Chapter 4

Diffraction and attenuation effects in Čerenkov and Smith-Purcell FELs

In Chapter 2, we presented a detailed two-dimensional (2D) analysis of the surface mode supported in a single slab configuration of the CFELs. The surface mode exponentially decays in the direction perpendicular to the dielectric surface, i.e., the \( x \)-direction, and propagates in the \( z \)-direction, which is the direction of the propagation of the electron beam. Under the 2D approximation, the surface mode was assumed to have a translational invariance along the horizontal direction, i.e., the \( y \)-direction, and it was therefore non-localized in the \( (y, z) \) plane. Using the results of 2D surface mode analysis from Chapter 2, we developed a 2D non-linear analysis to study the beam-wave interaction in a single slab based CFEL driven by a flat electron beam in Chapter 3. The electron beam in this analysis was assumed to have an infinite width along the horizontal direction and a vanishing thickness in the vertical direction. Analytical expressions for the small-signal gain and growth rate of CFEL were derived for this case. Although an arbitrary width \( \Delta y \) was chosen for the electron beam, the model presented in Chapter 3 essentially describes the interaction of a surface current and a surface wave having infinite extent along the \( y \)-direction. In a realistic situation, however, the electron beam size as well as the radiation beam size will be finite along the \( y \)-direction. Size of the radiation beam will increase due to diffraction. This will affect the overlap of the radiation beam with the electron beam, resulting
in reduction of the small-signal gain, as well as the saturated power obtained in the device. For an effective beam-wave interaction, one has to ensure that the electron beam envelope remains inside the radiation beam envelope over the entire interaction region. A realistic estimate of the radiation beam size requires the inclusion of diffraction effects and a detailed three-dimensional (3D) analysis of the surface mode.

One way to invoke 3D effects is to solve the electromagnetic Helmholtz wave equation by considering the diffraction in the surface mode. Andrew and Brau [98] used this technique to study the effect of diffraction on the growth rate in a single slab based CFEL. Growth rate was found to be decreasing on the accounts of the 3D effects as compared to the 2D analysis. The analysis in Ref. [98] is however performed for uniform electron beam having infinite vertical size, and hence, is not very useful to obtain the electron beam parameters in the vertical direction, which are critical to improve the performance of the system.

We have followed a different approach to consider the diffraction effects, where a 3D surface mode is constructed by combining plane waves propagating along different directions in the \((y,z)\) plane with suitable weight factor. The surface mode constructed in this way is localized in the horizontal direction and represents a realistic situation. The technique of localization of electromagnetic modes by using superposition of plane waves is a standard technique in laser optics [110, 140, 141]. Kim and Kumar [78, 90] used this approach to study the diffraction effects in SP-FELs, and worked out the requirements on the electron beam parameters for a THz SP-FEL [78, 89]. They observed that a SP-FEL system can produce copious amount of THz radiation if a specially designed electron beam with a flat transverse profile that allows the beam to travel very close to the grating surface, is used to drive the system. They also pointed out that these criteria were not met in the earlier experimental studies of SP-FELs [69]. As a consequence, the observed output power in these experiments has been low. Observation of low output power has also been mentioned in experimental studies on single slab based CFELs at ENEA Frascati Centre [63, 64] and at the Dartmouth college[65, 66]. In this chapter, we have examined the conditions under which performance of the CFELs can be improved. For this purpose, we have extended the work of Kim and Kumar [78, 90] on SP-FELs to CFELs, and
determined the requirements on the electron beam parameters, i.e., beam size, beam emittance, and beam current for the successful operation of a THz CFEL.

Another important effect that can deteriorate the system performance is the attenuation of the surface mode due to the losses present in the system. In a CFEL, the amplitude of the surface mode attenuates as it propagates inside a lossy dielectric slab supported over a metallic structure having finite conductivity. It is well known to the microwave community that the effect of attenuation increases with the frequency of the guided surface modes in dielectric-metal hybrid structures \[122\]. Despite this, the existing model on THz CFELs have always neglected such deleterious effect by assuming it to be insignificant. In our analysis, we have found that the dielectric and Ohmic losses can even prevent a CFEL from lasing, especially in the low-gain regime. For a low gain CFEL oscillator system, the small-signal gain has a cubic dependence on the length \(L\) \[142\]. Hence, one would like to increase the interaction length to obtain a higher gain \[142\]. However, at higher interaction length, attenuation effects increase as the power in the surface mode decays by a factor of \(e^{-\alpha L}\) for a round trip, where \(\alpha\) is the field attenuation coefficient. Thus one needs to optimize the system length by considering the attenuation effects due to the dielectric and Ohmic losses present in the system. The calculations for the attenuation coefficient due to the dielectric losses and Ohmic losses were presented in Chapter 2 for a single slab based CFEL. In this chapter, we have used these results to perform a detailed optimization study for a real world THz CFEL.

Considering the effects of attenuation and diffraction, we have established 3D coupled Maxwell-Lorentz equations for the CFELs. Following an approach similar to the approach discussed for CFELs, we have set up the 3D coupled Maxwell-Lorentz equations for the SP-FELs too. Although for the case of SP-FELs, the diffraction and attenuation effects have been studied in detail in Refs. \[78, 89, 90, 93\], a detailed derivation for the 3D coupled Maxwell-Lorentz equations was not presented, which is provided in this chapter. While discussing the 3D analysis of the CFELs and SP-FELs, we have highlighted important differences between the CFELs and SP-FELs in terms of diffraction effects, and explained the fundamental reason for these differences.
This chapter is organized as follows. In Sec. 4.1.1, we have discussed the effect of attenuation due to the dielectric losses and Ohmic losses on the performance of a single slab based CFEL. We determine the properties of the surface mode, including the effect of diffraction in Sec. 4.1.2. Next, in Sec. 4.1.3, we set up the 3D coupled Maxwell-Lorentz equations for a CFEL system. Considering the effect of diffraction, the requirements on the quality of electron beam for the successful operation of such devices become very stringent, which is discussed in Sec. 4.2 for the case of a CFEL. We also discuss the techniques to relax these stringent requirements, and also the methods for production of electron beam of required quality in the same section. In Sec. 4.3, we perform an optimization study of a THz CFEL to show that with achievable beam quality, it should be possible to generate copious amount of THz radiation in this device, even after including the 3D effects and the effects due to attenuation. Next, in Sec. 4.4.1, we describe essential features of the 3D surface mode analysis in SP-FELs [78, 89] and set up the 3D coupled Maxwell-Lorentz equations for the SP-FEL system. Here, we also highlight the differences in the analyses of a SP-FEL and a CFEL. Finally, we discuss the results and conclude our analysis in Sec. 4.5.

4.1 Attenuation and diffraction effects in Čerenkov FELs

4.1.1 Attenuation effects

In this section, we will consider the effect of attenuation of the surface mode on the performance of a single slab based CFEL. The field attenuation coefficient $\alpha$ is the sum of the dielectric attenuation coefficient $\alpha^d$ and the Ohmic attenuation coefficient $\alpha^e$, which can be obtained by adding the results of Eqs. (2.29) and (2.31), as discussed in Chapter 2. We find the following expression for the total attenuation coefficient of the surface mode in a CFEL:

$$
\alpha = \frac{\gamma_R k_0 Z_0 \tan \delta (2 - \epsilon \beta_R^2) + \beta_R \epsilon^3 k_0 (1 + a^2) (2 R_x + \beta_R k_0 Z_0 d \tan \delta)}{2 Z_0 [\gamma_R (1 + \epsilon^2 a^2) + \epsilon k_0 d (1 + a^2)]},
$$

(4.1)
where \( \tan \delta \) represents tangent loss of the dielectric medium, \( R_s = \sqrt{\mu_0 \omega / 2 \sigma_{\text{cond}}} \) is surface resistance of the metal, \( \mu_0 \) is the permeability of the free-space and \( \sigma_{\text{cond}} \) represents conductivity of the metal.

In a CFEL based on a positive refractive index dielectric, the surface mode will have a positive group velocity \( v_g \) and will be amplified as it co-propagates with the electron beam in the positive \( z \)-direction \(^{[98]}\). The longitudinal component of the electric field of a 2D non-localized surface mode supported in a CFEL is given by

\[
E_z(x, z, t) = E e^{i(k_0 z - \omega t)} e^{-\Gamma x},
\]

where \( E \) is the amplitude of the field at the location of the electron beam i.e., \( x = 0 \), \( k_0 \) is the propagation wavenumber in the \( z \)-direction, and \( \Gamma \) is the attenuation constant due to evanescent nature in the \( x \)-direction. The evolution of the amplitude \( E \) of the surface mode is mathematically described by Eq. (3.26), which after including the attenuation of the surface mode can be written as:

\[
\frac{\partial E}{\partial z} + \frac{1}{\beta_g c} \frac{\partial E}{\partial t} = \frac{Z_0 \chi}{2 \beta_g \gamma R} \frac{dI}{dy} e^{-2\Gamma y} e^{-\psi} - \alpha E,
\]

Here, the subscript \( R \) is meant for the resonant particle having velocity same as the phase velocity of the surface mode, \( \psi = k_0 z - \omega t \) is the electron phase, and the second term on the right hand side represents attenuation of the surface mode due to losses present in the dielectric and metallic structures. It should to be noted that in the oscillator configuration, the electromagnetic field will be attenuated as it propagates from the beginning to the end of the dielectric slab, and also during its backward propagation from the end point to the beginning of the dielectric slab. This results in a decay in the power of the surface mode by a factor of \( e^{-4\alpha L} \) during a round trip. The input field for the next pass is \( e^{-\alpha L} \) times the field in the previous pass, which is reflected from the mirror placed at the end point. The dynamical field equation given by Eq. (4.3) together with the Lorentz equations of motion given by Eqs. (3.18) and (3.19)
need to be solved under the above mentioned conditions to obtain the saturated power in the non-linear regime.

In the linear regime, the coupled Maxwell-Lorentz equations were solved without taking the effect of attenuation in Chapter 3 and an analytical expression was obtained for the small-signal gain, which is given by Eq. (3.35). Taking into account the effect of attenuation, there will be a single trip loss given by \(1 - e^{-2\alpha L}\) in addition to the gain described by Eq. (3.35). Note that in these calculations, we have assumed the losses to be small, i.e., \(2\alpha L \ll 1\). Taking the effect of attenuation, the growth rate of a CFEL system can be written as \(\mu - \alpha\), where \(\mu\) is given by Eq. (3.43). For an optimum performance of the system, one has to maximize the net gain and the net growth rate of the system. The gain and the growth rate will also reduce due to diffraction. This is because of the partial overlap of the diffracting radiation beam and the co-propagating electron beam. To estimate the size of the diffracting radiation beam, we need to perform a full 3D analysis of the surface mode, which is presented in the following section.

### 4.1.2 Diffraction effects

Now, we consider the effect of diffraction of the surface mode in the \(y\)-direction and construct the 3D localized surface mode supported in a single slab based CFEL system. As shown in Fig. 4.1, the dielectric slab is an open structure in the \(y\)-direction. Hence, the electromagnetic surface mode supported in this configuration is expected to behave like a freely propagating radiation beam and will undergo diffraction in the \((y, z)\) plane. The diffracting electromagnetic surface mode can be constructed by combining plane waves propagating at different angles in the \((y, z)\) plane, with suitable weight function \(A(k_y)\) in \(k_y\) as

\[
E_z(x, y, z, t) = \frac{1}{\sqrt{2\pi}} \int dk_y A(k_y) e^{ik_y y} e^{-i\omega t} e^{-\Gamma' x},
\]

where \(E_z\) is the longitudinal electric field, \(\Gamma'\) is the attenuation constant due to evanescent nature in the \(x\)-direction, when the wave is propagating in the \((y, z)\) plane with wavenumbers \(k_y\) and
Figure 4.1: Schematic of a 3D configuration of CFEL driven by a flat electron beam.

$k_z$ in the $y$-direction and the $z$-direction respectively. The surface mode constructed in this way will have a variation along the $y$-direction and will represent the generalized case of surface mode given by Eq. (4.2). We now invoke the paraxial wave approximation, i.e., $k_y \ll k_z$. Considering this, the electromagnetic surface mode given by Eq. (4.4) is mainly propagating in the $z$-direction and undergoes diffraction in the $y$-direction.

In the CFEL based on an uniform and isotropic dielectric medium, the optical properties of the surface mode will remain invariant under any arbitrary rotation of the propagating wave vector in the $(y, z)$ plane. In this situation, the 2D dispersion relation of the surface mode propagating along the $z$-axis can be easily generalized to the case where the surface mode is propagating along any arbitrary direction in the $(y, z)$ plane. For a given frequency $\omega$, if the phase velocity of the surface mode propagating along the $z$-axis is $v$, we obtain the following relation between $\omega, k_y$ and $k_z$ for a surface wave propagating in the $(y, z)$ plane:

$$\omega = v \sqrt{k_y^2 + k_z^2}. \quad (4.5)$$
The wavenumber in the longitudinal direction can be written as \( k_z = k_0 + \Delta k \), where \( k_0 = \omega/v \). By using the paraxial approximation \( (k_y \ll k_z) \), we obtained the following expression for \( k_z \):

\[
k_z = k_0 \left( 1 - \frac{k_y^2}{2k_0^2} \right).
\]  

(4.6)

Note that due to the property of isotropy in the \((y, z)\) plane, \( \Gamma = \Gamma' \). We can substitute Eq. (4.6) for \( k_z \) and \( \Gamma' = \Gamma \) in Eq. (4.4) to obtain the localized surface mode in a CFEL as

\[
E_z(x, y, z, t) = \frac{e^{-i\Gamma z}e^{i(k_zz-\omega t)}}{\sqrt{2\pi}} \int A(k_y) e^{-i(k_z^2/2k_0) - ik_yy} dk_y.
\]  

(4.7)

Above expression for the longitudinal field appears as a Fourier transform in \( k_y \) of the under-braced term. If we choose \( A(k_y) = e^{-k_y^2/2\sigma_{k_y}^2} \), the integration in Eq. (4.7) is Fourier transform of a Gaussian function. Gaussian functions belong to the distinct family of functions which are self-Fourier functions [143]. Hence, the resultant of integration in Eq. (4.7) is also a Gaussian function, and we obtain the intensity for the localized Gaussian mode at \( x = 0 \) as:

\[
\text{Intensity} : E_z \times E_z^* \propto e^{-\frac{z^2}{2\sigma_{k_y}^2} + \frac{2\sigma_y^2}{1 + 2\sigma_y^2/k_0^2}}.
\]  

(4.8)

We want to emphasize that this approach can easily be generalized for higher order modes by taking Gauss-Hermite functions for \( A(k_y) \), which are also self-Fourier functions and will give higher order Gauss-Hermite modes.

Next, we analyze the transverse properties of the localized surface mode. Using Eq. (4.8), we obtain an expression for the variation of rms optical beam size \( \sigma_y \) with \( z \) as

\[
\sigma_y^2(z) = \sigma_y^2(0) \left( 1 + \frac{z^2}{Z_R^2} \right).
\]  

(4.9)
Here, $\sigma_y(0)$ is the rms optical beam waist at $z = 0$ and $Z_R$ is the Rayleigh range, which is obtained as:

$$Z_R = \frac{4\pi\sigma_y^2(0)}{\beta_R \lambda}. \quad (4.10)$$

Another quantity of interest is the product of rms beam waist size and rms angular divergence $\sigma_\theta$, which is given by

$$\sigma_y(0) \times \sigma_\theta = \frac{\beta_R \lambda}{4\pi}. \quad (4.11)$$

Note that above expressions are similar to the standard expressions for the case of Gaussian mode propagating in free-space except that $\lambda$ is replaced with $\beta_R \lambda$. It is well known in optics that for a Gaussian mode propagating in a uniform, isotropic medium, $\lambda$ gets replaced with $\lambda/n$ in the above formulas, where $n$ is the refractive index of the medium. Using $n = c/v_p$, where $v_p = \beta_R c$ is the phase velocity of light in the medium, $\lambda/n$ is same as $\beta_R \lambda$. This is thus similar to our results obtained for a surface mode supported by a uniform, isotropic dielectric slab in a CFEL.

The present analysis for the localized surface mode will be used in Sec 4.2 to estimate the required parameters of the electron beam for efficient working of the CFEL.

### 4.1.3 3D Maxwell-Lorentz equations

Next, we will extend our 2D analysis of the beam-wave interaction in a CFEL for the 3D case by setting up the 3D Maxwell-Lorentz equations. For this purpose, we will perform the analysis of the 3D localized surface mode given by Eq. (4.4) in terms of its Fourier components, which are evolving due to their interaction with the corresponding Fourier components of the current density vector of a co-propagating electron beam. We start with the generalized expression for the sinusoidal component of the beam current density: $J = J(x, y) e^{i(k_0 z - \omega t)} + \langle e^{-i\phi} \rangle + \text{c.c.,}$, where $J(x, y)$ is the dc current density, $\langle e^{-i\phi} \rangle$ indicates bunching of the electron beam due to interaction
with the surface mode and c.c. represents the complex conjugate of quantity written on the right hand side. We can express the beam current density into its Fourier components as:

\[
J = \frac{e^{i(k_z z - w_0 t)}}{\sqrt{2\pi}} \int \tilde{J}(x, k_y) e^{ik_z^2/2k_0} (e^{-i\theta}) e^{-ik_y^2/2k_0} e^{ik_y y} d k_y + \text{c.c.} \tag{4.12}
\]

Note that we have cast the integral in a form such that the Fourier component of the electromagnetic field can be understood to be evolving with the Fourier component of the electron beam current density. For further calculations, we consider a flat electron beam for which \(J(x, y) = j(y)\delta(x)\) and its Fourier transform is written as \(\tilde{J}(x, k_y) = \tilde{j}(k_y)\delta(x)\). The longitudinal component of the electromagnetic field that evolves due to interaction with this current density is given by:

\[
E_z(x, y, z, t) = \frac{e^{-\Gamma x} e^{i(k_z z - w_0 t)}}{\sqrt{2\pi}} \int A(k_y, z, t) e^{-ik_y^2/2k_0} e^{ik_y y} d k_y + \text{c.c.} \tag{4.13}
\]

The amplitude \(A(k_y, z, t)\) of the surface mode will evolve due to interaction with the co-propagating electron beam and we have assumed it to be a slowly varying function of \(z\) and \(t\). The beam-wave interaction mechanism in a CFEL system, which describes the evolution of the 2D surface mode and the dynamics of the electron beam in the presence of this surface mode, has been discussed in detail in Chapter 3. Now, by following the same approach and realizing that the underbraced term in Eq. (4.13), which is the amplitude of the Fourier component of the electromagnetic field, is evolving due to interaction with the amplitude of the corresponding Fourier component of the current density denoted by the underbraced terms in Eq. (4.12), we obtain the following time dependent differential equation for the evolution of \(A(k_y, z, t)\):

\[
\frac{\partial A}{\partial z} + \frac{1}{\beta_s c} \frac{\partial A}{\partial t} = -\frac{Z_0 X}{2\beta_s \gamma_r} \tilde{j}(k_y) e^{ik_y^2/2k_0} e^{-2\Gamma x} (e^{-i\theta}) - \alpha A. \tag{4.14}
\]

By taking Fourier transform with respect to \(k_y\) in above equation and using the fact that \(A e^{-ik_y^2/2k_0}\) is the Fourier transform of the longitudinal surface field \(E\), we obtain the following dynamical
equation for the 3D surface mode:

\[ \frac{\partial E}{\partial z} - \frac{i}{2k_0} \frac{\partial^2 E}{\partial y^2} + \frac{1}{\beta_z c} \frac{\partial E}{\partial t} = \frac{-Z_0 \chi}{2 \beta_R \gamma} \frac{dI}{dy} e^{-2i \gamma_y (e^{-i\phi})} - \alpha E. \] (4.15)

Here, \( E \) is the amplitude of the longitudinal field \( E_z \) and \( dI/dy \) is the linear current density of the flat beam. Note that while deriving the above equation, we have assumed \( dI/dy \) to be constant, although it is a function of \( x \) and \( y \). The second term on the left hand side of above equation represents diffraction of the surface mode and allows us to study the transverse profile of the optical beam. In an approximate way, the effect of partial overlap between the electron beam and optical mode can be considered in the numerical solutions of 2D Maxwell-Lorentz equations by writing the linear current density \( dI/dy \) as \( I/\Delta y \), where \( \Delta y \) is the electron beam width, and is replaced with the effective optical beam width \( \Delta y_e \). The effective optical mode width \( \Delta y_e \) here has to be chosen by suitably matching the optical beam size given by Eq. (4.9) and the corresponding electron beam size. To find the electron beam size in the horizontal direction, we need to give a detailed description of the transverse profile of the electron beam, which is discussed in the following section.

## 4.2 Electron beam requirements and its production for the Čerenkov FELs

In this section, based on the analysis of the surface mode presented in Sec. 4.1.2, we will work out the electron beam requirements for successful operation of a CFEL. To perform these calculations, we will closely follow the approach given in Refs. [78, 89], where the requirements on the electron beam parameters have been determined for the successful operation of a SP-FEL system. Before discussing the transverse profile of the electron beam, we will first review some of the properties of the electron beam in phase space. The electron beam distribution in the four dimensional phase space \( (x, \varphi, y, \phi) \) is assumed to be Kapchinskij-Vladimirskij (KV) distribution [144], where \( x \) and \( y \) are the vertical and horizontal coordinates respectively, and
\( \varphi \) and \( \phi \) represents vertical and horizontal angles, respectively. The electron beam distribution is assumed to have half-widths \((\Delta x, \Delta \varphi, \Delta y, \Delta \phi)\) at the middle of the dielectric slab and the half widths are two times the rms values \((\sigma_x, \sigma_\varphi, \sigma_y, \sigma_\phi)\). Thus, \(\Delta x = 2\sigma_x\), \(\Delta \varphi = 2\sigma_\varphi\), \(\Delta y = 2\sigma_y\), and \(\Delta \phi = 2\sigma_\phi\). The geometric rms emittance in the \(y\)-direction is therefore given by \(\varepsilon_y^0 = \frac{1}{4\pi} \Delta y \Delta \phi\). The Courant-Snyder envelope \(\beta_y\), also known as the beta function in the \(y\)-direction, is defined as \(\beta_y = \frac{\sigma_y^2}{\varepsilon_y^0}\). Similar quantities are defined with the subscript \(x\) in the \(x\)-direction.

Let us first look for the requirements on the electron beam in the \(y\)-direction. The product of rms beam size \(\sigma_y(o)\) and divergence \(\sigma_\theta\) for the surface mode supported in the CFEL is given by \(\beta_R \lambda / 4\pi\). Now to ensure that electron beam envelope is within the envelope of optical beam, the rms unnormalized emittance is required to be less than this product. Applying this for the case of the CFEL, we get

\[
\varepsilon_y \leq \frac{\beta_R^2 \gamma_R \lambda}{4\pi},
\]

where \(\varepsilon_y = \beta_R \gamma_R \varepsilon_y^0\) is the normalized beam emittance in the \(y\)-direction. Next, the half width \(\Delta y\) of the electron beam, which is taken the same as the half width \(2\sigma_y\) of the optical beam, is chosen by requiring that the Rayleigh range \(Z_R\) is equal to the interaction length \(L\). This choice of \(Z_R\) ensures that the variation in the rms optical beam size over the interaction length is within 10\%, as can be seen by putting \(z = L/2\) (\(z = 0\) corresponds to middle of the dielectric slab and \(z = \pm L/2\) corresponds to the end points), and \(Z_R = L\) in Eq.(4.9). Now, by using Eq. (4.10), we find \(\sigma_y = \sqrt{\beta_R \lambda Z_R / 4\pi}\) and by inserting it in the above-mentioned condition, we obtain

\[
\Delta y = \frac{\sqrt{\beta_R \lambda L}}{\pi}.
\]

Let us now discuss the required electron beam parameters in the \(x\)-direction. In the view of the exponential factor \(e^{-2\Gamma h}\) in Eq. (3.35), where \(\Gamma = 2\pi / \beta_R \gamma_R \lambda\), it is desirable that the height \(h\) of the electrons should satisfy \(h \leq 1/2\Gamma\) for sufficient beam-wave interaction. Assuming that
the electron beam is propagating over the dielectric slab such that its centroid is at height \( h \) and its lower edge just touches the dielectric surface, we can take the half-width \( \Delta x \) of the electron beam same as \( h = 1/2 \Gamma \), and obtain

\[
\Delta x = \frac{\beta_R \gamma_R \lambda}{4 \pi}.
\]  

This implies that rms electron beam size \( \sigma_x = \Delta x / 2 = \beta_R \gamma_R \lambda / 8 \pi \) at the middle of the dielectric slab. The rms beam size in the \( x \)-direction at the end of the dielectrics slab is given by

\[
\sqrt{\varepsilon_x^0 \beta_x [1 + (L/2 \beta_x)^2]} \]  

[78]. In order to ensure that the variation in \( \sigma_x \) over the interaction length \( \leq 10\% \), we require \( \beta_x \geq L \). Using these two conditions and the relation that \( \varepsilon_x^0 = \sigma_x^2 / \beta_x \), we obtain

\[
\varepsilon_x \leq \frac{\beta_R^3 \gamma_R^3 \lambda^2}{64 \pi^2 L},
\]

where \( \varepsilon_x = \beta_R \gamma_R \varepsilon_x^0 \) is the normalized beam emittance in the \( x \)-direction. The condition on the normalized beam emittance \( \varepsilon_x \) comes out to be very stringent. As discussed in detail in the next section, a flat electron beam with transverse emittance ratio, \( \varepsilon_y / \varepsilon_x \approx 1000 \) is required for the operation of a practical THz CFEL. This value is roughly 10 times higher than the value achieved in a recent experiment [145, 146].

The stringent requirement on the emittance of a flat electron beam can be relaxed by introducing an external focusing by either using a wiggler field [147, 148] or by using a solenoid field [149–151]. Details of the two schemes are described in Ref. [89] for the case of a SP-FEL. Both the schemes are applicable for the case of the CFEL also. In the next two subsections, we will discuss these two schemes for a CFEL system.

### 4.2.1 Focusing of a flat electron beam by using a wiggler field

A flat electron beam can be focused in both the vertical and horizontal planes by using a wiggler field, as discussed and demonstrated by Booske et al. [152]. A flat electron beam for this
scheme can be generated by a novel phase space technique \cite{78, 153}, in which a round electron beam is first produced from a cathode placed in an axial magnetic field, and then the angular momentum of the beam is removed by using a set of quadrupoles. This gives a flat electron beam with transverse emittance ratio \cite{78}:

\[
\frac{\epsilon_y}{\epsilon_x} = \left( \frac{e B r_i^2}{mc^2 \epsilon_i} \right)^2,
\]  \hspace{1cm} (4.20)

where \( B \) represents the magnetic field at the cathode, \( r_i \) is the radius of the thermionic cathode, and \( \epsilon_i = \sqrt{\epsilon_x \epsilon_y} \) is the initial beam emittance of the round beam. The radius \( r_i \) is related to the initial emittance as \( r_i = 2 \epsilon_i / \sqrt{k_B T/mc^2} \) \cite{78}, where \( k_B \) is Boltzmann’s constant and \( T \) is the absolute temperature of the thermionic cathode. The magnetic field required to produce an electron beam with the desired transverse emittance ratio is evaluated by using Eq. (4.20) as \( B = k_B T/e\epsilon_x c \). Note that \( B \) is independent of \( \epsilon_y \). The current density \( J_i \) at the cathode for a given beam current \( I \) is \( J_i = I/\pi r_i^2 \) \cite{78}.

In Fig. 4.2, we have shown the schematic for focusing of the above mentioned flat beam in a CFEL by using a wiggler with a parabolic pole shape. In the presence of a wiggler magnetic field, the electron beam will be focused in both \( x \)- and \( y \)-directions. We need to find an electron beam matched to the focusing forces such that the beam size remains minimum and nearly uniform along the wiggler length. By neglecting the space charge effect in the envelope equation,
the matched rms beam sizes in the $x$- and $y$-directions are obtained as [89]:

$$
\sigma_{x,y} = 2^{1/4} \sqrt{\frac{\varepsilon_{x,y}}{a_u k_{x,y}}},
$$

(4.21)

Here, $a_u = eB_u/k_0mc$, $B_u$ represents the peak value of the magnetic field in the $x$-direction, along the $z$-axis, $k_u = 2\pi/\lambda_u$, $\lambda_u$ is the wiggler period, and $k_x$ and $k_y$ represent spatial frequency of the wiggler field in the $x$- and the $y$-direction respectively. We require $k_u^2 = k_x^2 + k_y^2$ to satisfy the Maxwell equations. In Eq. (4.21), we choose $\sigma_y = 1/2\Delta y$ and $\sigma_x = 1/2\Delta x$, where $\Delta y$ and $\Delta x$ are given by Eqs. (4.17) and (4.18) respectively, and find the appropriate value of $a_u$, $k_x$ and $k_y$ for a given values of emittances ($\varepsilon_x$, $\varepsilon_y$), such that the beam sizes are matched inside a wiggler and thus maintain a constant size throughout the wiggler. For a typical set of parameters of a CFEL, the focusing requirement in the vertical direction is very strong as compared to the horizontal direction. We can therefore choose $k_y = 0$ and $k_x = k_u$. It is clear from Eq. (4.21) that for a matched beam size, one can tolerate a larger vertical emittance by choosing a higher value of the peak wiggler magnetic field $B_u$. This helps us to relax the stringent requirement given by Eq. (4.19). Note that in case of external focusing in the vertical direction, we do not need to satisfy Eq. (4.19).

### 4.2.2 Focusing of a flat electron beam by using a solenoid field

In the second scheme, a solenoid magnetic field is used to focus a low energy flat electron beam. The required flat beam is generated by using an elliptically shaped, planar thermionic cathode with major axis $\Delta y_c = \Delta y$ and minor axis $\Delta x_c = \Delta x$. The normalized thermal emittances for the thermionic cathode are given by $(\varepsilon_x, \varepsilon_y) = 0.5(\Delta x_c, \Delta y_c) \sqrt{k_B T/mc^2}$, and current density at cathode, corresponding to current $I$, is given by $J_c = I/\pi \Delta x_c \Delta y_c$ [78]. For generating a flat beam, the vertical dimension of the cathode is very small compared to the horizontal dimension, and such a cathode is called a line cathode. The line cathode together with the dielectric slab is immersed inside the solenoid such that the electron beam is generated in the uniform field region of the solenoid. On the contrary, if line cathode is placed outside the solenoid in the
field-free region then the flat electron beam generated from such line cathode starts rotating as it enters into the solenoid field. To avoid the rotation of flat beam as it propagates over the dielectric slab, both the line cathode and the dielectric slab are placed inside the solenoid. The solenoid field strength required to focus a flat beam can be evaluated with the condition that the Larmor radius should be much smaller than the vertical rms beam size $\sigma_z$ [89], which gives us the following expression for the required axial magnetic field $B(0)$ near the cathode [89]:

$$B(0) \gg \frac{mc\varepsilon_x}{e\sigma_z^2}.$$  \hspace{1cm} (4.22)

The nonuniformity in the longitudinal on-axis magnetic field gives rise to a rotation $\theta$ to the flat beam, which is given by [89]

$$\theta(z) = \frac{z\omega_L \Delta B(z)}{3\beta_R e B(0)}.$$  \hspace{1cm} (4.23)

where $\Delta B(z) = B(z) - B(0)$ and $\omega_L$ is the Larmor frequency. Here, it is assumed that the cathode is placed at the centre of the solenoid ($z = 0$), where the field is maximum, and the variation of the quantities in the radial direction is assumed to be very slow. We have to ensure that the electron beam does not rotate significantly such that the flat beam nature is preserved.

Clearly, both the external focusing techniques allow us to tolerate larger emittance of the electron beam. A large emittance of the electron beam, however, gives rise to a spread in the trajectory angle $\theta$, and therefore in the longitudinal velocity given by

$$\frac{\Delta \beta_R}{\beta_R} = 1 - \cos \theta.$$  \hspace{1cm} (4.24)

Using $(1 - \cos \theta) \approx \frac{\theta^2}{2}$ and $\theta^2 = \varepsilon_x^2/\beta_R^2 \gamma_R^2 \sigma_z^2$ in the above equation, we obtain

$$\frac{\Delta \beta_R}{\beta_R} \approx \frac{\varepsilon_x^2}{2 \beta_R^2 \gamma_R^2 \sigma_z^2}.$$  \hspace{1cm} (4.25)

Focusing in the y-direction will also give similar contribution to the velocity spread. The effect of the longitudinal velocity spread is equivalent to an effective energy spread. The deleterious
effect of energy spread become prominent as we increase the interaction length to achieve a higher gain. The maximum energy spread that can be tolerated in a CFEL corresponds to the phase mismatch of $\pi$ between the electrons and the co-propagating surface mode at the exit of the interaction region, or equivalently $\Delta \beta R L / \beta R = \beta R \lambda / 2$. This condition gives us the maximum value of emittance which can be tolerated by the system as:

$$\varepsilon_x < \sigma_x \sqrt{\frac{\beta^3 R \gamma^2 R \lambda}{L}}.$$  \hspace{1cm} (4.26)

With the external focussing, we can increase the length $L$ of the dielectric slab to obtain a higher gain. However, increase in $L$ value will restrict the maximum emittance that can be tolerated, as indicated by Eq. (4.26). We need to choose an optimum value of $L$ for which the deleterious effects due to the energy spread are significantly less.

Finally, we summarize the procedure for optimization of the focusing strength and the emittance as follows: we first choose the vertical beam size from Eq. (4.18) for the given parameters of a CFEL, and then we choose the maximum focussing strength by using Eq. (4.21) (in the case of wiggler focusing) or using Eq. (4.22) (in the case of solenoid focusing) to attain the maximum tolerance on the vertical emittance, keeping in mind that the constraint is given by Eq. (4.26).

### 4.3 Optimization study of a THz Čerenkov FEL

In this section, we will perform an optimization study of a THz CFEL in accordance with the analysis given in the earlier sections. For an example case of a practical CFEL, we take the parameters of the Dartmouth experiment [66] discussed earlier in Chapters 2 and 3, where a CFEL device was tested with an operating frequency of 0.1 THz by using an electron beam having 1 mA beam-current and 30 keV ($\beta R = 0.33$) beam-energy. Two different materials, GaAs ($\epsilon = 13.1$) [120] and sapphire ($\epsilon$ varying from 9.6 to 10.0) [119] were used for the dielectric slab having slab thickness 350 $\mu$m, and a silver polished copper metal was used to support the
dielectric slabs. Among the above-mentioned dielectric materials, we have considered GaAs in our calculations for the reasons discussed earlier in Chapter 2. GaAs has a tangent loss \( \tan \delta = 2 \times 10^{-4} \) at room temperature, i.e., at 300 K. For these parameters, we find \( \lambda = 2.7 \text{ mm} \) (operating frequency = 0.1 THz), \( \beta_e = 0.23 \) and \( \chi = 181 \) per m. The conductivity of silver metal at 300 K is given by \( 6.3 \times 10^7 / \Omega \cdot \text{m} \) [154], for which the Ohmic attenuation coefficient, \( \alpha^e = 2.5 \text{ m}^{-1} \), as calculated by using Eq. (2.31). The dielectric attenuation coefficient is calculated by using Eq. (2.29) as \( \sigma^d = 0.9 \text{ m}^{-1} \) at 300 K. Note that the dielectric losses are less compared to the Ohmic losses. In the context of 3D analysis, the linear current density \( dI/dy \), which is needed to evaluate the gain and the growth rate of the system by using Eqs. (3.34) and (3.43) respectively, can be interpreted as the peak value at the middle of the electron beam distribution. We obtain the following expression for the linear current density [78]

\[
\frac{dI}{dy} = \frac{I}{\pi \Delta y/2},
\]

for the KV distribution discussed in Sec. 4.3. The effective electron beam width in the \( y \)-direction is thus taken as \( \pi \) times the rms optical beam waist size \( \sigma_y(0) \). We denote the effective electron beam width in the \( y \)-direction as \( \Delta y_e \), which we have evaluated by using Eq. (4.17) as \( \Delta y_e = \sqrt{\pi \beta_R \lambda L/4} \). The centroid of the electron beam is taken at height \( h = \Delta x = \beta_R \gamma_R \lambda / 4\pi \). The length of the dielectric slab was taken around 1 cm in the Dartmouth experiment. Note that in the previous chapter a larger interaction length (15 cm) was considered to achieve a reasonable gain, i.e., 56%. However, if we include the effect due to attenuation, we find a round trip loss \( 1 - e^{-4\alpha L} \) of around 87% for this length, which is higher than the gain and the system is unable to lase in these conditions. By taking the system length, i.e., \( L = 1 \text{ cm} \), we find the small signal gain of around 0.04%, which is also too low to overcome the losses present in the system as the round trip loss is around 13.0%. To get an appreciable gain, we take \( L = 5 \text{ cm} \) and increase the electron beam current from 1 mA to 35 mA. With the increased length, power loss due to attenuation also increases. To reduce the attenuation, we kept the silver metal and the dielectric slab at low temperature i.e., at 77 K, which is boiling point of liquid nitrogen. At
77 K, the tangent loss of GaAs is $2 \times 10^{-5}$ and the conductivity of silver is about $3.3 \times 10^{8}$ $\Omega^{-1}$m $^{-1}$ [154]. Now to find the optimum value of electron beam energy and dielectric thickness, we have kept the operating wavelength fixed, i.e., $\lambda = 2.7$ mm, and plotted the net gain and net growth rate as a function of beam energy and dielectric thickness respectively [155]. Note that we have to vary both the electron beam energy and the dielectric thickness simultaneously such that they satisfy dispersion relation to give $\lambda = 2.7$ mm. The plots of net gain and net growth rate as a function of electron beam energy are shown in Fig. 4.3. We obtain an optimum value of electron beam energy as 40 keV for which the net gain comes out to be 88% and net growth rate is obtained as 25.4 m$^{-1}$. Figure 4.4 shows plots of net gain and net growth rate as a function of dielectric slab thickness. The optimum value of slab thickness is obtained as 265 $\mu$m. For the 40 keV beam energy and 265 $\mu$m of slab thickness, the attenuation coefficient is calculated as 1.73 m$^{-1}$ at 77 K, which gives us a round trip loss of 29.3% over the 5 cm length.

The proposed system is a low-gain device, and has to be operated in the oscillator configuration. The Maxwell-Lorentz equations have been solved numerically to obtain the power in the surface mode supported in this configuration by using the leapfrog scheme [138] discussed in Chapter 3. The power builds up slowly and saturates at 3.6 W, as shown by the solid curve in Fig 4.5. The input electron beam power is 1.4 kW for the considered beam kinetic energy of 40
keV ($\beta_R = 0.374$, $\gamma_R = 1.078$), and beam current of 35 mA. For these parameters, we find the efficiency of the optimized CFEL as 0.26%. The analytic estimate for the upper bound of the efficiency $\eta_{eff}$ is given by Eq. (3.44) in Chapter 3, which represents the fraction of electron beam energy which appears in the form of outcoupled power plus the heat dissipated in the system. In the oscillator configuration, the outcoupled power is given by $P_{in}(1-R_m^2)$, where $P_{in}$ is the mean intra-cavity power and $R_m = 0.98$ is the reflection coefficient of the outcoupling mirror. Taking the effect of attenuation, there will be a round trip loss of power given by $P_{in}(1-e^{-4\alpha L})$. Considering these effects, efficiency of the system is obtained as

$$\eta_{eff} = \frac{P_{in}(1-R_m^2) + (1-e^{-4\alpha L})}{P_b}.$$ 

We define $\eta_{sim}$ as the efficiency observed in the simulation, which is also the actual efficiency representing the fraction of electron beam power that appears in the form of outcoupled power, i.e., $\eta_{sim} = P_{in}((1-R_m^2))/P_b$. This gives us $\eta_{sim} = [(1-R_m^2)/((1-R_m^2) + (1-e^{-4\alpha L}))]\eta_{eff}$. Thus, the upper bound for $\eta_{sim}$ is obtained by multiplying a factor of $(1-R_m^2)/((1-R_m^2) + (1-e^{-4\alpha L}))$ to Eq. (3.22) as

$$\eta_{sim}^{upper\ bound} = \frac{\beta_R^3 \gamma_R^3 A(1-R_m^2)}{L(\gamma_R - 1)[(1-R_m^2) + (1-e^{-4\alpha L})]}.$$  

(4.28)

For the prescribed parameters, we obtain an upper bound $\eta_{sim}^{upper\ bound}$ as 0.54%, which is in agreement with our numerical results. Figure 4.5 also shows the output power (dashed curve) of a CFEL, where Ohmic and dielectric losses are assumed to be zero. In this case, the CFEL
system gives 24.75 W output power on saturation with an efficiency of 1.77%, which is less than the analytically estimated value for the upper bound of 4%. Note that the presence of dielectric losses and Ohmic losses on the metallic surface severely affects the output power and efficiency of a CFEL system, and one has to optimize the system for minimum losses.

The requirements on electron beam sizes are evaluated by using Eqs. (4.17) and (4.18), as $\Delta y = 4.0 \text{ mm}$ in the $y$-direction and $\Delta x = 87 \mu\text{m}$ in the $x$-direction respectively. By using Eq. (4.19), we find that an electron beam with normalized vertical emittance $\varepsilon_y \leq 1.5 \times 10^{-5}$ m-rad is needed in the absence of any external focussing, which is a very stringent requirement. In the horizontal direction, the condition on beam emittance is quite relaxed as an electron beam with $\varepsilon_x \leq 3.3 \times 10^{-5}$ m-rad is required, which is calculated from Eq. (4.16). If we take $\varepsilon_y = 1.65 \times 10^{-5}$ m-rad, which is 2 times less than the maximum allowed value, then a flat electron beam with transverse emittance ratio $\varepsilon_y/\varepsilon_x = 1000$ is required for the operation of the CFEL. In the Dartmouth experiment [66], these conditions have been clearly violated, where a round electron beam having large vertical emittance was used to drive the Čerenkov FEL.

To relax the stringent requirement on the electron beam emittance, external focusing can be
provided by using a wiggler field as described in Sec. 4.2.1. Here, we will take an explicit example to perform the calculations for the analytical results discussed in Sec. 4.2.1. We assume a round electron beam with initial normalized emittance $\varepsilon_i = 1 \times 10^{-6}$ m-rad, which is easily achievable. A flat electron beam can be produced by using round to flat beam transformation as discussed earlier, and under this transformation $\varepsilon_f = \sqrt{\varepsilon_x \varepsilon_y}$. We choose the ratio of horizontal and vertical emittances as $100 : 1$, i.e., $\varepsilon_y = 10^{-5}$ m-rad and $\varepsilon_x = 10^{-7}$ m-rad. Taking the cathode temperature $1300$ K, this scheme requires an axial magnetic field $B = 71.87$ Gauss at the position of the cathode, which can be generated by using either a permanent magnet or an electromagnet [78]. The current density at the cathode, which is required to produce an electron beam of desired emittances as discussed above, is obtained by using the prescription given in Sec.4.2.2 as $J_T = 0.12$ A/cm$^2$ at $T = 1300$ K. This value of current density (current density upto $10$ A/cm$^2$) can be easily achieved in thermionic cathodes e.g., oxide, dispenser and M-type cathodes [156, 157], which can be operated for tens of thousands of hours at around $1300$ K. Next, we discuss the requirements of the wiggler parameters to focus the flat beam described above. For a matched beam size $\sigma_x = 43.5$ $\mu$m and vertical emittance $\varepsilon_y = 10^{-7}$ m-rad, we require $B_u = 1.3$ kG. This value of magnetic field can be obtained by using an array of regular pure permanent magnets in a Halbach configuration, which gives a peak field strength of $1.43 B_{rem} \exp(-\pi g_u/\lambda_u)$ [158], where $B_{rem}$ is the remnant field of the magnetic material and $g_u$ is the gap between the jaws of the wiggler. We have considered NdFeB as the magnetic material for which $B_{rem} = 1.1$ T [158]. We have considered two example cases for the wiggler gap, i.e., $g_u = 1$ mm and $g_u = 2$ mm, such that the beam transport is feasible. For $g_u = 1$ mm, we need a mini-wiggler with 1.25 mm period and for $g_u = 2$ mm, we need a wiggler period of 2.50 mm to obtain the required magnetic field of 1.3 kG. This type of mm or sub mm period wiggler can be fabricated by using laser micromachining of bulk permanent magnets as discussed in Ref. [159]. We would like to mention that when an electron beam is focused by a spatially modulated magnetic field as discussed above, it can cause a parametric instability of the beam envelope oscillations due to the mismatch of the beam parameters and the focusing field [160, 161]. These oscillations, if amplified enough, could cause emittance growth and lead to an unstable electron beam [160, 161]. In this case, the envelope equation for the flat beam
evolving in the presence of wiggler magnetic field can be used to obtain the following stability condition: 
\[ (eA_B / 2 \sqrt{2\pi mc} \beta R Y' R) < 0.5 \]  
This condition is known as the Mathieu stability condition of wiggler focusing [162]. We have checked that for the proposed wiggler field and wiggler period, the condition for the Mathieu stability of the wiggler focusing is satisfied. It has also been checked that the criterion for the maximum tolerance on beam emittance as given by Eq. (4.26) is easily met for the above discussed case.

Next, we discuss another possibility to relax the stringent requirement on the vertical beam emittance, where we can use a line cathode immersed in a solenoid field to produce a flat electron beam. This method has been discussed in detail in Sec. 4.2.2. To produce a flat electron beam with beam half widths \( \Delta x = 87 \, \mu m \) and \( \Delta y = 4.0 \, mm \), we need a line cathode with \( \Delta x_c = 87 \, \mu m \), and \( \Delta y_c = 4.0 \, mm \). At \( T = 1300 \, K \), we obtain \( \varepsilon_x = 2.2 \times 10^{-8} \, m-rad \) and \( \varepsilon_y = 9.7 \times 10^{-7} \, m-rad \) using equation \( (\varepsilon_x, \varepsilon_y) = 0.5(\Delta x_c, \Delta y_c) \sqrt{k_B T / mc^2} \). These values for \( (\varepsilon_x, \varepsilon_y) \) are quite acceptable. For a beam current of 35 mA, the current density at the line cathode is obtained as \( J_c = 3.2 \, A/cm^2 \). By using Eq. (4.22), we find that the solenoid magnetic field \( B(0) \) is required to be greater than 0.25 kG to focus such a flat electron beam. We choose \( B(0) = 1.0 \) kG. The Larmor radius is obtained as 10.3 \( \mu m \) for these parameters, which is significantly smaller than the rms beam size \( \sigma_x = 43.5 \, \mu m \). To keep the flat beam rotation less than 10 mrad over a length \( L = 5 \, cm \), we require the field uniformity \( \Delta B / B \) to be better than 0.4%, as calculated by using Eq. (4.23).

### 4.4 Attenuation and diffraction effects in Smith-Purcell FELs

The essential features of the analysis of surface mode supported in a SP-FEL system have been worked out earlier in Refs. [78, 88–93, 163] and are summarized in Sec. 2.4. Here, we further elaborate these results to highlight interesting differences in the analyses of CFEL and SP-FEL systems. We first review some important results of the 2D analysis of the beam-wave interaction in SP-FELs [88], which is similar to the 2D analysis of the CFELs presented in
Chapter 3. Then, we extend the results obtained in the earlier analysis of SP-FELs presented in Refs. [78, 88–90] to set up the 3D Maxwell-Lorentz equations for the system.

A schematic of a SP-FEL system is shown in Fig. 4.6, where a flat electron beam skims over the surface of a metallic reflection grating having length $L$, period $\lambda_g$, groove depth $d$ and groove width $w$. A metallic grating supports an electromagnetic field, which is a combination of Floquet space harmonics since the grating is a periodic structure. Under the 2D approximation, the longitudinal component of electric field can be written as:

$$E_z(x, z, t) = \sum_n E_ne^{i(k_x + nk_z)z - \omega t} e^{-\Gamma_n x},$$  \hspace{1cm} (4.29)

where $k_z$ is the propagation wavenumber in the $z$-direction for the $n = 0$ term, $k = \omega/c$, $k_x = 2\pi/\lambda_g$ and $\Gamma_n = \sqrt{(k_z + nk_x)^2 - k_x^2}$. The zeroth-order field component has a similar structure as given by Eq. (4.2) [88] and shows the strongest interaction with the electron beam since it is the only component that has phase velocity equal to the electron beam velocity [88]. Higher space harmonics will have a feeble interaction since the phase velocity is not matched with the beam velocity. However, one needs to include higher space harmonics in order to satisfy the boundary condition and to find the dispersion relation of the surface mode as discussed in Ref. [88]. Similar to the case of the CFEL described in the previous chapter, the electron beam
interacts with the co-propagating surface mode and the dynamical equation for the evolution of the amplitude of surface mode is given by [78, 88]

\[
\frac{\partial E}{\partial z} - \frac{1}{\beta_{s} c} \frac{\partial E}{\partial t} = \frac{Z_{0} \chi}{2 \beta_{R} \gamma_{R}} \frac{dI}{dy} e^{-2\gamma_{h}}(e^{-i\phi}) + \alpha E. \tag{4.30}
\]

Here, \( E \) is the amplitude of the zeroth-order longitudinal field. The calculation of \( \chi \) for a SP-FEL requires numerical evaluation of \( R \) as a function of growth rate for a given value of \((\omega, k_{0})\) of the surface mode, and details regarding the procedure for this calculation are described in Ref. [88]. Evaluation of the attenuation coefficient \( \alpha \) requires calculation of the heat dissipation at the metallic surfaces for the given surface mode, which has been discussed for the case of the SP-FEL in Ref. [99]. Note the difference in sign of terms containing \( \beta_{s}, \chi \) and \( \alpha \) in above equation as compared to Eq. (4.3) for the CFEL. This is due to the fact that the SP-FEL has negative group velocity for the surface mode, as described in Ref. [88], whereas the group velocity is positive in case of the CFEL system. The negative group velocity for the SP-FEL makes it a BWO and oscillations build up when the linear current density \( dI/dy \) exceeds a threshold value \( dI_{s}/dy \) [78, 88]:

\[
\frac{dI}{dy} > \frac{dI_{s}}{dy} = \mathcal{J}(\eta) \frac{I_{0} \beta_{R}^{4} / \gamma_{R}^{4}}{2 \pi \chi k L} e^{-2\gamma_{h}}. \tag{4.31}
\]

Here, \( \mathcal{J}(\eta) \) represents a dimensionless start current as a function of the loss parameter \( \eta = \alpha L \).

The calculations of \( \mathcal{J}(\eta) \) are given in Ref. [164].

After having briefly discussed the 2D analysis, we next discuss the localized surface mode and set up the 3D coupled Maxwell-Lorentz equations for the SP-FEL system.

### 4.4.1 Localized surface mode and 3D Maxwell-Lorentz equations

In order to construct the localized surface mode, we need to combine the plane waves propagating at different angles in the \((y, z)\) plane with a suitable weight factor. In order to perform this calculation, we need to know the 3D dispersion relation of the metallic grating, i.e., the
dependence of $\omega$ on $k_z$, for different values of $k_y$. The 3D dispersion relation for a rectangular metallic grating is obtained by Kumar and Kim [90] by deriving the condition for singularity in $R$. A remarkable observation in their analysis is that if we replace $\omega$ by $\sqrt{\omega^2 - c^2 k_y^2}$ in the expression for the reflectivity for the case $k_y = 0$; we obtain the reflectivity for the 3D case, where a finite value of $k_y$ is considered [90, 163]. This is because here we have electromagnetic field present only in one medium, i.e., the space above the surface of the reflection grating (including the grooves of the grating), which is in vacuum. Due to this feature, the expression for reflectivity has terms like $(\omega^2 - c^2 k_y^2)$ in the 2D case, which can be simply replaced with $(\omega^2 - c^2 k_y^2 - c^2 k_z^2)$ for the 3D case. This amounts to replacing $\omega$ in the 2D case by $\sqrt{\omega^2 - c^2 k_y^2}$, to determine the dispersion relation for the 3D case with finite $k_y$.

It is important here to note the difference between the dispersion relation of the SP-FEL and the CFEL systems. A careful observation of the 2D dispersion relation of a CFEL: $\sqrt{\varepsilon \omega^2 / c^2 - k_y^2} \times \tan(d \sqrt{\varepsilon \omega^2 / c^2 - k_y^2}) = \varepsilon \sqrt{k_y^2 - \omega^2 / c^2}$ indicates that the simple “replacement rule” as in the case of the SP-FEL system i.e., replacing $\omega$ in 2D dispersion relation with $\sqrt{\omega^2 - c^2 k_y^2}$ will not give us the 3D dispersion relation. This is because here, the electromagnetic field in a CFEL is present in vacuum as well as in the dielectric medium, unlike in the SP-FEL case; therefore terms like $(\omega^2 - c^2 k_y^2)$ as well as $(\omega^2 - c^2 k_y^2 / \varepsilon)$ appear in the 2D dispersion relation. In the case of the CFEL, the “isotropic nature” of the dielectric slab in $(y, z)$ plane facilitates us to analyze the diffraction in the surface mode as described in Sec. 4.1.2. The grating structure used in the SP-FEL system has grooves along the surface in the transverse direction and lacks isotropic behaviour in the $(y, z)$ plane. Due to this difference, the optical properties of the surface mode in the SP-FEL system are different as compared to the CFEL system, as elaborated in the following paragraph.

We now discuss the construction of localized surface mode in the SP-FEL. Due to the replacement rule: $\omega_{3D}(k_y) = \sqrt{\omega_{2D}^2 + (ck_y)^2}$, we write the longitudinal wavenumber $k_z$ as

$$k_z = k_0 + \left. \frac{\partial k}{\partial \omega} \right|_{k_y = 0} \Delta \omega,$$  \hspace{1cm} (4.32)
where \( \Delta \omega = \omega_{2D} - \omega_{3D} \) and the term \( \partial k / \partial \omega \) at \( k_\gamma = 0 \) is identified as \( -1/\beta_\epsilon c \). Using these results along with the paraxial approximation in Eq. (4.32), we obtain

\[
k_\epsilon = k_0 \left( 1 + \frac{k_\gamma^2}{2\beta R \beta_\epsilon k_0^2} \right)^3.
\]  

(4.33)

Here, we note the difference between above equation and corresponding equation [Eq. (4.6)] for the case of the CFEL. On the account of 3D effects, the magnitude of change in the longitudinal wavenumber \( |k_\epsilon - k_0| \) is given by \( \beta_R \lambda k_0^2 / 4\pi \) for the CFEL and \( \lambda k_0^2 / 4\pi \beta_\epsilon \) for the SP-FEL case respectively. It can be seen that in this term, \( \beta_R \lambda \) in case of the CFEL is replaced with \( \lambda / \beta_\epsilon \) for the case of the SP-FEL.

Next, by satisfying the wave equation for the electromagnetic field, we obtain the expression for \( \Gamma' \) as:

\[
\Gamma' = \Gamma \left( 1 + \frac{k_\gamma^2 (1 + \beta_R \beta_\epsilon)}{2\beta R \beta_\epsilon \Gamma^2} \right).
\]  

(4.34)

By following an approach similar to the one described in Sec. 4.1.2, the analysis for the localized surface mode supported by the grating structure is performed. The Rayleigh range for the optical surface mode is obtained as [78]:

\[
Z_R = \frac{4\pi \beta_\epsilon \sigma_y^2(0)}{\lambda},
\]  

(4.35)

where \( \sigma_y(0) \) is the rms beam size at the waist. Under paraxial approximation, the product of rms beam waist size and rms divergence is given by [78]

\[
\sigma_y(0) \times \sigma_\theta = \frac{\lambda}{4\pi \beta_\epsilon}.
\]  

(4.36)

Note that Eqs. (4.35) and (4.36) have dependence on the group velocity, while equivalent quantities in the CFEL system [Eqs. (4.10) and (4.11)] have dependence on the phase velocity of the surface mode. In these expressions, the term \( \beta_R \lambda \) in the case of the CFEL is replaced with \( \lambda / \beta_\epsilon \) in the case of the SP-FEL, as expected. We emphasize that this difference arises due to
a fundamental difference in the way the dispersion relation for the two systems gets modified for the 3D case, which we have explained. Due to this nature, it can be seen that the diffraction effects are more prominent in case of the SP-FEL as compared to the CFEL. The length $L$ of the grating in case of SP-FEL has to be kept small to maintain sufficient interaction of the surface mode with the co-propagating electron beam.

Next, the expression for $k_z$ and $\Gamma'$ can be used in Eq. (4.4) to set up the three-dimensional electromagnetic surface mode for the SP-FEL. By following the procedure described in Sec. 4.1.3, the following time dependent 3D differential equation for the evolution of the surface mode in a SP-FEL is obtained \cite{90}:

$$\frac{\partial E}{\partial z} + \frac{i}{2\beta_R^2 k_0} \frac{\partial^2 E}{\partial y^2} - \frac{1}{\beta_Z c} \frac{\partial E}{\partial t} = \frac{Z_0 \chi}{2\beta_R^2 \gamma_R} dy e^{-2\pi y/\lambda} (e^{-i\phi}) + \alpha E. \quad (4.37)$$

Note the difference in the second term of above equation as compared to the corresponding term in Eq. (4.15) for the CFEL. Here, a factor $\beta_R^2 \beta_Z$ appears, which shows large diffraction in the surface mode in an SP-FEL as compared to the CFEL. We would like to emphasize that although the diffraction term in the above equation has the same form as in the case of undulator based FEL \cite{42}, the free-space wavelength $\lambda$ appearing in this term for the undulator based FEL is replaced with $\beta_R \lambda$ in the CFEL and $\lambda/\beta_Z$ in the SP-FEL, and this is an important finding of our analysis.

### 4.5 Discussions and conclusion

In this chapter, we have presented a three-dimensional analysis of the surface mode in Čerenkov and Smith-Purcell FELs. Expressions have been derived for the electromagnetic field in a localized surface mode by suitably combining the plane wave solutions of Maxwell equations, propagating at different angles in the $(y, z)$ plane. A crucial input for this calculation was to have the information about the change in $k_z$, after we include the $\exp (ik_y)$-type dependence in the electromagnetic field, keeping the value of $\omega$ fixed. For the case of the CFEL, this was
simplified due to “isotropic nature” of the system in the \((y, z)\) plane, and in the case of the SP-FEL, this was simplified due to the “replacement rule” for the evaluation of reflectivity of the incident evanescent wave. Interestingly, the “isotropic nature” is not applicable for the SP-FEL case and the “replacement rule” is not applicable for the CFEL case.

A three-dimensional analysis of the surface mode allows us to include the effect of diffraction, which plays an important role in the performance of the CFEL and the SP-FEL systems. We have explained in the chapter that for an isotropic system, as in the case of the CFEL, the free-space wavelength \(\lambda\) in the diffraction term in the wave equation gets replaced with \(\beta_R \lambda\). On the other hand if the system is not isotropic, but the electromagnetic field is present only in vacuum, as in the case of the SP-FEL, \(\lambda\) gets replaced with \(\lambda/\beta_L\). Due to this difference, diffraction effects are observed to be prominent in the SP-FELs as compared to the CFELs. To incorporate the 3D effects in the analytical formulas for the gain and the growth rate of the CFEL system discussed in Chapter 3, we have taken the electron beam size to be same as the effective optical beam size, which has been evaluated by taking the 3D variations in the surface mode.

We also included the effect of the dielectric losses and losses due to finite conductivity of the metal, which play an important role when we increase the interaction length in order to increase the gain in a CFEL. Although all earlier analyses on the single slab CFELs have ignored this effect, it is important to take such realistic effects into account in a practical device, as is the case in any device using guided waves at high frequency. It is interesting to point out that even in the case of SP-FELs, the effect of attenuation was neglected in earlier studies, and its importance was realized in later studies \([99, 164]\). In order to reduce the loss due to finite conductivity of metal in a CFEL, we have proposed that the metallic base can be kept at low temperature, i.e., 77 K. We have optimized the parameters for a CFEL designed to operate at 0.1 THz and have shown that using a 40 keV electron beam with a current of 35 mA, an optimized CFEL oscillator can deliver an output power of 3.6 W at saturation with an efficiency of 0.26%.

Our overall approach to study CFELs is built on the earlier analyses given for SP-FELs in
Ref. [78, 89]. Like the SP-FEL [89], the requirements on the vertical beam emittance in a CFEL come out to be stringent and we have discussed two ways to relax the stringent requirements. In the first scheme, a wiggler magnetic field is used to focus a flat electron beam, which is produced by a novel phase-space technique discussed in Ref. [78]. This scheme requires a peak wiggler field of about 1.3 kG to focus a flat electron beam having transverse emittance ratio $\varepsilon_y/\varepsilon_x = 100$. Such a DC electron beam can be produced by employing a round to flat beam transformation to the initially round electron beam produced using a thermionic cathode such as LaB$_6$ as described in Ref. [78]. This technique of round to flat beam transformation has been demonstrated experimentally at Fermi National Accelerator Laboratory to generate an electron beam directly from a photoinjector with transverse emittance ratio of 100 [145]. In the second scheme, we used a solenoid field to focus a flat electron beam, which is produced by a line shaped tungsten cathode placed at the centre of a solenoid. The solenoid field is taken as 1 kG with field uniformity $\Delta B/B$ required to be better than 0.4% over a length of 5 cm. We would like to mention here that there could be problems with the transmission of a flat beam by using a uniform solenoid magnetic field at higher beam currents as discussed earlier in Refs. [152, 165], where it is pointed out that due to $E_x \times B(0)$ drift, where $E_x$ is the electric field from space charge, the flat beam gets a vertical kick in the opposite direction at its two edges, which results in the edge curling phenomenon. Beyond a certain threshold, this leads to instabilities like diocotron and/or filamentation instabilities [152], which can disrupt the flat nature of the electron beam resulting in significant interception of the beam. An analytic estimate for the threshold length $L_D$, after which the diocotron instability grows exponentially can be given by $L_D(\text{cm}) = 800 \beta_R^2 \gamma^3 B(0) \text{(kG)}/J_e \text{ (A/cm}^2 \text{)}$ [152]. It can be seen that for a given beam energy and focussing field strength, the diocotron instability is suppressed at low beam current densities $J_e$ or equivalently at reduced effective space charge. The considered electron beam in our analysis is not space charge dominated as the space charge term in the envelope
equation is small compared to the emittance term, i.e., \( I\Delta x^3 / 4\beta R \gamma R I \Delta e_x(x^2 + \Delta y) < 1 \) [144] for the \( x \)-direction. We have evaluated the left-hand side in this inequality and obtained its value as 0.86. The condition in the \( y \)-direction is less restrictive for our case. For the beam current density of 3.2 A/cm\(^2\) and axial magnetic field of 1.0 kG, we find the lower bound estimate for \( L_D \) as 44.1 cm. The proposed length of CFEL system (5 cm) is about 9 times less than the threshold growth length, hence, the diocotron instability due to \( E_s \times B(0) \) effect is not of concern in our system.

It is also important to mention here that for the thermionic cathode, we have taken only the thermal emittance into consideration. In reality, the beam emittance could be larger than this [166]. We have checked that with suitable change in the parameters, our schemes would still work. For the case of wiggler focusing, this would requires us to choose a smaller cathode size and hence, the beam current density at cathode would increase and also the magnetic field required at cathode for the flat beam production would increase. We have checked that even if the total emittance is twice the thermal emittance, the required current density and magnetic field required at cathode are increased by 4 times, and these values are still easily achievable. For the case of solenoid focussing, if we take the total emittance as twice the thermal emittance, the required minimum solenoid focussing field \( B(0) \) increases by a factor of two and therefore becomes 0.50 kG. In our calculation, we have considered a solenoid focussing field of 1 kG, which is still higher than 0.50 kG.

To summarize, we have performed a 3D analysis of the surface mode and set up 3D Maxwell-Lorentz equations for a CFEL and a SP-FEL system. Based on these results, we have made some interesting comparison between the analyses of these systems. We have optimized the parameters of a Čerenkov FEL by including the 3D effects and attenuation due to dielectric and Ohmic losses, and found that the device can produce copious THz radiation even after including these effects. Our analysis can be used for the detailed optimization of both the CFEL and the SP-FEL systems.