{L} \xi_n + \eta_n) \sum_{s=n_1}^{n} \frac{1}{\varphi_s} \]

for all \( n \geq n_2 \). Summing the last inequality from \( n_2 \) to \( n - 1 \), we obtain

\[ w_n \geq w_{n_2} + \sum_{s=n_2}^{n-1} (\mathcal{L} \xi_s + \eta_s) \sum_{k=n}^{s} \frac{1}{\varphi_k} - \sum_{s=n_2}^{n-1} \frac{\Delta r_s}{v_s^\beta + \sigma}. \] \hspace{1cm} \text{(2.3.2)}
Since \( \frac{\Delta r_n}{v_{n+\sigma_2}} > 0 \) for all \( n \geq n_2 \), \( \lim_{n \to \infty} \sum_{s=n_2}^{n-1} \frac{\Delta r_s}{v_{s+\sigma_2}} \) exists, and we prove that it is infinite. If not, then
\[
\lim_{n \to \infty} \sum_{s=n_2}^{n-1} \frac{\Delta r_s}{v_{s+\sigma_2}} = M_2 < \infty.
\]
From (2.3.1) and (2.3.2), we see that \( \lim_{n \to \infty} w_n = \infty \), which is a contradiction to the negativity of \( w_n \) for all \( n \geq n_2 \). Hence
\[
\lim_{n \to \infty} \sum_{s=n_2}^{n-1} \frac{\Delta r_s}{v_{s+\sigma_2}} = \infty.
\] (2.3.3)
Now using \( \Delta v_n \geq 0 \), we have
\[
\sum_{s=n_2}^{n-1} \frac{\Delta r_s}{v_{s+\sigma_2}} \leq \sum_{s=n_2}^{n-1} \Delta r_s \leq M_3 (r_n - r_{n_2}).
\] (2.3.4)
From (2.3.3) and (2.3.4), we conclude that
\[
\lim_{n \to \infty} r_n = \infty. \tag{2.3.5}
\]
Now from the definition of \( r_n \) and monotonicity of \( \{v_n\} \), we have
\[
r_n = v_n + \delta v_{n-\tau_1} + \gamma v_{n-\tau_2} \leq (1 + \delta + \gamma) v_{n+\tau_2}. \tag{2.3.6}
\]
From (2.3.5) and (2.3.6), we conclude that \( \lim_{n \to \infty} v_n = \infty \), and the proof is completed. \( \square \)

**Theorem 2.3.4.** Assume that the conditions \((H_1), (H_4), (H_5)\) hold. If
\[
\limsup_{n \to \infty} \sum_{s=n_0}^{n-1} (\xi_s + \mathcal{L}_1 \eta_s) \sum_{k=n_0}^{s} \frac{1}{\phi_k} = \infty
\]
for any \( \mathcal{L}_1 > 0 \), then every solution of (2.1.1) in the class \( \mathcal{M}^+ \) is unbounded.

**Proof.** The proof is similar to that of Theorem 2.3.3, and hence the details are omitted. \( \square \)

**Theorem 2.3.5.** Assume that the assumptions \((H_1), (H_4), (H_6)\) hold. If
\[
\limsup_{n \to \infty} \sum_{s=n_0}^{n-1} (\xi_s + \eta_s) \sum_{k=N}^{s} \frac{1}{\phi_k} = \infty,
\]
then every solution of (2.1.1) in the class \( \mathcal{M}^+ \) is unbounded.
2.4 Examples

In this section, we present some examples to illustrate the main results.

Example 2.4.1. Consider the second-order nonlinear neutral difference equation
\[
\Delta(2^n \Delta(v_n + \frac{1}{2}v_{n-1} + 2v_{n+1})) - \frac{2^{n+1} - 1}{(2^n - 4)^3}v_{n-2}^3 + \frac{1}{2^{2n-1}}v_{n+1} = 0, \ n \geq 3.
\] (2.4.1)

Here \(\varphi_n = 2^n\), \(\delta = \frac{1}{2}\), \(\gamma = 2\), \(\xi_n = -\frac{2^{n+1} - 1}{(2^n - 4)^3}\), \(\eta_n = \frac{1}{2^{2n-1}}\), \(\tau_1 = \tau_2 = 1\), \(\sigma_1 = 3\), \(\sigma_2 = 0\), \(\alpha = 3\) and \(\beta = 1\). It is easy to verify that all conditions of Theorem 2.2.1 are satisfied except condition (2.2.1). Therefore \(M^+\) may not be an empty set. We see that (2.4.1) has a solution \(\{v_n\} = \{\frac{2^n - 1}{2^n}\} \in M^+\) since it satisfies (2.4.1).

Example 2.4.2. Consider the second-order nonlinear neutral difference equation
\[
\Delta(\frac{1}{n} \Delta(v_n + 4v_{n-2} + 5v_{n+1})) + \frac{4}{n(n^2 - 1)}v_{n-1} + \frac{6}{n(n+1)(n+2)^3}v_{n+2}^3 = 0, \ n \geq 2.
\] (2.4.2)

Here \(\varphi_n = \frac{1}{n}\), \(\delta = 4\), \(\gamma = 5\), \(\xi_n = \frac{4}{n(n^2 - 1)}\), \(\eta_n = \frac{6}{n(n+1)(n+2)^3}\), \(\tau_1 = 2\), \(\tau_2 = 1\), \(\sigma_1 = 2\), \(\sigma_2 = 1\), \(\alpha = 1\) and \(\beta = 3\). It is easy to verify that all conditions of Theorem 2.2.2 are satisfied except the condition (2.2.5). Therefore \(M^+\) need not be an empty set. We see that (2.4.2) has a solution \(\{v_n\} = \{n\} \in M^+\) since it satisfies (2.4.2).

Example 2.4.3. Consider the second-order nonlinear neutral difference equation
\[
\Delta(\frac{1}{4n} \Delta(v_n + v_{n-1} + v_{n+1})) + \frac{3}{2^{2n+1}}v_{n-1} + \frac{1}{2^{2n+2}}v_{n+2} = 0.
\] (2.4.3)

Here \(\varphi_n = \frac{1}{4}\), \(\delta = \gamma = 1\), \(\xi_n = \frac{3}{2^{2n+1}}\), \(\eta_n = \frac{1}{2^{2n+2}}\), \(\tau_1 = \tau_2 = 1\), \(\sigma_1 = 2\), \(\sigma_2 = 1\), \(\alpha = 1\) and \(\beta = 1\). It is easy to verify that all conditions of Theorem 2.2.3 are satisfied except the
condition (2.2.10). Therefore \( M^+ \) need not be an empty. We see that (2.4.3) has a solution \( \{ v_n \} = \{ 2^n \} \in M^+ \) since it satisfies (2.4.3).

**Example 2.4.4.** Consider the second-order nonlinear neutral difference equation

\[
\Delta(n(n-1)\Delta(v_n - 3v_{n-1} + v_{n+1})) + \frac{n}{n-1}v_{n-1} + \frac{n}{(n+2)^3}v_{n+2}^3 = 0, \quad n \geq 2.
\] (2.4.4)

Here \( \varphi_n = n(n-1) \), \( \delta = -3 \), \( \gamma = 1 \), \( \xi_n = \frac{n}{n-1} \), \( \eta_n = \frac{n}{(n+2)^3} \), \( \tau_1 = \tau_2 = 1 \), \( \sigma_1 = 2 \), \( \sigma_2 = 1 \), \( \alpha = 1 \) and \( \beta = 3 \). It is easy to verify that all conditions of Theorem 2.2.5 are satisfied except the conditions \((H_7)\) and (2.2.11). Therefore \( M^+ \neq \phi \). In fact, it has a solution \( \{ v_n \} = \{ n \} \in M^+ \).

**Example 2.4.5.** Consider the second-order nonlinear neutral difference equation

\[
\Delta(4^n\Delta(v_n + \frac{1}{2}v_{n-1} + 4v_{n+2})) + \frac{7}{4}4^n v_n - 4^n v_{n+2} = 0, \quad n \geq 1.
\] (2.4.5)

Here \( \varphi_n = 4^n \), \( \delta = \frac{1}{2} \), \( \gamma = 4 \), \( \xi_n = \frac{7}{4}4^n \), \( \eta_n = -4^n \), \( \alpha = \beta = 1 \), \( \tau_1 = \tau_2 = 1 \), \( \sigma_1 = \sigma_2 = 1 \) and \( \alpha = \beta = 1 \). One can easily verify that all conditions of Theorem 2.2.8 are satisfied except condition \((H_{10})\). Therefore there may solution in the class \( M^- \). In fact, equation (2.4.5) has a solution \( \{ \frac{1}{2^n} \} \in M^- \).

**Example 2.4.6.** Consider the second-order nonlinear neutral difference equation

\[
\Delta(n\Delta(v_n + \frac{1}{4}v_{n-2} + \frac{1}{4}v_{n+1})) + nv_n + (5n + 2)v_{n+1}^3 = 0, \quad n \geq 1.
\] (2.4.6)

Here \( \varphi_n = n \), \( \delta = \gamma = \frac{1}{4} \), \( \xi_n = n \), \( \eta_n = (5n+2) \), \( \tau_1 = 2 \), \( \tau_2 = 1 \), \( \sigma_1 = 1 \), \( \sigma_2 = 0 \), \( \alpha = 1 \) and \( \beta = 3 \). Since \( \mathcal{R}_n = \frac{n(n-1)}{2} \). It is easy to see that all conditions of Theorem 2.2.10 are satisfied and hence the class \( \mathcal{O}S \) is non-empty. In fact \( \{ v_n \} = \{ (-1)^n \} \) is one such oscillatory solution of (2.4.6).
Example 2.4.7. Consider the difference equation (2.4.5). It is easy to see that all conditions of Theorem 2.3.2 are satisfied. Therefore (2.4.5) has a solution \( \{ v_n \} = \{ \frac{1}{\sqrt{n}} \} \in \mathcal{M}^- \) such that \( v_n \to 0 \) as \( n \to \infty \).

Example 2.4.8. Consider the difference equation (2.4.2). Then we see that all conditions of Theorem 2.3.4 are satisfied and hence (2.4.2) has a solution in \( \mathcal{M}^+ \) which is unbounded. In fact \( \{ v_n \} = \{ n \} \) is one such solution of (2.4.2) \( \in \mathcal{M}^+ \) and \( v_n \to \infty \) as \( n \to \infty \).

Example 2.4.9. Consider the difference equation (2.4.3). It can easily be seen that all conditions of Theorem 2.3.5 are satisfied. Therefore (2.4.3) has a solution \( \{ v_n \} = \{ 2^n \} \in \mathcal{M}^+ \) such that \( v_n \to \infty \) as \( n \to \infty \).

Conclusion:

In this chapter, we have obtained conditions for the nonexistence of solutions in the classes \( \mathcal{M}^+ \), \( \mathcal{M}^- \) and \( \text{WOS} \), and the existence of solutions in the class \( \text{OS} \). It would be interesting to extend the results of this chapter to the following equation

\[
\Delta \left( \varphi_n \Delta \left( v_n + \delta v_{n-\tau_1} + \gamma v_{n+\tau_2} \right) \right) + \xi_n v_{n-\sigma_1}^\alpha + \eta_n v_{n+\sigma_2}^\beta = \kappa_n
\]

where \( \{ \kappa_n \} \) is a real sequence.