As mentioned in section 1.5, this thesis will mainly focus on computation of different quantities, such as DPR, QNS and Debye Screening from the QGP phase. So, in this chapter we mainly discuss some preliminaries about the main ingredients used in this dissertation while computing the aforementioned observables. In section 2.1 we discuss basic formulation of QCD. Section 2.2 and 2.3 contains imaginary and real time formalism respectively as part of the formulation of finite temperature field theory. Basic formulations of HTLpt and GZ action are recalled in section 2.4 and 2.5. As a bridging between perturbative and non-perturbative methods, the basic structure of OPE is discussed in section 2.6. Section 2.7 provides a prelude to the magnetized medium. As an inseparable part of every computation in this dissertation, the formulation of correlation function and spectral function is studied in section 2.8. Following sections 2.9, 2.10 and 2.11 contain a detailed discussion about the importance and the formulation of DPR both in presence and in absence of an external magnetic field. This discussion will act as the foundation for the next three chapters (3, 4 and 5). We have also discussed different techniques of evaluation of DPR and their
scope in section 2.12. We conclude the chapter by briefly mentioning some generalities about QNS in section 2.13, which will act as the starting point of chapter 7.

2.1 Quantum Chromo Dynamics

As already mentioned in section 1.1, QCD is the governing theory of the strong interaction. QCD is a non-Abelian gauge theory which ideally belongs to the group $SU(3)_c$ when we consider that quark of a particular flavor can have three kinds of color associated with it ($N_c = 3$) and as mediator of strong interaction there are eight ($d_A = N_c^2 - 1 = 8$) types of non-Abelian guage fields or gluons.

The QCD Lagrangian is given by

$$L_{QCD} = \sum_f \bar{\psi}_f (i\slashed{D} - m_f) \psi_f - \frac{1}{4} F_{\mu\nu}^{c*} F_{\mu\nu}^{c},$$

where $F_{\mu\nu}^{c}$ is the field strength tensor for the non-Abelian case

$$F_{\mu\nu}^{c} = \partial_{\mu} A_{\nu}^{c} - \partial_{\nu} A_{\mu}^{c} + g f_{abc} A_{\mu}^{a} A_{\nu}^{b},$$

$A_{\mu}^{c}$ is the non-Abelian gauge field with color index $c$ and $D_{\mu} = \partial_{\mu} - ig T_{c} A_{\mu}$ is the covariant derivative. $T_{c}$’s are the generators of the non-Abelian $SU(3)_c$ group which satisfies the Lie algebra

$$[T_a, T_b] = i f_{abc} T_c,$$

with $f_{abc}$ as the antisymmetric structure constants of the group. Also, $f$ is the flavor index for quarks and in the 3-dimensional fundamental representation of $SU(3)_c$, $\psi_f$ can be
where red, blue and green are the three typical colors associated with a quark of mass $m_f$.

Appearance of the third term in Eq. (2.2) suggests that $F_{c}^{\mu\nu} F_{c}^{\mu\nu}$ term in the Lagrangian has terms like $g \partial_\nu A_\mu^a f^{abc} A_\nu^b A_\nu^c$ and $g^2 f^{abc} f^{ijk} A_\mu^b A_\nu^c A_\mu^j A_\nu^k$ which correspond to three and four point gluonic vertices in Feynman diagrams, respectively. This shows that QCD is a self-interacting theory in contrast with QED, where photons have no self interaction. To study different aspects of QCD in medium one needs to use finite temperature field theory.

In the next two sections, we will briefly discuss two such ways to incorporate the medium effects within the regime of strong interaction.

### 2.2 Imaginary time formalism

The imaginary time formalism was first proposed by Matsubara [59] which is achieved by the substitution $t = i\tau$ with $t$ being the Minkowski or real time and $\tau$ being the Euclidean or imaginary one. Physically it is realized by the Wick rotation in the complex time plane as shown in Fig. 2.1. Below we list some of the important points of ITF:

- The inverse temperature $\beta = 1/T$ is related with $\tau$ because the evolution operator $e^{\beta H}$ has the form of a time evolution operator ($e^{-i\mathcal{H}t}$) which implies $\beta = -it = \tau$ through analytic continuation.
Because of the compactness of $\beta$, $\tau$ becomes finite: $0 \leq \tau \leq \beta$. It amounts to the decoupling of space and time and the theory no longer remains Lorentz invariant.

The thermal Green’s function for $\tau > \tau'$ satisfies the following condition

$$G_\beta(\vec{x}, \vec{x}'; \tau, \tau') = \pm G_\beta(\vec{x}, \vec{x}'; \tau, \tau' + \beta),$$

which shows that within ITF the Dirac fields are anti-periodic in nature whereas the bosonic fields are periodic.

The Feynman rules in finite temperature field theory are exactly the same as in zero-temperature except that the imaginary time $\tau$ is now compact with an extent $1/T$. To go from $\tau$ to frequency space, one has to perform a Fourier series decomposition rather than a Fourier transform. That accounts for the only difference with zero-temperature Feynman
rules as loop frequency integrals are now replaced by loop frequency sums

\[ \int \frac{d^4 P}{(2\pi)^4} \rightarrow iT \sum_{p_0} \int \frac{d^3 p}{(2\pi)^3} \equiv \sum_{\omega_n} \int \frac{d^3 p}{(2\pi)^3}, \]  

(2.6)

over the discrete imaginary-time frequencies known as Matsubara frequencies

\[ p_0 = i\omega_n = 2n\pi T \quad \text{for bosons}, \]  

(2.7)

\[ p_0 = i\omega_n = (2n + 1)i\pi T + \mu \quad \text{for fermions}, \]  

(2.8)

which in turn implement the periodic and anti-periodic boundary conditions respectively.

Next we define the dimensionally regularized bosonic and fermionic sum-integrals, which will be extensively used later in chapter 4, as

\[ \sum \int_{P} \equiv \left( \frac{e^{\gamma_E} \Lambda^2}{4\pi} \right)^\epsilon iT \sum_{p_0=2n\pi T} \int \frac{d^3-2\epsilon p}{(2\pi)^3-2\epsilon}, \]  

(2.9)

\[ \sum \int_{\{P\}} \equiv \left( \frac{e^{\gamma_E} \Lambda^2}{4\pi} \right)^\epsilon iT \sum_{p_0=(2n+1)i\pi T + \mu} \int \frac{d^3-2\epsilon p}{(2\pi)^3-2\epsilon}, \]  

(2.10)

where \( 3 - 2\epsilon \) is the dimension of space, \( \gamma_E \approx 0.577216 \) the Euler-Mascheroni constant, \( \Lambda \) is an arbitrary momentum scale, \( P = (p_0, p) \) is the bosonic loop momentum, and \( \{P\} \) is the fermionic loop momentum. Because of the factor \( (e^{\gamma_E}/4\pi)^\epsilon \), after minimal subtraction of the poles in \( \epsilon \) due to ultraviolet divergences, \( \Lambda \) coincides with the \( \overline{MS} \) renormalization scheme.

Now the frequency sum in Eq. (2.6) for bosonic case is given by

\[ T \sum_{p_0} f(p_0 = i\omega_n = 2n\pi iT) = \frac{T}{2\pi i} \oint_{C} dp_0 \frac{\beta}{2} f(p_0) \coth \frac{\beta p_0}{2}, \]  

(2.11)
where the contour \( C \) can be deformed as \( C \equiv C_1 \cup C_2 \), as shown in Fig. (2.2a): The function \( \frac{\beta}{2} \coth \frac{\beta p_0}{2} \) has poles at \( p_0 = 2\pi n i T \) as shown by the crosses and is everywhere else bounded and analytic. So with the deformed contour, the frequency sum in Eq. (2.11) can be rewritten as

\[
T \sum_{p_0} f(p_0 = i\omega_n = 2n\pi i T) = \frac{1}{2\pi i} \int_{C_1} dp_0 f(p_0) \frac{1}{2} \coth \frac{\beta p_0}{2} + \frac{1}{2\pi i} \int_{C_2} dp_0 f(p_0) \frac{1}{2} \coth \frac{\beta p_0}{2} \\
= \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 f(p_0) \left[ \frac{1}{2} + n_B(p_0) \right] - \frac{1}{2\pi i} \int_{-i\infty-\epsilon}^{i\infty-\epsilon} dp_0 f(p_0) \left[ \frac{1}{2} + n_B(p_0) \right] \tag{2.12}
\]

where \( n_B(p) = 1/(\exp(\beta p) - 1) \) is Bose-Einstein distribution function, consisting of the medium effects.

One can also evaluate the frequency sum (2.6) for fermionic case in the similar manner
as

\[ T \sum_{p_0} f(p_0) = i \omega_n = (2n + 1) \pi iT + \mu = \frac{T}{2\pi i} \oint_{C'} dp_0 \frac{\beta}{2} f(p_0) \tanh \frac{\beta(p_0 - \mu)}{2} \] (2.13)

where the contour \( C' \) can be deformed in the similar manner as \( C' \equiv C_3 \cup C_4 \), as shown in the Fig. (2.2b).

The function \( \frac{\beta}{2} \tanh \frac{\beta(p_0 - \mu)}{2} \) has poles at \( p_0 = (2n + 1) \pi iT + \mu \), as shown by the crosses and is everywhere else bounded and analytic. With the deformed contour, the frequency sum in Eq. (2.13) can be rewritten as

\[
T \sum_{p_0} f(p_0) = i \omega_n = (2n + 1) \pi iT + \mu \\
= \frac{1}{2\pi i} \int_{C_3} dp_0 f(p_0) \frac{1}{2} \tanh \frac{\beta(p_0 - \mu)}{2} + \frac{1}{2\pi i} \int_{C_4} dp_0 f(p_0) \frac{1}{2} \tanh \frac{\beta(p_0 - \mu)}{2} \\
= \frac{1}{2\pi i} \int_{-i\infty + \mu + \epsilon}^{i\infty + \mu + \epsilon} dp_0 f(p_0) \left[ \frac{1}{2} - n_F(p_0 - \mu) \right] + \frac{1}{2\pi i} \int_{-i\infty + \mu - \epsilon}^{i\infty + \mu - \epsilon} dp_0 f(p_0) \left[ \frac{1}{2} - n_F(\mu - p_0) \right] \] (2.14)

where \( n_F(p) = 1/(\exp(\beta p) + 1) \) is Fermi-Dirac distribution function that takes into account the finite temperature effects.

### 2.3 Real time formalism

Real time formalism was introduced after ITF, first by Schwinger and Keldysh [60, 61] and then reformulated by Umezawa [62, 63]. Similarly as in ITF, in RTF also, the Boltzmann weight is compared with the time evolution operator. One can write \( e^{-\beta H} = e^{-iH(\tau - i\beta - \tau)} \), which can be thought to help the system evolve from time \( \tau \) to \( \tau - i\beta \), where \( \tau \) may take complex values.
The properties of thermal propagator imposes certain conditions which become restricted by the domain

\[-\beta \leq \text{Im}(\tau - \tau') \leq 0.\]  

(2.15)

Out of the various contour choices which satisfy this condition, ITF originates when time runs from 0 to \(-i\beta\). However, RTF would require one segment of the contour to run over the whole of real axis, with the two infinite intervals of time giving rise to a \(2 \times 2\) matrix as the propagator. In Fig. 2.3 a suitable time path is shown as it traverses the real axis from \(-i\) to \(+i\), then continues parallel to the imaginary axis up to \(i - i\beta/2\), going parallel to the real axis again up to \(-i - i\beta/2\) and finally again vertically to \(-i - i\beta\). This choice, shown
in Fig. 2.3, gives rise to a symmetric propagator for the case of scalar fields as:

\[
\begin{pmatrix}
D(\vec{x}, \vec{x}'; t, t') & D(\vec{x}, \vec{x}'; t, t' - i\beta/2) \\
D(\vec{x}, \vec{x}'; t - i\beta/2, t') & D(\vec{x}, \vec{x}'; t', t)
\end{pmatrix} = \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (X - X')} D(p, p_0)
\]  

(2.16)

where the momentum space scalar propagator is given by

\[
D(p, p_0) = \begin{pmatrix}
\Delta_F + 2\pi in_B \delta(P^2 - m^2) & 2\pi i \sqrt{n_B(1 + n_B)} \delta(P^2 - m^2) \\
2\pi i \sqrt{n_B(1 + n_B)} \delta(P^2 - m^2) & -\Delta_F + 2\pi in_B \delta(P^2 - m^2)
\end{pmatrix}
\]  

(2.17)

with \(n_B = n_B(\omega)\) is the Bose-Einstein distribution function and \(\Delta_F\) is the free scalar propagator. The fermionic propagator in RTF, will be used later in chapter 5. In the next section we will briefly discuss about one of the leading perturbative methods to study QCD at finite temperature and finite chemical potential, namely HTLpt.

### 2.4 HTL perturbation theory

The basic idea and the importance of HTLpt is already discussed in subsection 1.4.1. In this section we will briefly discuss the mathematical formulation of HTLpt as a foundation for chapters 4 and 7. After the idea of Hard Thermal Loop given by Braaten and Pisarski [64], HTLpt was gradually developed by Andersen, Braaten and Strickland [11, 65]. The HTLpt Lagrangian density can be written as a rearrangement of the in-medium perturbation theory for QCD. It reads as

\[
\mathcal{L} = (\mathcal{L}_{QCD} + \mathcal{L}_{HTL})|_{g \rightarrow \sqrt{g}} + \Delta \mathcal{L}_{HTL},
\]  

(2.18)
where $\Delta L_{HTL}$ is the HTL counterterm, $L_{QCD}$ is given in Eq. (2.1) and the added HTL term is [66]

$$L_{HTL} = (1 - \delta) i m_q^2 \bar{\psi} \gamma^\mu \left( \frac{x_\mu}{x \cdot D} \right)_x \psi - \frac{1}{2} (1 - \delta) m_D^2 \text{Tr} \left( G_{\mu \alpha} \left\langle \frac{x^\alpha x_\beta}{(x \cdot D)^2} \right\rangle_x G^{\mu \beta} \right),$$  \hspace{1cm} (2.19)

where $x^\mu = (1, \hat{x})$ is a light-like four-vector. The angular bracket indicates an average over the direction of the three dimensional unit vector $\hat{x}$. $m_D$ and $m_q$ can be recognized as the Debye screening mass and the thermal quark mass, respectively. They account for the screening effects, which we shall discuss in chapter 6 also. The one-loop running strong coupling, $g^2 = 4\pi\alpha_s$, is

$$g^2(T) = \frac{48\pi^2}{(33 - 2N_f) \ln \left( \frac{Q_0^2}{\Lambda_0^2} \right)},$$  \hspace{1cm} (2.20)

where $N_f$ is the number of quark flavors and $Q_0$ is the renormalization scale, which is usually chosen to be $2\pi T$ unless specified. We fix the scale $\Lambda_0$ by requiring that $\alpha_s(1.5 \text{ GeV}) = 0.326$, as obtained from lattice measurements [67]. For one-loop running, this procedure gives $\Lambda_0 = 176 \text{ MeV}$.

A HTLpt is formulated by treating $\delta$ as a formal expansion parameter. As we can see from Eq. (2.19) and Eq. (2.18), the HTLpt Lagrangian reduces to the QCD Lagrangian if we set $\delta = 1$. In HTLpt physical observables are calculated first by expanding in powers of $\delta$, then truncating at some specified order, and finally setting $\delta = 1$. For example, to obtain the one-loop or the leading order (LO) results, one has to expand the corresponding observable up to order $\delta^0$. Here we also note that HTLpt is gauge invariant order-by-order in the $\delta$ expansion. So, eventually, the results obtained are independent of the gauge-fixing parameter.
The NNLO HTLpt resummed result at finite T and chemical potential(s) is a renormalized and completely analytic expression that reproduces a host of lattice data for T > 250 MeV!

Figure 2.4: All thermodynamic results using 3-loop HTLpt compared with the lattice data. Results are quoted from [12] and most of them are used in the thesis of Najmul Haque [68].

In recent years, several novel results have been obtained using HTLpt, the most widespread result being the one-loop, two-loop and three-loop QCD equation of state and corresponding thermodynamic quantities [11, 12, 36, 65, 66, 69–85]. In addition, HTLpt has also been used to calculate several physical quantities which are relevant to the deconfined state of matter. These include the DPR [18, 86], QNS [87, 88], photon production rate [89], single
quark and quark anti-quark potentials [90–96], fermion [97], photon [98] and gluon [35,99] damping rate, jet energy loss [100–104], plasma instabilities [105, 106], and lepton asymmetry during leptogenesis [107, 108]. In Fig. 2.4 from one of our recent publication we demonstrate a set of thermodynamic quantities obtained in three-loop HTLpt, which agree remarkably well with the available LQCD data. In view of this we will use HTLpt along with the non-perturbative Gribov-Zwanziger action to evaluate the DPR in chapter 4 and the QNS in chapter 7. We will briefly introduce the GZ action and its consequences in the next section.

2.5 Gribov-Zwanziger action and its consequences

Gribov showed in 1978 [109] that in a non-Abelian gauge theory, fixing the divergence of the potential does not commute with the gauge fixing. Unfortunately, the solutions of the differential equations, which specify the gauge fixing with vanishing divergence, can have several copies (Gribov copies) or none at all. This is known as Gribov ambiguity. To resolve this ambiguity, the domain of the functional integral has to be restricted within a fundamental modular region, bounded by Gribov horizon. Following this, in 1989 Zwanziger [110] derived a local, renormalizable action for non-Abelian gauge theories which fulfills the idea of restriction. He also showed that by introducing this GZ action the divergences may be absorbed by suitable field and coupling constant renormalization. The GZ action is given by [111]

\[
S_{GZ} = S_0 + S_{\gamma G}; \quad (2.21)
\]

\[
S_0 = S_{YM} + S_{gf} + \int d^D x (\bar{\phi}^{ac}_{\mu} \partial_{\nu} D^{ab}_{\nu} \phi^{bc}_{\mu} - \bar{\omega}^{ac}_{\mu} \partial_{\nu} D^{ab}_{\nu} \omega^{bc}_{\mu}) + \Delta S_0; \quad (2.22)
\]

\[
S_{\gamma G} = \gamma^2_G \int d^D x g f^{abc} A^a_\mu (\phi^{bc}_{\mu} + \phi^{bc}_{\mu}) + \Delta S_\gamma; \quad (2.23)
\]
where \((\phi^{bc}_\mu, \bar{\phi}^{bc}_\mu)\) and \((\omega^{bc}_\mu, \bar{\omega}^{bc}_\mu)\) are a pair of complex conjugate bosonic and Grassmann fields respectively, introduced due to localization of the GZ action. \(S_{YM}\) and \(S_{gf}\) are the normal Yang-Mills and the gauge fixing terms of the action and \(D\) is the dimension of the theory. \(\Delta S_0\) and \(\Delta S_{\gamma G}\) are the corresponding counterterms of the \(\gamma G\) independent and dependent parts of the GZ action. \(\gamma G\) is called the Gribov parameter. In practice, \(\gamma G\) can be self-consistently determined using a one-loop gap equation and at asymptotically high temperatures it takes the following form\(^1\) [25, 113, 114]

\[
\gamma G = \frac{D - 1}{D} \frac{N_c}{4\sqrt{2}\pi g^2} T. \tag{2.24}
\]

We know that gluons play an important role in confinement. Using the GZ action [109, 110] the issue of confinement is usually tackled kinematically with the gluon propagator in covariant gauge taking the form [109, 110]

\[
D^{\mu\nu}(P) = \left[ \delta^{\mu\nu} - (1 - \xi) \frac{P^\mu P^\nu}{P^2} \right] \frac{P^2}{P^4 + \gamma_G^4}, \tag{2.25}
\]

where \(\xi\) is the gauge parameter. Inclusion of the term involving \(\gamma G\) in the denominator moves the poles of the gluon propagator off the energy axis so that there are no asymptotic gluon modes. Naturally, to maintain the consistency of the theory, these unphysical poles should not be considered in the exact correlation functions of gauge-invariant quantities. This suggests that the gluons are not physical excitations. In practice, this means that the inclusion of the Gribov parameter results in the effective confinement of gluons.

Though the Gribov ambiguity renders perturbative QCD calculations ambiguous, but the dimensionful Gribov parameter appearing above can acquire a well-defined meaning

---

\(^1\)Equation (2.24) is a one-loop result. In the vacuum, the two-loop result has been determined [112] and the Gribov propagator form (2.25) is unmodified. Only \(\gamma G\) itself is modified to take into account the two-loop correction. To the best of our knowledge, this would hold also at finite temperature.
if the topological structure of the $SU(3)$ gauge group is made to be consistent with the theory. Very recently, this has been argued and demonstrated by Kharzeev and Levin [115] by taking into account the periodicity of the $\theta$-vacuum [116] of the theory due to the compactness of the $SU(3)$ gauge group. The recent work of Kharzeev and Levin indicates that the Gribov term can be physically interpreted as the topological susceptibility of pure Yang-Mills theory and that confinement is built into the gluon propagator in Eq. (2.25), indicating non-propagation and screening of color charges at long distances in the running coupling. This also reconciles with the original view Zwanziger had regarding $\gamma_G$ being a statistical parameter [110].

2.6 Operator Product Expansion

In 1969, while dealing with the short distance behavior within strong interactions, Wilson proposed [117] that an ordinary product of two local fields $A(X)$ and $B(Y)$ can be expanded in the following form when the four vector $Y$ is near $X$, as

$$A(X)B(Y) = \sum_n W_n(X - Y) O_n(X), \quad (2.26)$$

where $O_n(X)$ are a set of local fields at $X$. The coefficient functions $W_n(X - Y)$, later termed as Wilson coefficients, involve powers of $(X - Y)$. Though the complete expansion of Eq. (2.26) would require an infinite number of local fields, but upon restricting upto a finite order in $(X - Y)$, the number of fields will be finite. This multipurpose proposal later became famous by the name of Operator Product Expansion.

Shifman-Vainshtein-Zakharov first argued [118, 119] that OPE is valid in presence of the non-perturbative effects [120]. By using OPE judiciously one can exploit both pertur-
bative and non-perturbative domains separately [121–123]. Unlike QED a favorable situation occurs particularly in QCD that allows us to do the power counting [124, 125]. OPE basically assumes a separation of large and short distance effects via condensates and Wilson coefficients. Also according to SVZ, the less effective nature of ordinary perturbation theory at relatively low invariant mass is a manifestation of the fact that nonperturbative vacuum condensates are appearing as power corrections in the OPE of a Green’s function. So in view of OPE, in the large-momentum (short-distance) limit, a two point current-current correlation function [118, 119, 126] can be expanded in terms of local composite operators and \( c \)-numbered Wilson coefficients as

\[
C_{\mu\nu}(P) \overset{z \to 0}{=} i \int e^{iP \cdot Z} d^4Z \langle T \{ J_\mu(Z) J_\nu(0) \} \rangle = \left( \frac{P_\mu P_\nu}{P^2} - g_{\mu\nu} \right) W(P^2, \nu^2) \langle O \rangle_D, \tag{2.27}
\]

provided \( P^2 \gg \Lambda^2 \), where \( \Lambda \) is the QCD scale and \( T \) is the time ordered product. \( D \) is the dimension of the composite operators (condensates) \( \langle O \rangle \) and those may have non-zero expectation values which are absent to all orders in perturbation theory. \( \nu \) is a scale that separates long and short distance dynamics. The power corrections appear through the Wilson coefficients \( W \) that contain all information about large momentum (short distance) physics above the scale \( \nu \), implying that those are free from any infrared and nonperturbative long distance effects. We note that for computing a correlator in vacuum(medium) one should first calculate it in a background of quark and gluonic fields and then average it with respect to these fields in the vacuum(medium) to incorporate the power corrections through relevant condensates. This is precisely the procedure we will follow in chapter 3.
2.7 Fermions in presence of constant magnetic fields

The Dirac Lagrangian density for fermions in a constant magnetic field $B$, anisotropic along the $z$ direction, has the following form,

$$\mathcal{L}_{\text{Dirac}} = \sum_f \bar{\psi}_f \left( i \tilde{D} - m_f \right) \psi_f,$$  \hspace{1cm} (2.28)

where

$$\tilde{D}_\mu = \partial_\mu + i q f A^\text{ext}_\mu,$$  \hspace{1cm} (2.29)

where $q_f$ is the absolute charge of the fermion. The external gauge field $A$ is given by

$$A^\text{ext}_\mu = \begin{cases} B \left( 0, -y, x, 0 \right) & \text{Symmetric gauge,} \\ B \left( 0, -y, 0, 0 \right) & \text{Landau gauge.} \end{cases}$$  \hspace{1cm} (2.30)

The equation of motion for $\psi$ can be deduced from Eq. (2.28), using

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \bar{\psi} \right)} \right) = 0,$$  \hspace{1cm} (2.31)

as

$$\left( i \tilde{D} - q_f A^\text{ext} - m_f \right) \psi_f = 0.$$  \hspace{1cm} (2.32)

Now, choosing stationary state solution for the wave function $\psi_f$ and putting the value

$^2 \tilde{D}_\mu$ is not to be confused with the $D_\mu$ used in Eq. (2.1).
of $A^\text{ext}_\mu$ from Eq. (2.30), one can get the energy spectrum as

$$E_n(p_z) = \pm \sqrt{p_z^2 + m_f^2 + 2nq_f B}$$

(2.33)

where $n = 0, 1, 2, \ldots$ are known as the degenerate Landau levels. As seen from Eq. (2.33), the transverse momenta $p_x$ and $p_y$ become quantized, in presence of an anisotropic magnetic field along the $z$ direction. The situation is pictorially depicted in Fig. 2.5. So, a fermion in the lowest Landau level will be quantized along the transverse direction, represented by the circle with the lowest radius among the concentric circles depicting the other Landau levels. These Landau levels can affect the quantum fluctuations of the charged fermions in the Dirac sea at $T = 0$ and thermal fluctuations at $T \neq 0$. We shall discuss more about Landau levels in chapter 5.

Figure 2.5: Orientation of Landau levels in presence of a magnetic field.
2.8 Correlation function and Spectral function

Correlation generally means a mutual relationship between multiple things. For our purpose, correlation functions are such kind of Green’s functions which describe how microscopic variables co-vary with response to one another. A CF in coordinate space \(X \equiv (t, \vec{x})\) can be represented as

\[
C_{AB}(X) = \langle T \hat{A}(X) \hat{B}(0) \rangle = \int \frac{d^4 Q}{(2\pi)^4} e^{iQ \cdot X} C_{AB}(Q),
\]

where \(T\) is the time ordering between the operators \(\hat{A}\) and \(\hat{B}\).

The vacuum properties of any particle changes while propagating through medium. Be it hot and dense medium or magnetized medium or both, its dispersion properties get modified via new attributes due to the presence of the medium. CF are extensively used in several branches of physics because all the modifications arising due to the presence of a medium are reflected in the CF of that particle. There are various kinds of CF in the literature. But we will extensively use the current-current CF which is directly related to most of the observables that will be computed in this thesis to characterize QGP. As we know, a many particle system can be characterized by studying both its static and dynamic properties. Static properties like the thermodynamic functions can be computed from the QCD equation of state itself. But to study the dynamical properties of such systems one typically provides an external perturbation to disturb the system slightly from equilibrium and correspondingly studies both the quantum fluctuations at \(T = 0\) and thermal fluctuations at \(T \neq 0\). Both of these fluctuations are related to various current-current CF. At
finite temperature the current-current CF is given by

\[
C(t, \vec{x}) = \left\langle \mathcal{T} J(t, \vec{x}) J^\dagger(0, \vec{0}) \right\rangle_\beta = iT \sum_{q_0} \int \frac{d^3 q}{(2\pi)^3} \frac{e^{iq_0 t - i\vec{q} \cdot \vec{x}}}{\beta} C(q_0, \vec{q}),
\]

(2.35)

where \( J(t, \vec{x}) = \tilde{\psi}(t, \vec{x}) \Gamma \psi(t, \vec{x}) \) with \( \Gamma = 1, \gamma_5, \gamma_\mu \) and \( \gamma_\mu \gamma_5 \) for scalar, pseudoscalar, vector and pseudovector channels respectively. \( \langle \rangle_\beta \) represents the thermal average. In Euclidean space putting \( t = i\tau \) and \( q_0 = i\omega_n \) we obtain

\[
C(\tau, \vec{x}) = T \sum_{\omega_n} \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{e^{-i(\omega_n \tau + \vec{q} \cdot \vec{x})}}{\beta} C(i\omega_n, \vec{q}),
\]

(2.36)

Fourier transform of which yields

\[
C(i\omega_n, \vec{q}) = \int_{0}^{\beta} d\tau \int d^3 \vec{x} e^{i(\omega_n \tau + \vec{q} \cdot \vec{x})} \left\langle \mathcal{T} J(\tau, \vec{x}) J^\dagger(0, \vec{0}) \right\rangle_\beta.
\]

(2.37)

Using the Kramers-Kronig relation, one can relate the real and imaginary parts of a complex function. Thus, following Eq. (2.37) we can write down the momentum space current-current CF as

\[
\text{Re} \ C(i\omega_n, \vec{q}) = \frac{1}{\pi} P \left( \int_{-\infty}^{\infty} d\omega \frac{\text{Im} \ C(i\omega_n = \omega + i\epsilon, \vec{q})}{\omega - i\omega_n} \right),
\]

(2.38)

where \( P \) represents the Cauchy principle value. Now, with the help of Eq. (2.38) and writing \( C(i\omega_n, \vec{q}) \) instead of \( \text{Re} \ C(i\omega_n, \vec{q}) \), we get the spectral representation of the momentum space current-current CF as

\[
C(i\omega_n, \vec{q}) = -\int_{-\infty}^{\infty} d\omega \frac{\rho(\omega, \vec{q})}{i\omega_n - \omega}.
\]

(2.39)
Here $\rho(\omega, \vec{q})$ is called the spectral function, which is extracted from the momentum space correlator by analytic continuation and can be defined as the following

$$\rho(\omega, \vec{q}) = \frac{1}{\pi} \text{Im} \, C(i\omega_n = \omega + i\epsilon, \vec{q})$$

identifying spectral function as the discontinuity or the imaginary part of a CF. As we know that in zero temperature imaginary parts play a crucial role in evaluating the decay rates. Likewise in medium by evaluating the discontinuities of a particular current-current CF, a SF generally carries all the information about the hadronic spectrum, coupled to the aforementioned current. This is the reason why SF is related with various physical quantities in hot and dense medium, like DPR, photon production, various susceptibilities, transport coefficients and damping rates. Importance of SF will be more and more lucid with the advancement of this dissertation.

![One loop photon self energy diagram](image)

Figure 2.6: One loop photon self energy diagram. The wavy line corresponds to photon.

The computation of SF is easily handled in the perturbative QCD using the Feynman diagrams. As a demonstration, in Fig 2.6 the electromagnetic polarization tensor or the one loop self energy diagram for photon is shown. In one loop level in QED it can be expressed
as

\[
\Pi_{\mu\nu}(P) = -i \sum_f q_f^2 \int \frac{d^4 K}{(2\pi)^4} \text{Tr}_c \left[ \gamma_\mu S(K) \gamma_\nu S(Q) \right],
\]

(2.41)

where \( P \) is the external momentum, \( K \) and \( Q = K - P \) are the loop momenta. \( \text{Tr}_c \) represents both color and Dirac traces whereas the \( \sum_f \) is over flavor.

![Figure 2.7: Discontinuity of one loop photon self energy diagram, from which the information about hadronic spectrum can be extracted.](image)

The two point current-current CF \( C_{\mu\nu}(P) \) is related to photon self-energy as

\[
q_f^2 C_{\mu\nu}(P) = \Pi_{\mu\nu}(P),
\]

(2.42)

with \( q_f \) is the electric charge of a given quark flavor \( f \). The electromagnetic spectral function for a given flavor \( f \), \( \rho_f(P) \), is extracted from the timelike discontinuity of the two point correlation function \( C_{\mu\nu}^f(P) \) as

\[
\rho_f(P) = \frac{1}{\pi} \text{Im} \left( C_{\mu\nu}^f(P) \right)_f = \frac{1}{\pi} \text{Im} \left( \Pi_{\mu\nu}^f(P) \right)_f / q_f^2,
\]

(2.43)
which is shown in Fig 2.7. It is also very clear from Fig. 2.7, how the spectral function can be used as an information carrier of the hadronic sector of any process.

However in LQCD the situation is quite different. Unfortunately the lattice techniques are solely applicable in Euclidean spacetime, while the spectral function is an inherently Minkowskian object. Though it can be obtained from the Euclidean correlator in principle, but the process of analytic continuation in the regime of lattice is ill-defined due to limited set of data. Because of this complication, in LQCD the spectral function is not defined by Eq. (2.40). Instead they proceed by the evaluation of Euclidean CF in the following way:

$$C(i\omega_n, \vec{q}) = \int_{-\infty}^{\infty} d\omega \frac{\rho(\omega, \vec{q})}{\omega - i\omega_n} = \int_{-\infty}^{\infty} d\omega \rho(\omega, \vec{q}) \left(1 + n_B(\omega)\right) \frac{e^{-\beta\omega} - 1}{i\omega_n - \omega}. \quad (2.44)$$

For bosons $\omega_n = 2n\pi T$ and one can further write

$$C(i\omega_n, \vec{q}) = \int_{-\infty}^{\infty} d\omega \rho(\omega, \vec{q}) \left(1 + n_B(\omega)\right) \frac{e^{\beta(\omega_n - \omega)} - 1}{i\omega_n - \omega} = \int_{0}^{\beta} d\tau' e^{i\omega_n \tau'} \int_{-\infty}^{\infty} d\omega \rho(\omega, \vec{q}) \left(1 + n_B(\omega)\right) e^{-\omega \tau'}. \quad (2.45)$$

Now, Fourier transform of the temporal part yields

$$C(\tau, \vec{q}) = T \sum_{n=-\infty}^{\infty} C(i\omega_n, \vec{q}) e^{-i\omega_n \tau} = \int_{-\infty}^{\infty} d\omega \rho(\omega, \vec{q}) \left(1 + n_B(\omega)\right) \int_{0}^{\beta} d\tau' \delta(\tau' - \tau) e^{-\omega \tau'}$$

$$= \int_{-\infty}^{\infty} d\omega \rho(\omega, \vec{q}) \left(1 + n_B(\omega)\right) e^{-\omega \tau}. \quad (2.46)$$
After breaking the limits in Eq. (2.46) and some straightforward mathematical manipulation one can get

\[ C(\tau, \vec{q}) = \int_{-\infty}^{\infty} d\omega \, \rho(\omega, \vec{q}) \frac{\cosh(\omega(\tau - \beta/2))}{\sinh(\omega\beta/2)}. \] (2.47)

Eq. (2.47) is then inverted to extract the SF vis-a-vis the spectral properties using a probabilistic Maximum Entropy method [21,22], which is also to some extent error prone [127]. MEM requires an ansatz for the spectral function and hence lower energy part of the spectral function becomes Ansatz dependent. More discussion about the LQCD calculations of spectral properties will be discussed in section 2.12.

### 2.9 Importance of DPR

As we know from the HIC timeline shown in Fig. 1.5, that the locally equilibrated plasma is short-lived in the collision. However, there are always some initial or final state interactions that may contaminate an observable one is interested in. In this respect the electromagnetic emissivity of the plasma in the form of real or virtual photon is particularly important. After a quark interacts with its antiparticle, real or virtual photos are created in the plasma. Real photon escapes unperturbed and virtual photon decays into a dilepton in the process. The momentum distribution of the dilepton depends on the momentum distribution of the quark and the antiquark which is eventually governed by the thermodynamic condition of the plasma. So the dilepton carries information about the transient plasma state at the moment of their production. Again, since the produced dilepton are colorless and interact only through the electromagnetic interaction their mean free path become quite large. So, it is very less probable that they would suffer further collisions after they are
produced. The very fact that they do not suffer from final state interactions and carry least contaminated information of the local equilibrium makes real or virtual photon production a desirable candidate for studying QGP. This is why, the DPR from QGP phase has been studied vividly in the last three decades [18, 26, 32, 49, 52, 56, 86, 128–151].

Even though the lepton pairs behave as free particles after production, they are produced in every stage of the collisions. So, disentangling the particular set of dileptons which originate from the QGP phase is quite tricky. To avoid this hindrance one usually talks about the total DPR all through the collision time. Thus ideally the observable to characterize QGP is the Dilepton spectrum which hydrodynamically takes care of the spacetime evolution of the DPR through the whole range of collision. But discussion of the dilepton spectrum is out of the scope of this dissertation and hence we will focus solely on the DPR in QGP phase in this thesis.

2.10 Formulation of DPR

Previously in section 1.5 we made a statement that the DPR [18] is directly related to the spectral function of the electromagnetic CF. In this section we will frame that relation.

In Fig. 2.8 a dilepton production process is shown. During the collision of two heavy nuclei, a quark ($q$) and an antiquark ($\bar{q}$) interact to produce a virtual photon $\gamma^*$, which in turn produces a $l\bar{l}$ pair as dilepton. The transition amplitude (Fig. 2.8) from an initial state $I$, composed of quarks and gluons to a final state $F$ of similar composition, along with the emission of a dilepton $l(P_1)$ and $\bar{l}(P_2)$ is given as

$$\langle F, l(P_1), \bar{l}(P_2)|S|I \rangle,$$  \hspace{1cm} (2.48)
where the scattering matrix operator $S$ is given by the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = (J_{l}^{\mu}(X) + J_{q}^{\mu}(X)) A_{\mu}(X),$$

(2.49)

of lepton and quark currents coupled to the electromagnetic field $A_{\mu}(X)$. This eventually yields

$$\langle F, l(P_{1}), \bar{l}(P_{2}) | S | I \rangle = e_{0} \bar{u}(P_{2}) \gamma_{\mu} v(P_{1}) \int d^{4}X e^{i Q \cdot X} \langle F | A_{\mu}(X) | I \rangle,$$

(2.50)

where $e_{0}$ is the unrenormalized charge and $\bar{u}(P_{2})(v(P_{1}))$ represents the incoming (outgoing) antilepton (leptons) in Fig. 2.8.

In ultra relativistic HIC the process of dilepton production rapidly thermalizes within a very short period of time. Thermalization discards the possibility of a single specific initial state and proposes an ensemble average over all possible initial states, each weighted by a Boltzmann factor. For high energetic dileptons produced by a single virtual photon with energy $q_{0} = E_{1} + E_{2}$ and spatial momentum $q = p_{1} + p_{2}$, the thermally averaged dilepton multiplicity in the local rest frame of the plasma is given by

$$N = \sum_{I} \sum_{F} |\langle F, l(P_{1}), \bar{l}(P_{2}) | S | I \rangle|^{2} e^{-\beta q_{0}} \frac{V d^{3}p_{1} V d^{3}p_{2}}{Z (2\pi)^{3} (2\pi)^{3}},$$

(2.51)

where, $Z = \text{Tr} \left[ e^{-\beta H} \right]$ is the canonical partition function. Now, after some straightforward mathematical steps following Eq. (2.50) and Eq. (2.51), the dilepton multiplicity per unit space-time volume can be obtained as [129]

$$\frac{dN}{d^{4}X} = 2\pi e^{2} e^{-\beta q_{0}} L_{\mu\nu} \rho^{\mu\nu} \frac{d^{3}p_{1}}{(2\pi)^{3} E_{1}} \frac{d^{3}p_{2}}{(2\pi)^{3} E_{2}}.$$
As expected from Fig. 2.8, the multiplicity consists of a hadronic part $\rho^{\mu\nu}$, a leptonic part $L_{\mu\nu}$ and a phase space part. The hadronic part is given by,

$$\rho^{\mu\nu}(q_0, \mathbf{q}) = -\frac{1}{\pi} \frac{e^{\beta q_0}}{e^{\beta q_0} - 1} \text{Im}[D^\mu_\nu(q_0, \mathbf{q})]$$

$$\equiv -\frac{1}{\pi} \frac{e^{\beta q_0}}{e^{\beta q_0} - 1} \frac{e^2}{Q^4} \text{Im}[C^\mu_\nu(q_0, \mathbf{q})], \quad (2.53)$$

where $C^\mu_\nu$ is the photonic two point current-current CF, whereas $D^\mu_\nu_R$ represents the photon propagator and they are related by the well known Dyson-Schwinger equation. [129, 130]. Also the notation $e_e^2$ represents the effective charge of the system.
Again, the leptonic part is given by the spin sum over the Dirac spinors as

\[
L_{\mu\nu} = \frac{1}{4} \text{Tr} \left[ \bar{u}(P_2) \gamma_{\mu} v(P_1) \bar{v}(P_1) \gamma_{\nu} u(P_2) \right]
\]

\[
= P_{1\mu} P_{2\nu} + P_{1\nu} P_{2\mu} - (P_1 \cdot P_2 + m_l^2) g_{\mu\nu}.
\]

(2.54)

So, putting together the hadronic and the leptonic part, the expression for the dilepton multiplicity comes out to be

\[
\frac{dN}{d^4X} = 2\pi e^2 e^{-\beta q_0} \rho_{\mu\nu} \frac{d^3p_1}{(2\pi)^3 E_1} \frac{d^3p_2}{(2\pi)^3 E_2} \int d^4 p_1 E_1 d^4 p_2 E_2 \delta^4(P_1 + P_2 - Q) L_{\mu\nu},
\]

(2.55)

After getting the expression of dilepton multiplicity, now the differential DPR can be obtained by differentiating the multiplicity by the four momentum \( Q \) of the virtual photon \( \gamma^* \) and is expressed as

\[
\frac{dN}{d^4X d^4Q} = 2\pi e^2 e^{-\beta q_0} \rho_{\mu\nu} \frac{d^3p_1}{(2\pi)^3 E_1} \frac{d^3p_2}{(2\pi)^3 E_2} \int d^4 p_1 E_1 d^4 p_2 E_2 \delta^4(P_1 + P_2 - Q) L_{\mu\nu},
\]

(2.56)

\[
= \frac{\alpha_{\text{em}}}{8\pi^4} e^{-\beta q_0} \rho_{\mu\nu} I_{\mu\nu},
\]

where,

\[
I_{\mu\nu}(Q) = \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \delta^4(P_1 + P_2 - Q) L_{\mu\nu},
\]

(2.57)

and \( \alpha_{\text{em}} \) is the electromagnetic fine structure constant.

The evaluation of the integral \( I_{\mu\nu} \) for nonzero leptonic mass is done in the Appendix A. As throughout this dissertation we will work with massless leptons, here we present the
expression of $I_{\mu\nu}$ for $m_l = 0$ as

$$I_{\mu\nu}(Q)\bigg|_{m_l = 0} = \frac{2\pi}{3} \left( Q_{\mu}Q_{\nu} - Q^2 g_{\mu\nu} \right) \quad (2.58)$$

Now, putting the expression of 2.56, we obtain the finite temperature differential DPR as

$$\frac{dN}{d^4X d^4Q} = \frac{\alpha_{em}}{8\pi^4} e^{-\beta q_0} \rho^{\mu\nu} \frac{2\pi}{3} \left( Q_{\mu}Q_{\nu} - Q^2 g_{\mu\nu} \right)$$

$$= \frac{\alpha_{em}}{12\pi^3} e^{-\beta q_0} \rho^{\mu\nu} \left( Q_{\mu}Q_{\nu} - Q^2 g_{\mu\nu} \right)$$

$$= \frac{\alpha_{em}}{12\pi^3} \frac{e^2}{e^\beta q_0 - 1} \frac{Q^2 g_{\mu\nu} - Q_{\mu}Q_{\nu}}{Q^4} \left( \frac{1}{\pi} \text{Im} \left[ C^{\mu\nu}(q_0, \mathbf{q}) \right] \right)$$

$$= \frac{\alpha_{em} e^2}{12\pi^3} \frac{n_B(q_0)}{Q^2} \left( \frac{1}{\pi} \text{Im} \left[ C_\mu^{\mu}(q_0, \mathbf{q}) \right] \right) \quad (2.59)$$

The invariant mass of the lepton pair is $M^2 = Q^2 = q_0^2 - q^2$. The dilepton rate in Eq. (2.59) is valid only at leading order in $\alpha_{em}$ but to all orders in strong coupling constant $\alpha_s$. The lepton masses are neglected in Eq. (2.59).

Now if we consider a two-flavor case, $N_f = 2$,

$$e^2_c = \frac{1}{N_f} \sum_f e^2_f = \frac{5}{18} e^2 = \frac{5 \times 4\pi \alpha_{em}}{18}.$$ $$\therefore \frac{dN}{d^4X d^4Q} = \frac{5\alpha_{em}^2 n_B(q_0)}{54\pi^2} \left( \frac{1}{\pi} \text{Im} \left[ C_\mu^{\mu}(q_0, \mathbf{q}) \right] \right). \quad (2.60)$$
2.11 Dilepton rate in presence of strong external constant magnetic field

As mentioned in chapter 1, in non-central HIC an anisotropic magnetic field is expected to be generated in the direction perpendicular to the reaction plane, due to the relative motion of the heavy-ions themselves (participants and spectators). In chapter 5 of this thesis the electromagnetic spectral properties of a hot magnetized medium will be explicitly discussed. As a stepping stone to that computation, the modification in DPR in the presence of an external magnetic field will be discussed in this section. The dilepton production from a magnetized hot and dense matter can generally be dealt with three different scenarios [144, 145]:

- Only the quarks move in a magnetized medium but not the final lepton pairs,
- Both quarks and leptons move in a magnetized medium,
- Only the final lepton pairs move in the magnetic field.

Based on our recent effort Ref [32], we discuss below all three cases one by one.

2.11.1 Quarks move in a magnetized medium but not the final lepton pairs

The first case is shown in Fig. 2.9. We hereby emphasize that this scenario is interesting and extremely relevant to noncentral heavy-ion collisions, especially for the scenario of fast decaying magnetic field [30, 31] and also for lepton pairs produced late or at the edges of
hot and dense magnetized medium so that they are unaffected by the magnetic field. In this scenario only the hadronic part $\rho^{\mu\nu}$ in Eq. (2.52) will be modified by the background constant magnetic field whereas the leptonic tensor $L_{\mu\nu}$ and the phase space factors will remain unaffected. So, the dilepton rate for massless ($m_l = 0$) leptons can then be written from Eq. (2.59) as [32]

$$
\frac{dN^m}{d^4X d^4Q} = \frac{\alpha_{\text{em}} e^2}{12\pi^3} \frac{n_B(q_0)}{Q^2} \left( \frac{1}{\pi} \text{Im} \left[ \tilde{C}_\mu(Q_{||}, Q_{\perp}) \right] \right),
$$

(2.61)

where $\tilde{C}(Q_{||}, Q_{\perp})$ represent the modified CF in presence of external anisotropic magnetic field. Detailed computations of the modification in CF for the case of both strongly and weakly magnetized hot medium are discussed in chapter 5.

### 2.11.2 Both quark and lepton move in magnetized medium

This scenario, shown in Fig. 2.10, is expected to be the most general one. To consider
such a scenario the usual DPR given in Eq. (2.59) has to be supplemented with the appropriate modification of the hadronic and leptonic tensor along with the phase space factors in a magnetized medium. Later in chapter 5 this scenario will be dealt with sufficiently strong magnetic field, which focuses only on the lowest landau level. So, in this subsection though we will briefly outline the modifications in the DPR for any landau level, but for simplicity the explicit calculation will be done only for LLL. We list below all the required modifications

- The phase space factor in presence of magnetized medium gets modified as

\[
\frac{d^3p}{(2\pi)^3 E} \to \frac{|eB|}{(2\pi)^2} \sum_{n=0}^{\infty} \frac{dp_z}{E}.
\]  

(2.62)

where \(d^2P_\perp = 2\pi|eB|\), \(e\) is the electric charge of the lepton and \(\sum_{n=0}^{\infty}\) is over all the landau levels. For strong magnetic field one is confined within LLL and \(n = 0\) only. The factor \(|eB|/(2\pi)^2\) is the density of states in the transverse direction and true for
• The hadronic part gets modified as before. The unmagnetized electromagnetic CF $C$ gets modified to magnetized $\tilde{C}$. The modification of electromagnetic CF and SF for LLL will be discussed in detail in chapter 5.

• In presence of constant magnetic field the spin of fermions is aligned along the field direction and the usual Dirac spinors $u(P)$ and $v(P)$ in Eq. (2.54) get modified \cite{152,153} by $\mathcal{P}_n u(\tilde{P})$ and $\mathcal{P}_n v(\tilde{P})$ with $\tilde{P}^\mu = (p^0, 0, 0, p^3)$. $\mathcal{P}_n$ is the projection operator at the $n$th LL. For LLL it takes a simple form

$$\mathcal{P}_0 = \frac{1 - i\gamma_1\gamma_2}{2}. \quad (2.63)$$

Now, the modification in the leptonic part in presence of a strong magnetic field can be carried out as

$$L_{\mu\nu}^m = \frac{1}{4} \sum_{\text{spins}} \text{Tr} \left[ \bar{u}(\tilde{P}_2) \mathcal{P}_0 \gamma_{\mu} \mathcal{P}_0 \bar{v}(\tilde{P}_1) \mathcal{P}_0 \gamma_{\nu} \mathcal{P}_0 u(\tilde{P}_2) \right]$$

$$= \frac{1}{4} \text{Tr} \left[ (\tilde{P}_1 + m_l) \left( \frac{1 - i\gamma_1\gamma_2}{2} \right) \gamma_{\mu} \left( \frac{1 - i\gamma_1\gamma_2}{2} \right) \times \right.$$  

$$\left. (\tilde{P}_2 - m_l) \left( \frac{1 - i\gamma_1\gamma_2}{2} \right) \gamma_{\nu} \left( \frac{1 - i\gamma_1\gamma_2}{2} \right) \right]$$

$$= \frac{1}{2} \left[ P_{1\mu} P_{2\nu} + P_{1\nu} P_{2\mu} - ((P_1 \cdot P_2)_n + m_l^2) (g_{\mu\nu} - g_{\mu\nu} - g_{1\mu} g_{1\nu} - g_{2\mu} g_{2\nu}) \right].$$

• Similarly as in the previous section here also we define an integral as

$$I_{\mu\nu}^m(Q) = \int \frac{d^3 p_1}{2E_1} \int \frac{d^3 p_2}{2E_2} \delta^4(P_1 + P_2 - Q) \times \left[ P_{1\mu} P_{2\nu} + P_{1\nu} P_{2\mu} - ((P_1 \cdot P_2)_n + m_l^2) (g_{\mu\nu} - g_{\mu\nu} - g_{1\mu} g_{1\nu} - g_{2\mu} g_{2\nu}) \right]. \quad (2.64)$$
The evaluation of this integral is done in Appendix B yielding

\[ I_{\mu\nu}^m(Q) = \frac{4\pi}{(Q_{\parallel}^2)^2} F_2(m_l, Q_{\parallel}^2) \left( Q_{\mu}^2 Q_{\nu}^2 - Q_{\parallel}^2 g_{\mu\nu} \right), \quad (2.65) \]

where from Appendix B we know that

\[ F_2(m_l, Q_{\parallel}^2) = |eB|m_l^2 \left( 1 - \frac{4m_l^2}{Q_{\parallel}^2} \right)^{-\frac{1}{2}}. \quad (2.66) \]

Putting all these together, we finally obtain the dilepton production rate from Eq. (2.59) for LLL as \[^{[32]}\]

\[ \frac{dN^m}{d^4X d^4Q} = \frac{\alpha_{\text{em}} e^2}{4\pi^3} \frac{n_B(q_0)}{Q_{\parallel}^2 Q_{\perp}^4} F_2(m_l, Q_{\parallel}^2) \left( \frac{1}{\pi} \text{Im} \left[ \tilde{C}_\mu(Q_{\parallel}, Q_{\perp}) \right] \right). \quad (2.67) \]

### 2.11.3 Only the final lepton pairs move in the magnetic field

Fig. 2.11 represents the final scenario. The production rate for this relatively rare case requires modification of the leptonic tensor in a magnetized medium but the hadronic part vis-a-vis the CF remains unmagnetized. Since we have already discussed the most general case, now it can easily be obtained as

\[ \frac{dN^m}{d^4X d^4Q} = \frac{\alpha_{\text{em}} e^2}{2\pi^3} n_B(q_0) Q_{\parallel}^2 Q_{\perp}^4 F_2(m_l, Q_{\parallel}^2) \left( \frac{1}{\pi} \text{Im} \left[ C_\mu(Q) \right] \right). \quad (2.68) \]
2.12 Evaluation of DPR

After providing a basic introduction to DPR in the three previous sections (2.9, 2.10 and 2.11), in this section, different ways of evaluating DPR will be discussed. Here we would also like to mention that the next three chapters (chapter 3, 4 and 5) of this thesis will be devoted to detailed computation of DPR via three different methods under three different circumstances.

As explicitly stated in section 2.9, dileptons are produced in every stage of HIC. One of the ways to differentiate between them from the final yield is to classify them with respect to their invariant mass $M$. The high mass dileptons are mostly produced due to collision between hard partons and they are not particularly very informative about QGP. This is because the Drell-Yan processes [19] and charmonium decays [20, 154] are the major processes in that regime. On the other hand the low mass dilepton production is enhanced [155] compared to all known sources of electromagnetic decay of the hadronic

Figure 2.11: Dilepton production when quarks remain unmagnetized but the final lepton pair move in a magnetized medium. Bold fermionic line represents magnetized lepton pair.
particles and the contribution of a radiating QGP. So, the low mass dileptons (≤ 1 GeV) possibly indicate, some nonhadronic sources and the intricacies are discussed in the literature in a more phenomenological way [56, 86, 142]. There also exists dilepton production [156] in the intermediate range of invariant mass (say 1 – 3 GeV) with optimized contribution from the QGP phase, which is not dominated by hadronic processes, but still can be treated via perturbative methods. We emphasize that the higher order perturbative calculations [139, 149–151] of the dilepton rate do not converge in a small strong coupling (g) limit. This is because the temperatures attained in recent heavy-ion collisions are not so high to make perturbative calculations applicable. However, the leading order perturbative quark-antiquark annihilation is the only dilepton rate from the QGP phase that has been used extensively in the literature. Nevertheless, this contribution is very appropriate at large invariant mass but not in low and intermediate invariant mass. In this mass regime one expects that the nonperturbative contributions could be important and substantial.

The non-perturbative effects of QCD are taken care of by the LQCD computations. As mentioned in chapter 1, LQCD has very reliably computed the nonperturbative effects associated with the bulk properties (thermodynamics and conserved density fluctuations) of the deconfined phase, around and above the pseudocritical transition temperature. Efforts have also been made in lattice within the quenched approximation of QCD [49–53, 141] and in full QCD [157, 158] for studying the structure of vector correlation functions and their spectral representations as discussed in section 2.8. Nevertheless, such studies have provided only critically needed information about various transport coefficients both at zero [50, 51] and finite [53] momentum, and the thermal dilepton rate [49, 52, 141]. Employing a free-field spectral function as an ansatz, the spectral function in the quenched approximation of QCD was obtained earlier and found to approach zero in the low-energy limit [52]. In the same work, the authors found that the lattice dilepton rate approached
zero at low invariant masses \[52\]. In a more recent LQCD calculation with larger lattice size, the authors used a Breit-Wigner form for low-energies plus a free-field form for high-energies as their ansatz for the spectral function \[49\]. The low-energy BW form of their ansatz gave a finite low-energy spectral function and low-mass dilepton rate. Nevertheless, the above discussion indicates that because of its limitations the computation of low and intermediate mass dilepton rate in LQCD is indeed a difficult task and it is also not very clear if there are structures in the low-mass dilepton rate similar to those found in the HTLpt calculation \[18\].

Given the uncertainty associated with lattice computation of dynamical quantities, e.g. spectral functions, dilepton rate, and transport coefficients, it is desirable to have some alternative approaches to evaluate the DPR as well as include the non-perturbative effects that can be handled analytically in a similar way as in resummed perturbation theory. A few such approaches are available in the literature: one approach is a semi-empirical way to incorporate non-perturbative aspects by introducing a gluon condensate in combination with the Green’s functions in momentum space, which has been proposed in e.g. Refs. \[159–164\]. An important aspect of the phase structure of QCD is to understand the effects of different condensates, which serve as order parameters of the broken symmetry phase. These condensates are non-perturbative in nature and their connection with bulk properties of QCD matter is provided by LQCD. The gluon condensate has a potentially substantial impact on the bulk properties, e.g., on the equation of state of QCD matter, compared to the quark condensate. In this approach, the effective \(n\)-point functions are related by Slavnov-Taylor identities which contain gluon condensates in the deconfined phase as hinted from lattice measurements in pure-glue QCD \[165\]. The dispersion relations with dimension-four gluon condensates in medium exhibit two massive modes \[159\] (a normal quark mode and a plasmino mode) similar to HTL quark dispersion relations. This feature
leads to sharp structures (van Hove singularities, energy gap, etc.) in the dilepton production rates \([140, 161]\) at zero momentum, qualitatively similar to the HTLpt rate \([18]\).

In chapter 3 a similar approach will be applied to incorporate the non-perturbative effects via dimension-four gluon and quark condensates within OPE to compute the intermediate mass DPR.

Using quenched LQCD, Refs. \([166, 167]\) calculated the Landau-gauge quark propagator and its corresponding spectral function by employing a two pole ansatz corresponding to a normal quark and a plasmino mode following the HTL dispersion relations \([18]\). In a very recent approach \([168]\), a Schwinger-Dyson equation has been constructed with the aforementioned Landau-gauge propagator obtained using quenched LQCD \([166, 167]\) and a vertex function related through ST identity. Using this setup the authors computed the dilepton rate from the deconfined phase and found that it has the characteristic van-Hove singularities but does not have an energy gap. In chapter 4 DPR will be evaluated following another such recent observation using the Gribov-Zwanzigor action to include the confining and nonperturbative effects in the gluon propagator.

### 2.13 QNS - some generalities

Apart from DPR or photon production, screening and fluctuations of conserved quantities have been considered as an important probe for the transient QGP Phase. In contrast with the hadronic phase, where charges are in integer units, in the deconfined QGP phase charges are associated with individual quarks in fractional units. This leads to the difference in charge fluctuations between the two phases, which are related to the corresponding susceptibilities. The QNS is of direct experimental relevance because it is related with the charge fluctuation of a system through the number fluctuation. It can be interpreted
as the response of the conserved quark number density \( n \), with infinitesimal variation in the quark chemical potentials \( \mu + \delta \mu \). In QCD thermodynamics it is defined as the second order derivative of pressure \( P \) with respect to quark chemical potential \( \mu \). But again, using the fluctuation-dissipation (FD) theorem, the QNS for a given quark flavor can also be defined from the time-time component of the current-current correlator in the vector channel [87, 88, 169–173].

The preceding discussion suggests the formulation of QNS in the following way. QNS can be generally expressed as

\[
\chi_q(T) = \left. \frac{\partial n}{\partial \mu} \right|_{\mu \rightarrow 0} = \left. \frac{\partial^2 P}{\partial^2 \mu} \right|_{\mu \rightarrow 0} = \int d^4X \langle J_0(0, \vec{x})J_0(0, \vec{0}) \rangle = \beta \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{-2}{1 - e^{-\beta \omega}} \text{Im} \Pi_{00}(\omega, \vec{0}),
\]

where \( J_0 \) is the temporal component of the vector current and \( \Pi_{00} \) is the time-time component of the vector correlator or self-energy with external four-momenta \( Q \equiv (\omega, \vec{q}) \). The above relation in Eq. (2.69) is known as the thermodynamic sum rule [172, 173] where the thermodynamic derivative with respect to the external source \( \mu \), is related to the time-time component of static correlation function in the vector channel. We will discuss more about QNS in chapter 7 where we will calculate it within HTLpt using the non-perturbative GZ action.