Chapter 1

Preliminaries

In this chapter we present some basic definitions and results regarding different algebraic structures and categories arising out of these structures used in the sequel. For the concepts in lattice we follow Birkhoff ([4]), Gratzer ([10]), T.S. Blyth ([6]) and Halmos ([8]). Regarding semigroup theory, we follow J.M. Howie ([15]), P.A. Grillet ([12]) and Clifford and Preston ([7]). For rings and modules we follow Musili ([24]), Artin ([2]) and Serge Lang ([18]). For the definitions and results regarding category and cross-connections, we follow S.Mclane ([17]) and K.S.S. Nambooripad ([25]).

1.1 Lattices

Here we recall definitions and basic results regarding partially ordered sets and lattices.

**Definition 1.1.1** ([6], page 1). If $L$ is a nonempty set then by a partial order on $L$ we mean a binary relation on $L$ that is reflexive, antisymmetric and transitive. We usually denote a partial order by the symbol $\leq$. Thus $\leq$ is a partial order on $L$ if and only if

$(1) \ \forall a \in L, a \leq a$ (reflexive);
(2) \( \forall a, b \in L, \) if \( a \leq b \) and \( b \leq a \) then \( a = b \) (antisymmetric); and

(3) \( \forall a, b, c \in L, \) if \( a \leq b \) and \( b \leq c \) then \( a \leq c \) (transitive).

**Example 1.1.1** (cf.[6], Example 1.3). On the set \( \mathbb{N} \) of natural numbers the relation of divisibility is a partial order.

Let \( (L, \leq) \) be a poset and \( B \subseteq L \).

- \( a \in L \) is called an upper bound of \( B \) \( \iff \forall b \in B : b \leq a. \)

- \( a \in L \) is called a lower bound of \( B \) \( \iff \forall b \in B : a \leq b. \)

- The greatest amongst the lower bounds, whenever it exists, is called the infimum of \( B \), and is denoted by \( \inf B \).

- The least upper bound of \( B \), whenever it exists, is called the supremum of \( B \), and is denoted by \( \sup B \).

**Definition 1.1.2** ([10]). A lattice is a poset \( (L, \leq) \) such that \( \sup \{a, b\} \) and \( \inf \{a, b\} \) exist for all \( a, b \in L \). A sublattice of \( L \) is a nonempty subset \( K \) of \( L \) such that \( K \) is closed under join and meet of \( L \).

**Example 1.1.2** (cf.[6], Example 2.8). Let \( V \) be a vector space and \( \text{Sub}V \) denotes the set of subspaces of \( V \) then \( \langle \text{Sub}V; \cap, +, \subseteq \rangle \) is a lattice.

A subset \( I \) of a lattice \( L \) is called an ideal if it is a sublattice of \( L \) and \( x \in I \) and \( a \in L \) imply that \( x \land a \in I. \) An ideal \( I \) of \( L \) is proper if \( I \neq L. \) The principal ideal \( L(x) \) of \( L \) generated by \( x \in L \) is \( L(x) = \{y \in L | y \leq x\} \). It is the smallest ideal of \( L \) containing \( x \). A lattice in which every subset has meet and join is a complete lattice. If \( (L, \leq) \) is a lattice, so is its dual \( (L, \geq) \).

If a lattice \( L \) contains the smallest (greatest) element with respect to \( \leq \), then this uniquely determined element is called the zero element
(one element), denoted by 0 (1). 0 and 1 are called universal bounds.

The principal ideal \( L(x) \) of \( L \) generated by \( x \in L \) can also be denoted as the interval \([0, x] \).

**Definition 1.1.3** (cf.[6], page 77). A lattice \( L \) with 0 and 1 is called complemented if for all \( a \) in \( L \), there exists at least one element \( b \) such that \( a \lor b = 1 \) and \( a \land b = 0 \). Then \( b \) is called the complement of \( a \). A lattice \( L \) is called relatively complemented if given \( a \leq x \leq b \), an element \( y \) exists such that \( x \land y = a \) and \( x \lor y = b \). A lattice \( L \) is called modular if for every \( a, b, c \in L \), \( a \leq c \Rightarrow (a \lor b) \land c = a \lor (b \land c) \).

A lattice \( L \) called distributive if for all \( a, b, c \in L \),

\[
a \lor (b \land c) = (a \lor b) \land (a \lor c) \text{ or } \]  
\[
a \land (b \lor c) = (a \land b) \lor (a \land c). \]

A complemented distributive lattice is called Boolean lattice. In a Boolean lattice complement of each element is unique.

**Example 1.1.3** (cf.[6], Example 6.2). Lattice of all subspaces of a vector space is a complemented modular lattice, however the complement is not unique.

**Example 1.1.4** ([28], page 72). The principal right [left] ideals of a regular ring form relatively complemented modular lattice.

**Example 1.1.5** ([8], page 8). The power set of any set \( X \), \( (P(X), \cap, \cup, ^c) \) is a Boolean lattice.

**Definition 1.1.4** ([8]). An ideal of a Boolean lattice \( L \) is a set \( I \subseteq L \) such that

1. \( 0 \in I \),
2. if \( a \in I \) and \( b \in I \), then \( a \lor b \in I \), and
3. if \( a \in I \) and \( b \in L \), then \( a \land b \in I \).
The principal ideal of $L$ generated by $a$ in $L$ is $L(a) = [0, a]$.

**Definition 1.1.5** (cf.[8], page 202). *Complete ideal* in a Boolean lattice $L$ is an ideal $I$ of $L$ such that if $\{a_i\}$ is a family in $I$ with a supremum $a$ in $L$, then $a \in I$.

Principal ideals are examples of complete ideals.

**Definition 1.1.6** ([8], page 89). Let $A$ and $B$ be Boolean lattices. A (Boolean) homomorphism is a mapping $f : A \to B$ such that, for all $p, q \in A$:

1. $f(p \land q) = f(p) \land f(q)$,
2. $f(p \lor q) = f(p) \lor f(q)$ and
3. $f(a^c) = f(a)^c$.

**Theorem 1.1.1** (cf.[8], Theorem 21). The class of all complete ideals in a Boolean lattice $L$ is itself a complete Boolean lattice with respect to the distinguished Boolean elements and operations defined by

(1) $0 = \{0\}$,

(2) $1 = L$,

(3) $M \land N = M \cap N$,

(4) $M \lor N = \bigcap \{I : I$ is a complete ideal in $L$ and $M \cup N \subseteq I\}$,

(5) $M^c = \{p \in L : p \land q = 0$ for all $q \in M\}$.

### 1.2 Semigroups

The formal study of semigroups began in the early twentieth century. A semigroup is a nonempty set $S$ with a binary operation from $S \times S \to S$ as $(x, y) \to xy$ such that $x(yz) = (xy)z$ for all $x, y, z \in S$. A subset $T$
of a semigroup $S$ is a subsemigroup of $S$ if $T$ is a semigroup with respect to the restriction of the binary operation of $S$ to $T$. A semigroup with an identity element is called a monoid. A semigroup $S$ is commutative if the product in $S$ is commutative. An element $e \in S$ is said to be an idempotent if $e^2 = e$. The set of idempotents of $S$ is denoted as $E(S)$.

**Example 1.2.1** ([15]). $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$ are semigroups with respect to addition and multiplication of natural numbers.

**Example 1.2.2** (cf.[15], page 6). The set of all maps from a set $X$ into $X$ with the binary operation as composition of maps is a semigroup which is called the *full transformation semigroup* on $X$.

**Example 1.2.3** (cf.[15], page 16). The set of all binary relations on a set $X$ is a semigroup denoted by $B_X$ with the operation ‘$\circ$’ defined as, for all $\rho, \sigma \in B_X$,

$$\rho \circ \sigma = \{(x, y) \in X \times X : (\exists z \in X)(x, z) \in \rho \text{ and } (z, y) \in \sigma\}.$$ 

An element $\phi$ of $B_X$ is called a *partial map* of $X$ if $|x\phi| = 1$ for all $x$ in $\text{dom}\phi$, that is, if, for all $x, y_1, y_2 \in X$,

$$[(x, y_1) \in \phi \text{ and } (x, y_2) \in \phi] \Rightarrow y_1 = y_2.$$ 

The set of all partial maps of $X$ denoted as $\mathcal{PT}_X$ is a subsemigroup of $B_X$ with the same operation as in $B_X$, called partial transformation semigroup. $T_X$ is also a subsemigroup of $B_X$.

**Definition 1.2.1** ([14]). An element $a$ of a semigroup $S$ is called *regular* if there exists an element $x$ in $S$ such that $axa = a$. The semigroup $S$ is called *regular semigroup* (von Neumann regular) if all its elements are regular.

**Example 1.2.4** ([15]). The full transformation semigroup $T_X$ on a set $X$ is regular and $\mathcal{PT}_X$ is a regular subsemigroup of $T_X$. 

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Definition 1.2.2 (cf. [25], page 45). A left translation in a semigroup $S$ is a mapping $\lambda : S \to S$ such that $\lambda(xy) = \lambda(x)y$ for all $x, y \in S$. If $\lambda$ and $\mu$ are left translations then so is $\lambda\mu$. Dually a right translation in $S$ can be defined.

Ideals and Green’s Relations

Green’s relations are five equivalence relations that characterise the elements of a semigroup in terms of the principal ideals they generate, are important tools for analyzing the ideals of a semigroup and related notions of structure. The relations are named after James Alexander Green, who introduced them in a paper in 1951. Instead of working directly with a semigroup $S$, we define Green’s relations over the monoid $S^1$ (see [15]).

Let $S$ be a semigroup. $I \subseteq S$ is called left [right] ideal of a semigroup $S$ if $SI \subseteq I$, $IS \subseteq I$. The principal left ideal of a semigroup $S$ generated by $a$ is $S^1a = \{sa | s \in S^1\}$ where $S^1$ is the semigroup $S$ with an identity adjoined if necessary. That is, $S^1a$ is $Sa \cup \{a\}$. Dually principal right ideal also can be defined.

The Green’s relations on a semigroup $S$ written as $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are defined as follows: For elements $a$ and $b$ of $S$,

$$a\mathcal{L}b \iff S^1a = S^1b,$$

$$a\mathcal{R}b \iff aS^1 = bS^1$$

$$a\mathcal{J}b \iff S^1aS^1 = S^1bS^1$$

where $S^1a$, $aS^1$ and $S^1aS^1$ are principal left, right and two sided ideals generated by $a$ respectively. The Green’s relations $\mathcal{D}$ and $\mathcal{H}$ are defined as

$$\mathcal{D} = \mathcal{L} \lor \mathcal{R},$$

$$\mathcal{H} = \mathcal{L} \land \mathcal{R}.$$
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For commutative semigroups all the Green’s relations coincide. \( \mathcal{L} \)-class, \( \mathcal{R} \)-class, \( \mathcal{H} \)-class, \( \mathcal{D} \)-class, \( \mathcal{J} \)-class containing the element \( a \) are denoted by \( \mathcal{L}_a, \mathcal{R}_a, \mathcal{H}_a, \mathcal{D}_a, \mathcal{J}_a \) respectively. Partial orders are defined on the quotient sets \( S/\mathcal{L}, S/\mathcal{R}, S/\mathcal{J} \) as follows:

\[
\mathcal{L}_a \leq \mathcal{L}_b \iff S^1a \subseteq S^1b \\
\mathcal{R}_a \leq \mathcal{R}_b \iff aS^1 \subseteq bS^1 \\
\mathcal{J}_a \leq \mathcal{J}_b \iff S^1aS^1 \subseteq S^1bS^1.
\]

We see that \( S/\mathcal{L}(S/\mathcal{R}, S/\mathcal{J}) \) is isomorphic to the partially ordered set of all principal left (right, two-sided) ideals of \( S \) ordered by inclusion.

**Proposition 1.2.1** (cf.[15], Proposition 2.1.1). Let \( a, b \) be elements of a semigroup \( S \). Then \( a \mathcal{L} b \) if and only if there exist \( x, y \in S^1 \) such that \( xa = b, yb = a \). Also \( a \mathcal{R} b \) if and only if there exists \( u, v \in S^1 \) such that \( au = b, bv = a \).

Let \( S \) be a semigroup. A relation \( R \) on the set \( S \) is called left compatible (with the operation on \( S \)) if

\[
(\forall s, t, a \in S), \ (s, t) \in R \Rightarrow (as, at) \in R,
\]

and right compatible if

\[
(\forall s, t, a \in S), \ (s, t) \in R \Rightarrow (sa, ta) \in R.
\]

It is called compatible if

\[
(\forall s, t, s', t' \in S), \ [(s, t) \in R \text{ and } (s', t') \in R] \Rightarrow (ss', tt') \in R.
\]

A left [right] compatible equivalence is called a left [right] congruence and a compatible equivalence relation is called a congruence (cf.[14]). Thus it can be seen that \( \mathcal{L} \) is a right congruence and \( \mathcal{R} \) is a left congruence.
Definition 1.2.3 ([12]). A fundamental semigroup $S$ is a semigroup in which the equality on $S$ is the only congruence contained in $\mathcal{H}$, that is semigroups having no non-trivial idempotent separating congruences.

The fundamental semigroups were first introduced by Munn in 1966 ([23]).

1.3 Rings

A ring is an algebraic structure with operations that generalize the arithmetic operations of addition and multiplication. A ring is a basic structure in algebra and by a ring we always mean an associative ring with identity.

Definition 1.3.1 (cf.[24], Definition 1.1.1). A nonempty set $R$ together with two binary operations called addition ($+$) and multiplication ($\cdot$) on $R$ is called a ring, if

1. $(R,+)$ is an abelian group,

2. $(R,\cdot)$ is a semigroup and

3. Distributive laws hold.

Thus the theory of rings is a combination of a semigroup and an abelian group structure usually written as $(R,+,\cdot)$. In the ring $(R,+,\cdot)$, if the semigroup $(R,\cdot)$ has an identity, it is unique and is denoted by 1 and is called the identity element or the unity of $R$. A ring $R$ is said to be commutative if the semigroup$(R,\cdot)$ is commutative. Subring of a ring $R$ is a non-empty subset $S$ of $R$ such that $(S,+)$ is a subgroup of $(R,+)$ and $(S,\cdot)$ is a subsemigroup of $(R,\cdot)$. 
Example 1.3.1 (cf.[24], page 5). The set of all integers($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), real numbers ($\mathbb{R}$) and complex numbers ($\mathbb{C}$) are commutative rings with unity under the standard operations of addition and multiplication. The subsets $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ are all subrings of $\mathbb{C}$.

Example 1.3.2 (cf.[24], Definition 1.8.3, page 24). Gaussian integers $\mathbb{Z}[i]$ where $i \in \mathbb{C}$ defined by $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$ is a ring.

Example 1.3.3 (cf.[24], page 15). Let $n \in \mathbb{N}$. The set of all $n \times n$ matrices over $\mathbb{R}$ is a ring with respect to usual addition and multiplication of matrices.

Definition 1.3.2 (cf.[24], Definition 3.1.1). A mapping $f : R \to S$ of rings $R$ and $S$ is called a homomorphism if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$. An isomorphism of rings is a bijective homomorphism.

An integral domain $R$ is a nonzero ring having no zero divisors. That is if $ab = 0$, then $a = 0$ or $b = 0$ and also $1 \neq 0$ in $R$.

A ring $R$ is a division ring if every nonzero element of $R$ has a multiplicative inverse in $R$. A commutative division ring is called a field.

Definition 1.3.3 (cf.[24], Definition 4.1.1, page 107). Let $a, b \in R, a \neq 0$ where $R$ is a ring. We say that $a$ divides $b$ or $a$ is a divisor of $b$ and written $a | b$ if there exists $c \in R$ such that $b = a \cdot c$.

Definition 1.3.4 (cf.[24], Definition 4.2.1). A commutative integral domain $R$ (with or without unity) is called a Euclidean domain if there is a map $d : R^* \to \mathbb{Z}^+$ where $R^* = R \setminus \{0\}$ such that:

1. $\forall a, b \in R^*, a | b \Rightarrow d(a) \leq d(b)$

2. Given $a \in R, b \in R^*$, there exists $q, r \in R$ (depending on $a$ and $b$) such that $a = qb + r$ with either $r = 0$ or else $d(r) < d(b)$. 

**Example 1.3.4** ([24], Example 4.2.3). The ring of integers $\mathbb{Z}$, any field $F$ and ring of Gaussian integers $\mathbb{Z}[i]$ are Euclidean domains.

**Definition 1.3.5** ([28]). $(R, +, \cdot)$ is a (von Neumann) regular ring if it is a ring with the multiplicative part a regular semigroup (see Definition 1.2.1).

**Example 1.3.5** ([28]). Every field is von Neumann regular. The ring of $n \times n$ matrices $M_n(F)$ is regular with entries from some field $F$.

An element $e \in R$ is said to be idempotent if $e.e = e$. $E(R)$ denotes the set of all idempotents of $R$. The principal ideals of a regular ring are idempotent generated and form a relatively complemented modular lattice ([28]).

**Definition 1.3.6** ([24], Definition 1.7.1). A ring $R$ with identity is called **Boolean ring** if every element is an idempotent.

Boolean ring corresponds to Boolean lattice and vice versa ([8]). A subset of a Boolean lattice is a Boolean ideal if and only if it is an ideal in the corresponding Boolean ring [8]. Note that a Boolean ring is von Neumann regular ring, necessarily commutative and has cardinal number a power of 2.

**Example 1.3.6** ([24], Example 1.7.2). Let $X$ be any non-empty set and $\mathcal{P}(X)$ be the power set of $X$ with addition and multiplication on $\mathcal{P}(X)$ defined by $A + B = A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ and $AB = A \cap B$ is a Boolean ring.

**Ideals and Green’s relations in rings**

Let $R$ be a ring. The **left** (right) **ideal** $I$ of $R$ is an additive abelian group such that $RI \subseteq I[I\mathbb{R} \subseteq I]$. Let $a \in R$. The **principal left** (right) **ideal** generated by $a$ is $(a)_l = Ra$ and $(a)_r = aR$. If $I$ is simultaneously both left and right ideal of $R$, we say that $I$ is a **two-sided ideal**. Suppose $I$
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and $J$ are both left or right or two sided ideals of ring $R$. Their sum $I + J$ is defined as

$$I + J = \{x + y | x \in I, y \in J\}$$

it is the smallest ideal containing both $I$ and $J$.

Their intersection $I \cap J$ is the usual intersection of sets:

$$I \cap J = \{x | x \in I \text{ and } x \in J\}.$$

Following the Green’s relations in semigroups, analogous versions of Green’s relations have been defined for rings (cf. [29]).

1.4 Modules

A module is one of the fundamental algebraic structures in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of a ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module. Just as the linear transformations between vector spaces, we have homomorphisms between modules.

**Definition 1.4.1** (cf.[24], Definition 5.1.1). Let $R$ be any ring. A left $R$-module $M$ is an abelian group $(M, +)$ together with a map from $R \times M \rightarrow M$ as $(a, x) \mapsto ax$ called the scalar multiplication such that

1. $a(x + y) = ax + ay$ for all $a \in R$ and $x, y \in M$
2. $(a + b)x = ax + bx$ for all $a, b \in R$ and $x \in M$
3. $(ab)x = a(bx)$ for all $a, b \in R$ and $x \in M$

A left $R$-module $M$ is called **unitary left $R$-module** if $1 \cdot x = x$ for all $x \in M$. 


Similarly one can define the right $R$-module as an additive abelian group with scalar multiplication on the right. If $R$ is commutative, the notions of left and right modules coincide.

Let $M$ be an $R$-module. A nonempty subset $N$ of $M$ is called an $R$-submodule of $M$ if

1. $N$ is an additive subgroup of $M$, i.e., $x, y \in N \Rightarrow x - y \in N$

2. $N$ is closed for arbitrary scalar multiplication, i.e., $x \in N, a \in R \Rightarrow ax \in N$.

Suppose $M$ is an $R$-module and $P, Q$ are both submodules of $M$. Then the sum of the submodules $P + Q = \{x + y | x \in P, y \in Q\}$ is the smallest $R$-submodule containing both $P$ and $Q$. Their intersection $P \cap Q = \{x | x \in P \text{ and } x \in Q\}$ is the intersection of $P$ and $Q$ in the usual sense.

A homomorphism of $R$-modules $M$ and $N$ is a map $f : M \rightarrow N$ which is compatible with the laws of composition

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(ax) = af(x),$$

for all $x, y \in M$ and $a \in R$. A bijective homomorphism is called isomorphism. The kernel of a homomorphism $f : M \rightarrow N$ is a submodule of $M$; denoted by $\ker f = \{x \in M | f(x) = 0\}$ and image of $f$ is a submodule of $N$.

**Remark 1.4.1** (cf.[24], page 144-145). The direct product of two $R$-modules is again an $R$-module. For a collection of $R$-modules $\{M_i\}_{i \in I}$, the direct product $\Pi_{i \in I} M_i$ is the product of the underlying sets $M_i$ with $R$-module structure given by component-wise addition and scalar multiplication.

The direct product $\Pi_{i \in I} M_i$ is equipped with a collection of projection maps $\{\pi_i : \Pi_{i \in I} M_i \rightarrow M_i\}_{i \in I}$ given by $\pi_i((m_i)_{i \in I}) = m_i$ for all $i \in I$. Each $\pi_i$ is an $R$-module homomorphism.
The direct sum \( \bigoplus_{i \in I} M_i \) is a submodule of the direct product \( \prod_{i \in I} M_i \) consisting of elements \((m_i)_{i \in I}\) such that all but a finitely many \(m_i\) are zero.

The direct sum \( \bigoplus_{i \in I} M_i \) is equipped with a collection of injection maps \( \{ p_i : M_i \to \prod_{i \in I} M_i \}_{i \in I} \) given by \( p_i(m) = (m_i)_{i \in I} \) where for all \( j \neq i, m_j = 0 \) and \( m_i = m \), for all \( m \in M_i \). Each \( p_i \) is an \( R \)-module homomorphism.

**Example 1.4.1 ([2]).** If \( R \) is a field \( F \), then \( F \)-module is an \( F \)-vector space. Unitary modules over \( \mathbb{Z} \) are simply abelian groups. A ring \( R \) can be considered to be both a left \( R \)-module and a right \( R \)-module.

**Definition 1.4.2** (cf.[24], Definition 5.8.3). A nonzero module \( M \) is called simple module if it has only trivial submodules (0) and \( M \). A field is a simple module viewed as a module over itself. A module is called semisimple if it is a direct sum of simple modules.

The module \( M_n(D) \) is semisimple for division ring \( D \). Every simple module is semisimple. Note that the ring \( \mathbb{Z} \) is not a semisimple module over itself but \( \mathbb{Z}_n \) with \( n \), a square free integer is a semisimple module over \( \mathbb{Z} \). If \( M \) is semisimple \( R \)-module, then every submodule and every quotient module of \( M \) are semisimple.

**Remark 1.4.2.** Let \( M \) be a semisimple module and \( M = \bigoplus_{i \in I} M_i \) where \( M_i \) are simple modules and let \( W \) be a submodule of \( M \), then \( M = W \bigoplus W' \) where \( W = \bigoplus_{i : M_i \subset W} M_i \), hence \( W \) is semisimple. Also \( W' = \bigoplus_{i : M_i \cap W = 0} M_i \). As \( M/W \cong W' \), it is semisimple and it is the complement of \( W \). The submodules of a semisimple module form complemented modular lattice with respect to intersection as meet and sum as join.

**Lemma 1.4.1** (cf.[24], Shur’s Lemma). Suppose \( M \) and \( N \) are
two simple $R$-modules. Then any $R$-module homomorphism ($R$-linear map) $f : M \to N$ is either 0 or an isomorphism. In particular, the endomorphism ring $\text{End}_R(M)$ is a division ring.

The first isomorphism theorem for modules states that if $\theta : M \to N$ is an $R$-module homomorphism between two $R$-modules $M$ and $N$ then the induced homomorphism $\bar{\theta} : M|_{\ker \theta} \to \text{im} \theta$ is an isomorphism. For semisimple modules $M|_{\ker \theta} \cong (\ker \theta)^c$ and hence $(\ker \theta)^c \cong \text{im} \theta$.

1.5 Cross-connection of complemented modular lattice

Here we describe Grillet’s method of cross-connection on regular posets (cf.[12]) and K.S.S. Nambooripad and F.J. Pastijn’s method of cross-connection on complemented modular lattices (cf.[27]).

**Definition 1.5.1** (cf.[12], page 278). The **ideal** of a partially ordered set $X$ is a subset $Y$ of $X$ such that $x \leq y \in Y$ implies $x \in Y$. The **principal ideal** $X(x)$ of $X$ generated by $x \in X$ is $\{y \in X | y \leq x\}$; it is the smallest ideal of $X$ containing $x$.

**Definition 1.5.2** (cf.[12], page 278). Let $X$ be a partially ordered set, a mapping $f : X \to X$ is a **normal mapping** if it has the following three properties:

1. $f$ is order preserving;
2. the range $\text{im} f$ of $f$ is a principal ideal of $X$;
3. for each $x \in X$ there exists $y \leq x$ such that $f$ maps $X(y)$ isomorphically upon $X(xf)$.

In particular, if $f$ is normal, then there exists at least one element $b \in X$ such that $f$ is an isomorphism of $X(b)$ onto $X(a) = \text{im} f$. We denote by $M(f)$ the set of all elements $b \in X$ with this property.
Example 1.5.1. Consider the partially ordered set $P$ below,

\[
\begin{array}{ccc}
| & & |\\
 a & b & \\
| & & |\\
 d & & c
\end{array}
\]

The map $f : P \to P$ defined by $f : a \to b, b \to b, c \to c, d \to d$ is a normal mapping with $M(f) = \{a, b\}$.

The set of all normal mappings from $X$ to $X$, denoted by $N(X)$ is a semigroup under composition. The elements of $N(X)$ will be written as right operators and the elements of its dual $N^\text{op}(X)$ will be written as left operators. Idempotent normal mappings are called normal retractions and a principal ideal $X(a)$ is called normal retract if principal ideal $X(a) = \text{ime}$ where $e$ is some normal retraction. $X$ is called regular poset if every principal ideal of $X$ is a normal retract.

Example 1.5.2 (cf.[12], page 278). If $S$ is a regular semigroup, $\mathcal{L}$ and $\mathcal{R}$ are Green’s relations; then $\Lambda = S/\mathcal{L}$ and $I = S/\mathcal{R}$ are regular posets.

Definition 1.5.3 (cf.[12]). An equivalence relation $\rho$ on a poset $P$ is said to be normal if there exists a normal mapping $f \in N(P)$ such that $ker f = ff^{-1} = \rho$.

The poset (under the reverse of inclusion) of all normal equivalences on $P$ is denoted by $P^\rho$ such that when $P$ is regular, then so is $P^\rho$ ([13]). With each $\rho \in P^\rho$ we may associate the subset $M(\rho)$ defined by $M(\rho) = M(f)$, where $f$ is any normal mapping with $ker f = \rho$ and $a \in M(\rho)$ iff $P(a)$ intersects every $\rho$-class in exactly one element. Then $P(a) \cap \rho(x)$ contains a single element which is minimal in its $\rho$-class and the mapping $\varepsilon(\rho, a)$ which sends each $x$ in $P$ to the unique element in $P(a) \cap \rho(x)$ is a normal retraction with $ker \varepsilon(\rho, a) = \rho$ and
\[ \text{im} \varepsilon_p(\rho, a) = P(a). \quad \varepsilon_p(\rho, a) \text{ is called the projection along } \rho \text{ upon } P(a) \] (cf. [13]).

**Proposition 1.5.1** (cf. [27], Proposition 1). Let \( I \) and \( \Lambda \) be regular posets and \( f : I \to \Lambda \) be a normal mapping. For \( \sigma \in \Lambda^o \), define \( f^o(\sigma) = \text{ker}(f \varepsilon_\Lambda(\sigma, u)) = \sigma f^{-1} \) where \( u \in M(\sigma) \). Then \( f^o : \Lambda^o \to I^o \) is a normal mapping such that \( \text{im} f^o = I^o(\text{ker} f) \) and \( M(f^o) = \{ \rho \in \Lambda^o | b \in M(\rho) \} \) where \( \text{im} f = \Lambda(b) \). If \( P, Q \) and \( R \) are regular partially ordered sets, and if \( f : P \to Q \) and \( g : Q \to R \) are normal mappings then \((fg)^o = f^o g^o\).

**Definition 1.5.4** (cf. [6], page 7). An order preserving mapping \( f : P \to Q \) of posets \( P \) and \( Q \) is said to be residuated if there exists an order preserving mapping \( f^+ : Q \to P \) such that \( f \cdot f^+ \geq id_P \) and \( f^+ \cdot f \leq id_Q \). The mapping \( f^+ \) is called the residual of \( f \).

**Example 1.5.3** (cf. [6], Example 1.17). If \( E \) is any set and \( A \subseteq E \) then for the power set \( P(E) \) of \( E \), \( \lambda_A : P(E) \to P(E) \) defined by \( \lambda_A(X) = A \cap X \) is residuated with residual \( \lambda_A^+ \) given by \( \lambda_A^+(Y) = Y \cup A^c \).

**Example 1.5.4** (cf. [6], Example 1.20). If \( S \) is a semigroup, define a multiplication on the power set \( P(S) \) of \( S \) by

\[
XY = \begin{cases} 
\{ xy | x \in X, y \in Y \} & \text{if } X, Y \neq \emptyset; \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

Then multiplication by a fixed subset of \( S \) is a residuated mapping on \( P(S) \).

The set \( ResP \) of all residuated maps of \( P \) is a semigroup and \( f \to f^+ \) is a dual isomorphism of \( ResP \) onto the semigroup \( Res^+P \) of all residuals of elements of \( ResP \). An \( f \in ResP \) is totally range closed if \( f \) maps principal ideals onto principal ideals. Observe that a residuated map that is also normal must be totally range closed. Further \( f \in ResP \) is strongly range closed if \( f \) and \( f^+ \) are totally range closed.
transformations of \( P \) and \( P^{\text{op}} \) respectively where \( P^{\text{op}} \) is the dual of \( P \).

The set \( B(P) \) of all strongly range closed transformations of \( P \) is a subsemigroup of \( \text{ResP} \) and \( f \rightarrow f^+ \) is an isomorphism of \( B(P) \) onto \( B(P^{\text{op}}) \). If \( f \in \text{ResP} \) and if both \( f \) and \( f^+ \) are normal, then \( f \) is binormal mapping and \( f \in B(P) \).

In [27] it is described that if \( I \) and \( \Lambda \) are regular posets and \( \Gamma : \Lambda \rightarrow I^o, \Delta : I \rightarrow \Lambda^o \) are order preserving mappings, then \( (f,g) \in N(I) \times N(\Lambda)^{\text{op}} \) is compatible with \((\Gamma,\Delta)\) if the following conditions hold:

1. \( \text{im} f = I(x), \text{im} g = \Lambda(y) \Rightarrow \text{ker} f = \Gamma(y), \text{ker} g = \Delta(x), \)

2. the following diagrams commute:

\[
\begin{array}{ccc}
I & \xrightarrow{\Delta} & \Lambda^o \\
\uparrow f & & \uparrow g^o \\
I & \xrightarrow{\Delta} & \Lambda^o \\
\end{array}
\quad
\begin{array}{ccc}
\Lambda & \xrightarrow{\Gamma} & I^o \\
\uparrow g & & \uparrow f^o \\
\Lambda & \xrightarrow{\Gamma} & I^o \\
\end{array}
\]

The definition of cross-connection is given in the following theorem:

**Theorem 1.5.1** (cf.[27], Theorem 2). Let \( I, \Lambda \) be regular partially ordered sets and let \( \Gamma : \Lambda \rightarrow I^o, \Delta : I \rightarrow \Lambda^o \) be order preserving mappings. Then \([I,\Lambda;\Gamma,\Delta]\) is a cross-connection if and only if the following conditions are satisfied:

1. \( x \in M(\Gamma(y)) \Leftrightarrow y \in M(\Delta(x)), x \in I, y \in \Lambda, \)

2. if \( x \in M(\Gamma(y)), \) then the pair

\[
(\varepsilon_I(\Gamma(y),x),\varepsilon_\Lambda(\Delta(x),y))
\]

is compatible with \((\Gamma,\Delta)\).

**Proposition 1.5.2** ([12], Proposition 2.3). Let \([I,\Lambda;\Gamma,\Delta]\) be a cross-connection between two regular posets \( I \) and \( \Lambda \). Then \( U = \)
$U(I, \Lambda; \Gamma, \Delta)$ consisting of all the pairs $(f, g) \in N(I) \times N(\Lambda)^{op}$ that are compatible with $(\Gamma, \Delta)$ is a fundamental regular semigroup.

In particular if the regular poset becomes a complemented modular lattice $L$, the cross-connection of complemented modular lattices $L$ and its dual $L^{op}$ is described below (cf. [27]).

**Theorem 1.5.2** (cf. [6], Theorem 6.21, page 97). If $L$ is a lattice then a residuated mapping $f : L \to L$ is totally range closed if and only if

$$f[f^+(x) \land y] = x \land f(y), \quad (\forall x, y \in L);$$

and is dually totally range closed if and only if

$$f^+[f(x) \lor y] = x \lor f^+(y), \quad (\forall x, y \in L).$$

**Proposition 1.5.3** (cf. [27], Proposition 3). Let $L$ be a complemented modular lattice, let $a \in L$ and let $a^c$ be a complement of $a$ in $L$. Then $(a; a^c) : L \to L, x \mapsto (x \lor a) \land a^c$ is a binormal idempotent mapping such that $(a; a^c)^+ : L^{op} \to L^{op}, y \mapsto (y \land a^c) \lor a$ is the residual of $(a; a^c)$. Further

$$ker(a; a^c) = \Delta(a) = \{(x, y) | x \lor a = y \lor a\}$$

$$ker(a; a^c)^+ = \Gamma(a^c) = \{(x, y) | x \land a^c = y \land a^c\}$$

and $M(\Gamma(a)) = M(\Delta(a)) = \{a^c | a^c$ is a complement of $a$ in $L\}$.

Let $L$ be a lattice with 0 and 1. For each $a \in L$ the relation

$$\Gamma(a) = \{(x, y) | x \land a = y \land a\}$$

is an equivalence relation on $L$, and the mapping $\Gamma : L \to Eq(L)$, $a \mapsto \Gamma(a)$ is an order preserving embedding of $L$ into the poset $Eq(L)$ of all equivalence relations on $L$ ordered under the reverse of inclusion.
1.5. Cross-connection of complemented modular lattice

Note that $\Gamma(a) = \ker f_a$, where $f_a : L \to L, x \mapsto x \land a$ is a normal retraction of $L$. Hence $\Gamma(a) \in L^o$ for all $a \in L$ and

$$\Gamma : L \to L^o, a \mapsto \Gamma(a) \quad (1.2)$$

is an order preserving embedding of $L$ into $L^o$. Above proposition shows that if $L$ is a complemented modular lattice, $\Gamma$ is an order preserving embedding of $L$ into $(L^{op})^o$ also. Dually,

$$\Delta(a) = \{(x, y) | x \lor a = y \lor a\} \quad (1.3)$$

is a normal equivalence on $L^{op}$ and

$$\Delta : L^{op} \to (L^{op})^o, a \mapsto \Delta(a) \quad (1.4)$$

is an order preserving embedding of $L^{op}$ into $(L^{op})^o$. Again by the above proposition we see that if $L$ is a complemented modular lattice, then $\Delta$ is also an order preserving embedding of $L^{op}$ into $L^o$.

**Theorem 1.5.3** ([27], Theorem 6). Let $L$ be a lattice with 0 and 1, and define $\Gamma$ and $\Delta$ as defined by equations 1.1, 1.2, 1.3 and 1.4. Then the following are equivalent:

(i) $L$ is a complemented modular lattice,

(ii) $\Delta$ is an order embedding of $L^{op}$ into $L^o$,

(iii) $\Gamma$ is an order embedding of $L$ into $(L^{op})^o$,

(iv) $[L^{op}, L; \Gamma, \Delta]$ is a cross-connection.

If these conditions are satisfied, then the fundamental regular semigroup $U = U(L^{op}, L; \Gamma, \Delta)$ is given by

$$U = \{(f^+, f) | f \in B(L)\}.$$
1.6 Categories

A small category is a category in which the class of objects and class of morphisms are both sets and all categories considered here are small categories. We commence our discussion of the theory of categories with the axiomatic definition of a category and then concentrate on certain types of categories such as preadditive, additive, abelian etc. A detailed survey of the categories with subobjects, factorization etc. and the properties of the ideal categories of a regular semigroup are provided here (cf.[25]). In this thesis all morphisms are written in the order of their composition i.e., from left to right.

Definition 1.6.1 (cf.[17], page 7). A category $\mathcal{C}$ consists of the following data:

1. objects denoted by $a, b, c, ...$ and arrows (morphisms) $f, g, h, ...$

2. for each arrow $f$ there are given objects: $\text{dom}(f), \text{cod}(f)$ called the domain and codomain of $f$. We write: $f : a \to b$ to indicate that $a = \text{dom}(f)$ and $b = \text{cod}(f)$

3. given arrows $f : a \to b$ and $g : b \to c$, that is, with $\text{cod}(f) = \text{dom}(g)$ then there exists an arrow: $f \cdot g : a \to c$ called the composite of $f$ and $g$

4. for each object $a$ there is given an arrow: $I_a : a \to a$ called the identity arrow of $a$.

These data are required to satisfy the following laws:

5. associativity: $f \cdot (g \cdot h) = (f \cdot g) \cdot h$, $\forall f : a \to b, g : b \to c, h : c \to d$

6. unit: $f \cdot I_b = f = I_a \cdot f$, $\forall f : a \to b$.

A category is anything that satisfies this definition. For a category $\mathcal{C}$, we denote by $\mathcal{v}\mathcal{C}$ the set of objects of $\mathcal{C}$ and for $a, b \in \mathcal{v}\mathcal{C}$ the set of morphisms from $a$ to $b$ is denoted by $\mathcal{C}(a,b)$ or $\text{hom}(a,b)$ and is called homset.
Example 1.6.1 (cf.[17], page 12). • **Set**: Category of sets with maps,

• **Vct\(_K\)**: Category of vector spaces over a field \( K \) with linear mappings,

• **Grp**: Category of groups with group homomorphisms,

• **Ab**: Category of abelian groups with group homomorphisms,

• **Rng**: Category of rings with ring homomorphisms,

• **\( R - \text{mod} \)**: Category of left \( R \)-modules with module homomorphisms.

A subcategory \( \mathcal{C}' \) of a category \( \mathcal{C} \) is a collection of some of the objects and some of the arrows of \( \mathcal{C} \), which includes with each arrow \( f \) both the object \( \text{dom} f \) and the object \( \text{cod} f \), with each object \( s \) its identity arrow \( I_s \) and with each pair of composable arrows \( s \to s' \to s'' \) their composite. If \( \mathcal{C}'(a, b) = \mathcal{C}(a, b) \), then \( \mathcal{C}' \) is called full subcategory of \( \mathcal{C} \). For example, \( \text{Ab} \) is a full subcategory of \( \text{Grp} \).

A functor is a homomorphism of categories and is defined as follows:

**Definition 1.6.2** (cf.[17], page 13). For categories \( \mathcal{C} \) and \( \mathcal{B} \) a functor \( F : \mathcal{C} \to \mathcal{B} \) with domain \( \mathcal{C} \) and codomain \( \mathcal{B} \) consists of two suitably related functions: The object function \( F \) which assigns to each object \( c \) of \( \mathcal{C} \) an object \( F(c) \) of \( \mathcal{B} \) and the arrow function which assigns to each arrow \( f : c \to c' \) of \( \mathcal{C} \) an arrow \( F(f) : F(c) \to F(c') \) of \( \mathcal{B} \), in such a way that

\[
F(I_c) = I_{F(c)}, F(f \cdot g) = F(f) \cdot F(g),
\]

the later whenever the composite \( f \cdot g \) is defined in \( \mathcal{C} \). \( F \) is called covariant functor.

A simple example is the powerset functor \( \mathcal{P} : \text{Set} \to \text{Set} \).
Chapter 1. Preliminaries

A functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful if for each $c, c' \in \mathcal{C}$ the restriction of $F$ to $\mathcal{C}(c, c')$ is injective. $F$ is called full if for each $c, c' \in \mathcal{C}$, $F$ maps $\mathcal{C}(c, c')$ onto $\mathcal{D}(F(c), F(c'))$. An isomorphism of categories is a full and faithful functor $F$ in which $vF$ is a bijection.

**Definition 1.6.3** (cf.[25], page 36). Let $\mathcal{C}, \mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is the category with an object is a pair $(c, d)$ of object $c$ of $\mathcal{C}$ and $d$ of $\mathcal{D}$; a morphism $(c, d) \to (c', d')$ of $\mathcal{C} \times \mathcal{D}$ is a pair $(f, g)$ of arrows $f : c \to c'$ and $g : d \to d'$. The composition of morphisms are defined component wise. A *bifunctor* or a functor in two variables is a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ (where $\mathcal{A}$ is another category).

A *natural transformation* is a morphism of functors and is defined as follows:

**Definition 1.6.4** (cf.[17], page 40). Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\tau : F \to G$ is a function which assigns to each object $c$ of $\mathcal{C}$ an arrow $\tau_c : F(c) \to G(c)$ of $\mathcal{D}$ in such a way that every arrow $f : c \to c'$ in $\mathcal{C}$ yields a diagram which is commutative.

\[
\begin{array}{ccc}
F(c) & \xrightarrow{\tau_c} & G(c) \\
F(f) \downarrow & & \downarrow G(f) \\
F(c') & \xrightarrow{\tau_{c'}} & G(c')
\end{array}
\]

**Definition 1.6.5** (cf.[17], page 40). Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, then there is an associated category denoted by $[\mathcal{C}, \mathcal{D}]$ in which every functor from $\mathcal{C}$ to $\mathcal{D}$ is an object and every natural transformation between two such functors is a morphism. Any subcategory of $[\mathcal{C}, \mathcal{D}]$ is called a functor category. The category $\mathcal{C}^*$ denote the functor category $[\mathcal{C}, \text{Set}]$ and $\mathcal{C}^*$ is regarded as a dual of $\mathcal{C}$.

A morphism $f$ in a category $\mathcal{C}$ is a monomorphism if for $g, h \in \mathcal{C}$,
\(gf = hf\) implies \(g = h\); that is \(f\) is a monomorphism if it is right cancellable. Dually a morphism \(f \in \mathcal{C}\) is an epimorphism if \(f\) is left cancellable.

A morphism \(f \in \mathcal{C}(c, c')\) is called a *split monomorphism* if there exists a morphism \(g \in \mathcal{C}(c', c)\) such that \(fg = 1_c\). That is \(f\) has a right inverse. A morphism \(f \in \mathcal{C}(c, c')\) is called a *split epimorphism* if \(f\) has a left inverse. Two monomorphisms \(f, g \in \mathcal{C}\) are equivalent if there exists \(h, k \in \mathcal{C}\) with \(f = hg\) and \(g = kf\). The fact that \(f\) and \(g\) are monomorphisms imply that \(h\) is an isomorphism and \(k = h^{-1}\).

An object \(a\) is terminal in \(\mathcal{C}\) if for each object \(b\) there is exactly one arrow \(b \to a\). An object \(c\) is initial object if to each object \(b\) there is exactly one arrow \(c \to b\).

**Definition 1.6.6** ([17], page 20). A *zero object* or *null object* \(z\) in \(\mathcal{C}\) is an object which is both initial and terminal.

For any two objects \(a\) and \(b\) the unique arrows \(a \to z\) and \(z \to b\) have a composite \(O^b_a : a \to b\) called the *zero morphism* from \(a \to b\). The zero object is unique up to isomorphism and the notion of zero arrow is independent of the choice of the zero object.

**Example 1.6.2** ([17]). Zero module is the zero object in the category \(\mathbf{R} - \text{mod}\). Trivial group is the zero object in the category \(\mathbf{Grp}\).

**Definition 1.6.7** (cf. [17], page 70). An *equalizer* of \(f, g : b \to a\) in \(\mathcal{C}\) is an arrow \(e : d \to b\) such that \(e \cdot f = e \cdot g\) with that to any \(h : c \to b\) with \(h \cdot f = h \cdot g\) there is a unique \(h' : c \to d\) with \(h' \cdot e = h\).

Dually *coequalizer* of \(f, g : a \to b\) is an arrow \(u : b \to d\) such that \(f \cdot u = g \cdot u\); and if \(h : b \to c\) has \(f \cdot h = g \cdot h\), then \(h = u \cdot h'\) for a unique arrow \(h' : d \to c\).

Let \(\mathcal{C}\) has a zero object. A *kernel* of an arrow \(f : a \to b\) is defined to be an equalizer of the arrows \(f, O : a \Rightarrow b\). A kernel is necessarily
a monomorphism. Dually cokernel of \( f : a \to b \) is coequalizer of the arrows \( f, O : a \rightrightarrows b \). A cokernel is necessarily an epimorphism.

**Definition 1.6.8** (cf.\([17]\), page 68). The *product* of two objects \( a \) and \( b \) of category \( C \) is written \( a \times b \) or \( a \Pi b \) with two arrows \( p_1 : a \Pi b \to a \), \( p_2 : a \Pi b \to b \) called the projections of the product \( a \Pi b \) such that for any \( c \in C \) with given arrows \( f : c \to a \) and \( g : c \to b \), there is a unique \( h : c \to a \Pi b \) with \( h \cdot p_1 = f \) and \( h \cdot p_2 = g \).

Dually *coproduct* is written as \( a + b \) or \( a \amalg b \) with two arrows \( q_1 : a \to a \amalg b \), \( q_2 : b \to a \amalg b \) called the injections of the coproduct \( a \amalg b \) such that for any \( d \in C \) with given arrows \( f : a \to d \) and \( g : b \to d \), there is a unique \( h : a \amalg b \to d \) with \( q_1 \cdot h = f \) and \( q_2 \cdot h = g \).

*Biproduct* of a finite collection of objects, in a category with zero objects, is both a product and a coproduct. For example, in the category of \( R \)-modules \( R - \text{mod} \), the direct product of two \( R \)-modules is a biproduct.

**Definition 1.6.9** (cf.\([17]\), page 192). A *preadditive category* (or *\( Ab \)-category*) \( A \) is a category in which each homset \( A(b, c) \) is an additive abelian group and composition of arrows is bilinear relative to this addition and \( A \) has zero object.

A preadditive category with biproduct for each pair of its objects, is called *additive category*.

**Definition 1.6.10** (cf.\([17]\), page 198). An *abelian category* \( A \) is a preadditive category satisfying:

1. \( A \) has biproducts,
2. every arrow in \( A \) has a kernel and a cokernel,
3. every monomorphism is a kernel, and every epimorphism is a cokernel.

**Example 1.6.3** ([17], page 199). \( R - \text{Mod} \) and \( \text{Mod} - R \), the
categories of left and right $R$-modules with $R$-module homomorphisms are abelian categories with the usual kernels and cokernels.

**Category with Subobjects**

In the following we recall subobject relation in categories and provides some results regarding categories with subobjects from [25].

A preorder $P$ is a category such that, for any $p, p' \in vP$; the homset $P(p, p')$ contains at most one morphism. In this case, the relation $\subseteq$ on the class $vP$ defined by $p \subseteq p' \iff P(p, p') \neq \phi$ is a quasi-order on $vP$. In a preorder, $p$ and $p'$ are isomorphic if and only if $P(p, p') \neq \phi \neq P(p', p)$. Therefore $p \subseteq p'$ is a partial order if and only if $P$ does not contain any nontrivial isomorphisms. Equivalently, the only isomorphisms of $P$ are identity morphisms and in this case $P$ is said to be a strict preorder.

**Definition 1.6.11** (cf.[25], Definition 1, page 18). Let $C$ be a category and $P$ be a subcategory of $C$. Then $(C, P)$ is called a category with subobjects if the following hold:

1. $P$ is a strict preorder with $vP = vC$
2. every $f \in P$ is a monomorphism in $C$
3. if $f, g \in P$ and if $f = hg$ for some $h \in C$, then $h \in P$.

In a category with subobjects if $f : a \to b$ is a morphism in preorder $P$ then $f$ is said to be an inclusion, we denote this inclusion by $j(a, b)$.

If there is a morphism $e : b \to a$ such that $j(a, b) \cdot e = I_a$, then $e$ is called a retraction from $b \to a$ and is denoted by $e(b, a)$.

In case a retraction from $b$ to $a$ exists then the inclusion $j(a, b) : a \to b$ is a split inclusion.

Any monomorphism equivalent to an inclusion is called an embedding. Clearly every inclusion is an embedding.

Let $C$ be a category with subobjects. A morphism $f \in C$ has fac-
torization if

\[ f = p \cdot m \]

where \( p \) is an epimorphism and \( m \) is an embedding (see cf.\([25]\), page 21). A category \( C \) is said to have the factorization property if every morphism of \( C \) has a factorization.

Thus, if \( C \) has the factorization property, then any morphism \( f \) in \( C \) has at least one factorization of the form \( f = qj \), where \( q \) is an epimorphism and \( j \) is an inclusion. Factorizations of this type are called canonical factorizations.

A normal factorization of a morphism \( f \) in \( C \) is a factorization of the form

\[ f = euj \]

where \( e \) is a retraction, \( u \) is an isomorphism and \( j \) is an inclusion.

A morphism \( f \) in a category with subobjects is said to have an image if it has a canonical factorization \( f = xj \), where \( x \) is an epimorphism and \( j \) is an inclusion with the property that whenever \( f = yj' \) is any other canonical factorization, then there exists an inclusion \( j'' \) such that \( y = xj'' \). A category is said to have images if every morphism in \( C \) has an image. In this case, the codomain of \( x \) is said to be the image of \( f \).

When the morphism \( f \) has an image we denote the unique canonical factorization of \( f \) by \( f = f^o j_f \), where \( f^o \) is the unique epimorphic component and \( j_f \) is the inclusion of \( f \).

**Definition 1.6.12** ([25]). Let \( C \) be a category with subobjects, images, every morphism in \( C \) has normal factorizations in which the inclusion splits. For \( d \in vC \), a cone with vertex \( d \) is a collection of maps \( \gamma : vC \rightarrow d \) from the base \( vC \) to \( d \) satisfying the following:

1. \( \gamma(c) \in C(c, d) \) for all \( c \in vC \),

2. if \( c' \subseteq c \) then \( j(c', c) \gamma(c) = \gamma(c') \).
Definition 1.6.13 ([25] and [31]). Let \( \mathcal{C} \) be a category with sub-objects, in which inclusion splits and every morphism has normal [balanced] factorization. Then a normal [balanced] cone in \( \mathcal{C} \) is a cone with at least one component isomorphism [balanced morphism]. For a cone \( \gamma \in \mathcal{C} \), the M-set [B-set] of \( \gamma \) is defined by

\[
M_\gamma = \{ c \in \mathcal{V}_\mathcal{C} : \gamma(c) \text{ is an isomorphism} \} \quad \text{and} \quad B_\gamma = \{ c \in \mathcal{V}_\mathcal{C} : \gamma(c) \text{ is a balanced morphism} \}
\]

respectively.

The vertex \( d \) of the cone \( \gamma \) is usually denoted as \( c_\gamma \).

In [25] it is described that the set of all normal cones in the category \( \mathcal{C} \) is a regular semigroup denoted by \( \mathcal{T} \mathcal{C} \) with respect to the operation; for \( \gamma, \beta \in \mathcal{T} \mathcal{C} \)

\[
\gamma \cdot \beta = \gamma * \beta(c_\gamma)^\circ.
\]

That is for every \( a \in \mathcal{V}_\mathcal{C} \),

\[
(\gamma \cdot \beta)(a) = \gamma(a) \cdot \beta(c_\gamma)^\circ.
\]

Let \( E(\mathcal{T} \mathcal{C}) \) be the set of all idempotent normal cones in \( \mathcal{C} \) and \( \mathcal{B} \mathcal{C} \) be the concordant semigroup of all balanced cones in category \( \mathcal{C} \) (cf.[31]).
Definition 1.6.14 ([25]). A normal category is a pair \((C, P)\) satisfying the following:

1. \((C, P)\) is a category with subobjects
2. every inclusion in \(C\) splits
3. Any morphism in \(C\) has a normal factorization
4. for each \(a \in \text{Obj}(C)\) there is a normal cone \(\gamma\) with vertex \(a\) and \(\gamma(a) = I_a\).

Proposition 1.6.1 ([22], Proposition 1.2.6). Let \(C\) and \(D\) be two isomorphic normal categories, then \(TC\) is isomorphic to \(TD\) as semigroups.

Ideal categories of regular semigroup

Let \(S\) be a regular semigroup. The category of principal left ideals \(\mathbb{L}(S)\) is defined as \(\text{vL}(S) = \{Se : e \in E(S)\}\)

\(\mathbb{L}(S)(Se, Sf) = \{\rho : Se \to Sf : (st)\rho = s(t\rho)\text{ for all }s, t \in Se\}\)

Dually, the category of right ideals \(\mathbb{R}(S)\) is defined as follows:

\(\text{vR}(S) = \{eS : e \in E(S)\}\)

\(\mathbb{R}(S)(eS, fS) = \{\lambda : eS \to fS : \lambda(st) = (\lambda s)t\text{ for all }s, t \in eS\}\).

By Lemma 12 of [25], \(\mathbb{L}(S)\) is the category whose vertex set is the set of all principal left ideals and whose morphism set is the set of right translations as defined above. Let \(\rho(e, u, f) = \rho_u|_{Se}\) where \(e, f \in E(S); u \in eSf\). Then we have the following:

1. For every \(e, f \in E(S)\) and \(u \in eSf\), \(\rho(e, u, f) \in \mathbb{L}(S)(Se, Sf)\). Moreover the map \(\rho(e, u, f) \mapsto u\) is a bijection of \(\mathbb{L}(S)(Se, Sf)\) onto \(eSf\).
2. \( \rho(e, u, f) = \rho(e', v, f') \) if and only if \( eLe', fL'f', u \in eSf, v \in e'Sf' \) and \( v = e'u \).

3. If \( \rho(e, u, f) \) and \( \rho(g, v, h) \) are composable morphisms in \( \mathbb{L}(S) \) (so that \( fLg, u \in eSf \) and \( v \in gSh \)), then

\[
\rho(e, u, f) \rho(g, v, h) = \rho(e, uv, h).
\]

In particular \( \mathbb{L}(S) \) is a category with subobjects, in which every inclusion splits and every morphism has a normal factorization.

**Lemma 1.6.1** (cf.[25], Lemma 15, page 50). Let \( S \) be a regular semigroup, \( a \in S \) and \( f \in E(L_a) \). Then the map

\[
\rho^a(Se) = \rho(e, ea, f)
\]

is a normal cone in \( \mathbb{L}(S) \) with vertex \( Sa \) and such that

\[
M \rho^a = \{ Se : e \in E(R_a) \}.
\]

Moreover, \( \rho^a \) is an idempotent normal cone in \( T\mathbb{L}(S) \) if and only if \( a \in E(S) \). \( E(L_a)[E(R_a)] \) is the set of all idempotents in the \( L[R] \)-class of \( a \).

Thus it is seen that given a regular semigroup \( S \), the category \( \mathbb{L}(S) \) described above is a normal category. Further we have the following proposition.

**Proposition 1.6.2** (cf.[25], Proposition 13, page 48). Let \( S \) be a regular semigroup and the category of principal left ideals of \( S \), \( \mathbb{L}(S) \) is a normal category. Let \( \rho = \rho(e, u, f) : Se \to Sf \) be a morphism in \( \mathbb{L}(S) \). We have the following:

1. The morphism \( \rho(e, u, f) \) is a monomorphism iff \( \rho(e, u, f) \) is injective and this is true iff \( eR \)u.
2. \(\rho(e, u, f)\) is an epimorphism if it is surjective and this is true iff \(uLf\).

3. If \(Se \subseteq Sf\), then \(j(Se, Sf) = \rho(e, e, f)\) and \(\rho(f, fe, e) : Sf \to Se\) is a retraction.

**Theorem 1.6.1** (cf. [25], Theorem 19, page 53). Let \(\mathcal{C}\) be a normal category. Define \(F\) on objects and morphisms of \(\mathcal{C}\) as follows. For \(c \in \mathfrak{vC}\), let

\[vF(c) = (TC)\epsilon\]

where \(\epsilon \in E(TC)\), with \(c_\epsilon = c\); and for a morphism \(f \in \mathcal{C}(c, d)\), let

\[F : f \mapsto \rho(\epsilon, \epsilon^* f^o, \epsilon') : (TC)\epsilon \to (TC)\epsilon'\]

where \(\epsilon, \epsilon' \in E(TC)\), with \(c_\epsilon = c, \epsilon_\epsilon = d\) and \((\epsilon^* f^o)(a) = \epsilon(a) \cdot f^o\) for \(a \in \mathfrak{vC}\). Then \(F : \mathcal{C} \to \mathbb{L}(TC)\) is an isomorphism of normal categories.

From this theorem it is clear that for a normal category \(\mathcal{C}\) the set of all normal cones \(TC\) is a regular semigroup and its left ideal category \(\mathbb{L}(TC)\) is a normal category. Similarly the right ideal category \(\mathbb{R}(TC)\) is also a normal category.

**Theorem 1.6.2** (cf. [25], Theorem 16, page 51). Let \(S\) be a regular semigroup and \(S_\rho\) be the set of all right translations on \(S\). Then \(\mathbb{L}(S)\) is a normal category. Moreover there exists a homomorphism \(\bar{\rho} : S \to T\mathbb{L}(S)\) and an injective homomorphism \(\phi : S_\rho \to T\mathbb{L}(S)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
S & \xrightarrow{\rho} & S_\rho \\
\downarrow & & \phi \\
S & \xrightarrow{\bar{\rho}} & T\mathbb{L}(S)
\end{array}
\]

Similar results holds for the category \(\mathbb{R}(S)\) whose vertex set is the
set of all principal right ideals and morphisms set is the set of all left translations.

**Normal dual**

Let $\mathcal{C}$ be a normal category and $\mathcal{T}\mathcal{C}$ be the regular semigroup of normal cones. The poset of right ideals of $\mathcal{T}\mathcal{C}$ can be represented as a poset of certain set valued functors, called $H$-functors. For each $\gamma \in \mathcal{T}\mathcal{C}$, define $H$-functor, $H(\gamma; -)$ on objects and morphisms of $\mathcal{C}$ as follows:

\[
H(\gamma; c) = \{\gamma \star f^\circ : f \in \mathcal{C}(c, c)\}
\]

\[
H(\gamma; g) : \gamma \star f^\circ \mapsto \gamma \star (fg)^\circ.
\]

Let $\gamma, \gamma' \in \mathcal{T}\mathcal{C}$. If $H(\gamma; -) = H(\gamma'; -)$ then $M_\gamma = M_{\gamma'}$ (cf.[25]). In view of this result, we may write $MH(\gamma; -)$ for $M_\gamma$.

**Theorem 1.6.3** ([25], Theorem 11, page 44). Let $\mathcal{C}$ be a normal category and $\gamma, \gamma' \in \mathcal{T}\mathcal{C}$. Then

\[
\gamma \mathcal{L} \gamma' \iff c_\gamma = c_{\gamma'}.
\]

\[
\gamma \mathcal{R} \gamma' \iff H(\gamma, -) = H(\gamma', -).
\]

\[
\gamma \mathcal{D} \gamma' \iff c_\gamma \cong c_{\gamma'}.
\]

**Definition 1.6.15** (cf.[25], Definition 4, page 55). If $\mathcal{C}$ is a normal category, then the *normal dual* of $\mathcal{C}$, denoted by $N^*\mathcal{C}$ is the full subcategory of $\mathcal{C}^*$ with objects $H$-functors $H(\varepsilon, -) : \mathcal{C} \to \textbf{Set}$, where $\varepsilon$ is an idempotent normal cone, that is

\[
vN^*\mathcal{C} = \{H(\varepsilon; -) : \varepsilon \in E(\mathcal{T}\mathcal{C})\}
\]

and the morphisms are appropriate natural transformations between such functors.
The following lemma describes morphisms of $N^*\mathcal{C}$ in terms of those of $\mathcal{C}$.

**Lemma 1.6.2** ([25], Lemma 21, page 56). To every morphism $\sigma : H(\epsilon; -) \to H(\epsilon'; -)$ in $N^*\mathcal{C}$, there is a unique $\hat{\sigma} : c_{\epsilon'} \to c_\epsilon$ in $\mathcal{C}$ such that the following diagram commutes.

\[
\begin{array}{ccc}
H(\epsilon; -) & \xrightarrow{\eta_\epsilon} & \mathcal{C}(c_\epsilon, -) \\
\downarrow \sigma & & \downarrow \mathcal{C}(\hat{\sigma}, -) \\
H(\epsilon'; -) & \xrightarrow{\eta_{\epsilon'}} & \mathcal{C}(c_{\epsilon'}, -)
\end{array}
\]

In this case, the component of the natural transformation $\sigma$ at $c \in \mathcal{C}$ is the map given by $\sigma(c) : \epsilon * f^o \mapsto \epsilon' * (\hat{\sigma} f)^o$. In particular, $\sigma$ is the inclusion $H(\epsilon; -) \subseteq H(\epsilon'; -)$ if and only if $\epsilon = \epsilon' * \hat{\sigma}$. Moreover, the map $\sigma \mapsto \hat{\sigma}$ is a bijection of $N^*\mathcal{C}(H(\epsilon; -), H(\epsilon'; -))$ onto $\mathcal{C}(c_{\epsilon'}, c_\epsilon)$.

**Lemma 1.6.3** ([25], Lemma 22, page 57). Let $\epsilon, \epsilon' \in E(\mathcal{T}\mathcal{C})$. Then the map $\lambda(\epsilon, \gamma, \epsilon') \longmapsto \tilde{\gamma}$ where $\gamma \in \epsilon'(\mathcal{T}\mathcal{C})\epsilon$ and

\[
\tilde{\gamma} = \gamma(c_{\epsilon'}) \circ (c_\gamma, c_\epsilon)
\]  

(1.6)

is a bijection of $\mathbb{R}(\mathcal{T}\mathcal{C})(\epsilon(\mathcal{T}\mathcal{C}), \epsilon'(\mathcal{T}\mathcal{C}))$ onto $\mathcal{C}(c_{\epsilon'}, c_\epsilon)$.

**Lemma 1.6.4** ([25], Lemma 24, page 58). Let $\gamma \in \epsilon'(\mathcal{T}\mathcal{C})\epsilon$ and $\gamma' \in \epsilon''(\mathcal{T}\mathcal{C})\epsilon'$. Assume that $\tilde{\gamma}, \tilde{\gamma}'$ and $\hat{\gamma}' \cdot \hat{\gamma}$ are morphisms defined by Equation 1.6. Then $\tilde{\gamma}' \cdot \hat{\gamma} = \tilde{\gamma}' \cdot \tilde{\gamma}$.

**Theorem 1.6.4** ([25], Theorem 25, page 58). Let $\mathcal{C}$ be a normal category. Define $G$ on objects and morphisms of $\mathcal{C}$ as follows:

\[\nu G(\epsilon(\mathcal{T}\mathcal{C})) = H(\epsilon; -)\]

and for $\lambda = \lambda(\epsilon, \gamma, \epsilon') : \epsilon(\mathcal{T}\mathcal{C}) \to \epsilon'(\mathcal{T}\mathcal{C})$, let $G(\lambda)$ be the natural
1.7. Cross-connection of normal categories

In this section we describe the cross-connection of normal categories given by K.S.S. Nambooripad in [25].

Let $C$ be a category with subobjects. Then an ideal denoted by $\langle c \rangle$ is the full subcategory of $C$ whose objects are subobjects of $c$ in $C$. It is the principal ideal generated by $c$.

**Definition 1.7.1** (cf.[25], Definition 1, page 62). A local isomorphism between two normal categories $C$ and $\mathcal{D}$ is a functor $F : C \to \mathcal{D}$ which is inclusion preserving, fully faithful and for each $c \in vC$, $F|_{\langle c \rangle}$ is an isomorphism of the ideal $\langle c \rangle$ onto $\langle F(c) \rangle$.

If $C$ and $\mathcal{D}$ are normal categories a local isomorphism $\Gamma : \mathcal{D} \to N^*C$ is called a connection of $\mathcal{D}$ with $C$.

**Definition 1.7.2** (cf.[25], Definition 5, page 86). A cross-connection between two normal categories $C$ and $\mathcal{D}$ is a triplet $(\mathcal{D}, C; \Gamma)$ where $\Gamma : \mathcal{D} \to N^*C$ is a local isomorphism such that for every $c \in vC$, there is some $d \in v\mathcal{D}$ such that $c \in M\Gamma(d)$ where $M\Gamma(d)$ is the $M$-set of normal cone with vertex $d$. 

transformation making the following diagram commutative.

\[
\begin{array}{ccc}
H(\epsilon; -) & \xrightarrow{\eta_{\epsilon}} & C(c_\epsilon, -) \\
G(\lambda) & \downarrow & \downarrow C(\tilde{\gamma}, -) \\
H(\epsilon'; -) & \xrightarrow{\eta_{\epsilon'}} & C(c_{\epsilon'}, -)
\end{array}
\]

Then $G : \mathbb{R}(\mathcal{T}C) \to N^*C$ is an isomorphism of normal categories.

In the light of the above discussion we can conclude that for a normal category $C$, its dual $N^*C$ is also a normal category.
Proposition 1.7.1 ([25], Proposition 12, page 77). Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a connection between normal categories $\mathcal{C}$ and $\mathcal{D}$ and let $\mathcal{C}_\Gamma$ be the subcategory of $\mathcal{C}$ such that

$$v\mathcal{C}_\Gamma = \{ c \in \mathcal{C} : c \in MT(d) \text{ for some } d \in v\mathcal{D} \}$$

Then $\mathcal{C}_\Gamma$ is an ideal in $\mathcal{C}$.

Theorem 1.7.1 ([25], Theorem 15, page 81). Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a connection between normal categories $\mathcal{C}$. Then there exists a connection $\Gamma^* : \mathcal{C}_\Gamma \to N^*\mathcal{D}$ such that, for $c \in v\mathcal{C}_\Gamma$ and $d \in v\mathcal{D}$, $c \in MT(d)$ if and only if $d \in MT^*(c)$.

Definition 1.7.3. Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a connection between normal categories $\mathcal{C}$. The functor $\Gamma^* : \mathcal{C}_\Gamma \to N^*\mathcal{D}$ defined as in the above theorem is called the dual of the connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$.

Thus given a cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, there is a dual cross-connection $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$ denoted by $(\mathcal{C}, \mathcal{D}; \Gamma^*)$.

Remark 1.7.1 ([25]). Given a cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, since $N^*\mathcal{C} \subseteq \mathcal{C}^*$ ($\mathcal{C}^*$ is the category of all functors from $\mathcal{C}$ to $\textbf{Set}$), by category isomorphisms we get a unique bifunctor $\Gamma(-,-) : \mathcal{C} \times \mathcal{D} \to \textbf{Set}$ defined by

$$\Gamma(c, d) = \Gamma(d)(c) \text{ and }$$

$$\Gamma(f, g) = (\Gamma(d)(f))(\Gamma(g)(c')) = (\Gamma(g)(c))(\Gamma(d')(f))$$

for all $(c, d) \in v\mathcal{C} \times v\mathcal{D}$ and $(f, g) : (c, d) \to (c', d')$. Similarly corresponding to $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$, we have $\Gamma^*(-,-) : \mathcal{C} \times \mathcal{D} \to \textbf{Set}$ defined by

$$\Gamma^*(c, d) = \Gamma^*(c)(d) \text{ and }$$

$$\Gamma^*(f, g) = (\Gamma^*(c)(g))(\Gamma^*(f)(d')) = (\Gamma^*(f)(d))(\Gamma^*(c')(g))$$

for all $(c, d) \in v\mathcal{C} \times v\mathcal{D}$ and $(f, g) : (c, d) \to (c', d')$. 
Theorem 1.7.2 ([25], Theorem 4, page 67). Given cross-connection 
\((\mathcal{D}, \mathcal{C}; \Gamma)\), there is a natural isomorphism \(\chi_\Gamma\) between the bifunctors \(\Gamma(-,-)\) and \(\Gamma^*(-,-)\) such that \(\chi_\Gamma : \Gamma(c,d) \to \Gamma^*(c,d)\) is a bijection. \(\chi_\Gamma\) is known as the duality associated with \(\Gamma\).

Let \(\mathcal{C}\) and \(\mathcal{D}\) be \(RR\)–normal categories and \(\Gamma : \mathcal{D} \to N^*\mathcal{C}\) is a cross-connection. \(\Gamma^* : \mathcal{C} \to N^*\mathcal{D}\) be its dual cross-connection. Define

\[E_\Gamma = \{(c,d) : c \in \mathcal{C}_\Gamma, d \in \mathcal{D} and c \in M\Gamma(d)\}\].

For each \((c,d) \in E_\Gamma\), \(\gamma(c,d)\) denotes the unique cone in \(\mathcal{C}\) such that

\[c_{\gamma(c,d)} = c and \Gamma(d) = H(\gamma(c,d); -)\].

Similarly for each \((c,d) \in E_\Gamma\), there is a unique cone \(\gamma^*(c,d)\) in \(\mathcal{D}\) such that

\[c_{\gamma^*(c,d)} = d and \Gamma^*(c) = H(\gamma^*(c,d); -)\].

Let \((c,d) \in \mathcal{C}_\Gamma \times \mathcal{D}\) and choose \(c' \in \mathcal{C}_\Gamma\) and \(d' \in \mathcal{D}\) such that \((c,d'), (c',d) \in E_\Gamma\). Then every cone in \(\Gamma(c,d)\) can be represented as \(\gamma(c',d) \ast f^o\) with \(f \in \mathcal{C}(c',c)\) and every element of \(\Gamma^*(c,d)\) can be written as \(\gamma^*(c,d') \ast g^o\) with \(g \in \mathcal{D}(d',d)\). Hence for every \((c,d) \in \mathcal{C}_\Gamma \times \mathcal{D}\) and \(\gamma(c',d) \ast f^o \in \Gamma(c,d)\), we have natural isomorphism

\[\chi_{\Gamma(c,d)}(\gamma(c',d) \ast f^o) = \gamma^*(c,d') \ast g^o\]

where \((c,d), (c',d') \in E_\Gamma\) and \(f \in \mathcal{C}(c,c'), g \in \mathcal{D}(d',d)\) are such that the following diagram commutes.

\[
\begin{array}{ccc}
\Gamma(d') & \xrightarrow{\eta_{(c,d')}} & \mathcal{C}(c, -) \\
\Gamma(g) \downarrow & & \downarrow \mathcal{C}(f, -) \\
\Gamma(d) & \xrightarrow{\eta_{(c',d)}} & \mathcal{C}(c', -)
\end{array}
\]
**Definition 1.7.4** (cf.[25], Section 5.1, page 97). Let \( \Gamma \) be a cross-connection of \( \mathcal{D} \) with \( \mathcal{C} \). Define

\[
U_\Gamma = \bigcup \{ \Gamma(c, d) : (c, d) \in \mathcal{vC} \times \mathcal{vD} \} \tag{1.7}
\]

\[
U_\Gamma^* = \bigcup \{ \Gamma^*(c, d) : (c, d) \in \mathcal{vC} \times \mathcal{vD} \} \tag{1.8}
\]

**Proposition 1.7.2** (cf.[25], Proposition 31, page 99). For any cross-connection \( \Gamma : \mathcal{D} \to N^* \mathcal{C} \), \( U_\Gamma \) is a regular subsemigroup of \( \mathcal{T}_\mathcal{C} \) such that \( \mathcal{C} \) is isomorphic to \( \mathbb{L}(U_\Gamma) \). \( U_\Gamma^* \) is a regular subsemigroup of \( \mathcal{T}_\mathcal{D} \) such that \( \mathcal{D} \) is isomorphic to \( \mathbb{L}(U_\Gamma^*) \).

**Definition 1.7.5** (cf.[25], page 100). For a cross-connection \( \Gamma : \mathcal{D} \to N^* \mathcal{C} \), we shall say that \( \gamma \in U_\Gamma \) is linked to \( \delta \in U_\Gamma^* \) if there is a \( (c, d) \in \mathcal{vC} \times \mathcal{vD} \) such that \( \gamma \in \Gamma(c, d) \) and \( \delta = \chi_\Gamma(c, d)(\gamma) \).

**Theorem 1.7.3** ([25], Theorem 32, page 101). Let \( \Gamma : \mathcal{D} \to N^* \mathcal{C} \) be a cross-connection. Then

\[
\tilde{S}_\Gamma = \{(\gamma, \delta) \in U_\Gamma \times U_\Gamma^* : (\gamma, \delta) \text{ is linked} \}
\]

is a regular semigroup with the binary operation defined by

\[
(\gamma, \delta) \circ (\gamma', \delta') = (\gamma \cdot \gamma', \delta' \cdot \delta)
\]

for all \( (\gamma, \delta), (\gamma', \delta') \in \tilde{S}_\Gamma \). Then \( \tilde{S}_\Gamma \) is a sub direct product of \( U_\Gamma \) and \( U(\Gamma^*)^{\text{op}} \) and is called the cross-connection semigroup determined by \( \Gamma \).

Now consider a regular semigroup \( S \). From the above section it is clear that for a regular semigroup \( S \) the categories \( \mathbb{L}(S) \) and \( \mathbb{R}(S) \) of principal left and right ideals of \( S \) are normal categories. The set of all normal cones \( \mathcal{T}\mathbb{L}(S) \) and \( \mathcal{T}\mathbb{R}(S) \) are regular semigroups.

**Proposition 1.7.3** (cf.[25], Proposition 1, page 63). For any
regular semigroup $S$, $FS_\rho : \mathbb{R}(S) \to \mathbb{R}(\mathcal{TL}(S))$ defined by

$$FS_\rho(eS) = \rho^e(\mathcal{TL}(S))$$

and

$$FS_\rho(\lambda(e, u, f)) = \lambda(\rho^e, \rho^u, \rho^f)$$
is a local isomorphism. Dually, $FS_\lambda : \mathbb{L}(S) \to \mathbb{L}(\mathcal{TR}(S))$ defined as

$$FS_\lambda(Se) = (\mathcal{TR}(S))\lambda^e$$

and

$$FS_\lambda(\rho(e, u, f)) = \rho(\lambda^e, \lambda^u, \lambda^f)$$
is also a local isomorphism.

**Theorem 1.7.4** (cf. [25], Theorem 2, page 65). Let $S$ be a regular semigroup. For $fS \in v\mathbb{R}(S)$ and $\lambda = \lambda(e, u, f)$ in $\mathbb{R}(S)$, let $\Gamma S$ be defined on objects and morphisms of $\mathbb{R}(S)$ by:

$$v\Gamma S(fS) = H(\rho^f; -), \Gamma S(\lambda) = \eta_{\rho^e} \mathbb{L}(S)(\rho(f, u, e), -) \eta_{\rho^f}^{-1}.$$  

Then $\Gamma S$ is a local isomorphism from $\mathbb{R}(S)$ to $N^*\mathbb{L}(S)$. Dually, $\Gamma^*S$, defined on objects and morphisms of $\mathbb{L}(S)$ by

$$v\Gamma^*S(Se) = H(\lambda^e; -), \Gamma^*S(\rho) = \eta_{\lambda^f} \mathbb{R}(S)(\lambda(e, u, f), -) \eta_{\lambda^e}^{-1}.$$  

for all $Se \in v\mathbb{L}(S)$ and $\rho = \rho(f, u, e) \in \mathbb{L}(S)$, defines a local isomorphism.

Since $\Gamma S : \mathbb{R}(S) \to N^*\mathbb{L}(S) \subseteq \mathbb{L}(S)^*$, by the category isomorphisms, there is a unique bifunctor $\Gamma S(-, -) : \mathbb{L}(S) \times \mathbb{R}(S) \to \mathbf{Set}$. $\Gamma S(-, -)$ is defined on objects and morphisms as follows:

$$\Gamma S(Se, fS) = \Gamma S(fS)(Se);$$

$$\Gamma S(\rho, \lambda) = \Gamma S(fS)(\rho) \Gamma S(\lambda)(Se') = \Gamma S(\lambda)(Se) \Gamma S(f'\rho)(\rho)$$.
for all \((Se, fS) \in \mathcal{L}(S) \times \mathbb{R}(S)\) and \((\rho, \lambda) : (Se, fS) \rightarrow (Se', f'S)\).

**Theorem 1.7.5** (cf.[25], Theorem 4, page 67). Let \(S\) be a regular semigroup. Then there is a natural isomorphism \(\chi_S\) from \(\Gamma S(\cdot, \cdot)\) to \(\Gamma^* S(\cdot, \cdot)\) whose components are defined by

\[
\chi_S(Se, fS) : \rho^f \star \rho(f, u, e)^o \mapsto \lambda^e \star \lambda(e, u, f)^o
\]

for each \((Se, fS) \in v(\mathcal{L}(S) \times \mathbb{R}(S))\).

**Theorem 1.7.6** (cf.[25], Theorem 17, page 87). Let \(S\) be a regular semigroup. Then \(\Gamma S\) is a cross-connection of \(\mathbb{R}(S)\) with \(\mathcal{L}(S)\). The dual cross-connection is \(\Gamma^* S = (\Gamma S)^*\).

Note that any regular semigroup induces a cross-connection and any cross-connection of normal categories induces a regular semigroup \(\tilde{\Sigma}\) as in Theorem 1.7.3.