CHAPTER 8

HYPER TM-ALGEBRAS

8.1 INTRODUCTION

Hyper Structure in the algebraic relationship provides a generalized form of a combination of conditions or a combination of algebras or a combinations of ideals, with stringent set of conditions. Kang (2011) provides the conditions for such hyper algebra ($K$-algebra), together with implications-fuzzy hyper $K$-algebra. Jun and Shim (2002) provide the conditions for fuzzy hyper ideal for BCK-ideal and fuzzy implicative hyper BCK ideal along with hyper BCK-algebra. Borzooei and Bakhshi (2004) deal with weak hyper ideals, positive implicative BCK-ideals, and hyper subalgebra under different conditions.

Jun and Roh (2006) have given the conditions for fuzzy weak $k$-ideals and weak implicative hyper $K$-ideals. In this chapter, conditions for weakly positive implicative hyper TM-ideals have been established and proved with certain examples. ($\alpha, \beta$)-fuzzy hyper TM-algebras have also been studied with different conditions, level sets and their relationships.
8.2 HYPER TM-ALGEBRAS

Definition 8.2.1

A Hyper structure is a non-empty set $H$ together with a mapping $\circ : H \times H \to P^*(H)$ where $P^*(H)$ is the set of all non-empty subsets of $H$. If $A, B \in P^*(H)$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Here “$\circ$” is called the hyper operation.

Definition 8.2.2

A hyper TM-algebra $(H, \circ, 0)$, it is meant that a non-empty set $H$ endowed with a hyper operation “$\circ$” and a constant “0” satisfying the following axioms.

(1) $(x \circ y) \circ (x \circ z) \ll z \circ y$

(2) $x \circ 0 = \{x\}$

(3) $x \ll y, y \ll x$ implies $x = y$, for all $x, y, z \in H$ where $x \ll y$ is defined by $0 \in x \circ y$, and for every $A, B \subseteq H$, $A \ll B$ is defined for all $a \in A$, there exists $b \in B$ such that $a \ll b$.

Proposition 8.2.3

In any hyper TM-algebra $H$, the following hold:

(i) $0 \circ 0 = \{0\}$

(ii) $0 \ll x$

(iii) $x \ll x$
(iv) \( A \ll A \)

(v) \( A \subseteq B \Rightarrow A \ll B \)

(vi) \( A \ll \{0\} \Rightarrow A = \{0\}\)

(vii) \( x \circ 0 \ll \{y\} \Rightarrow x \ll y \)

(viii) \((A \circ B) \circ C = (A \circ C) \circ B, A \circ B \ll A, 0 \circ A \ll \{0\}\).

Proof

The proof follows from the Propositions 3.2.4, 3.2.5 and 3.2.6.

Example 8.2.4

Let \((H, *, 0)\) be a TM-algebra. Define \(x \circ y = \{x \ast y\}\), for all \(x, y \in H\).

Now,
\[
(x \circ y) \circ (x \circ z) = \{(x \circ y) \ast (x \circ z)\} \\
= \{(x \ast y) \ast (x \ast z)\} \\
= \{z \ast y\} \\
\ll z \circ y
\]

Hence \((H, 0)\) is a hyper TM-algebra.

Table 8.1 is a special case of this example. Here the set \(H = \{0, 1, 2, 3\}\)

<table>
<thead>
<tr>
<th>(\circ)</th>
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Definition 8.2.5

A non-empty subset \( S \) of a hyper TM-algebra \( H \) is said to be a hyper subalgebra of \( H \) if \( x \circ y \subseteq S \) for all \( x, y \in S \).

Definition 8.2.6

Let \( H \) be a TM-algebra. Then \( H \) is called a

(i) positive implicative hyper TM-algebra if
\[
(x \circ y) \circ z = (x \circ z) \circ (y \circ z)
\]
for all \( x, y, z \in H \) and

(ii) implicative hyper TM-algebra if
\[
x \circ (y \circ x) = x
\]
for all \( x, y \in H \).

Definition 8.2.7

Let \( I \) be a non-empty subset of a hyper TM-algebra \( H \) and \( 0 \in I \). Then \( I \) is

(i) a weak hyper TM-ideal of \( H \) if
\[
x \circ y \subseteq I \text{ and } y \in I \text{ imply } x \in I \text{ for all } x, y \in H
\]

(ii) a hyper TM-ideal of \( H \) if
\[
x \circ y \ll I \text{ and } y \in I \text{ imply } x \in I, \text{ for all } x, y \in H \text{ and}
\]

(iii) a strong hyper TM-ideal of \( H \) if
\[
(x \circ y) \cap I \neq \emptyset \text{ and } y \in I \text{ imply } x \in I, \text{ for all } x, y \in H.
\]
Definition 8.2.8

A non-empty subset $I$ of a hyper TM-algebra $H$ is called a weakly positive implicative hyper TM-ideal of $H$ if it satisfies

(i) $0 \in I$

(ii) $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$ imply $(x \circ z) \circ z \subseteq I$ for all $x, y, z \in H$.

8.3 FUZZY WEAKLY POSITIVE IMPLICATIVE HYPER TM-IDEALS

Definition 8.3.1

A fuzzy set $\mu$ in a hyper TM-algebra $H$ is called a fuzzy hyper TM-ideal of $H$ if

(i) $x \ll y \Rightarrow \mu(x) \geq \mu(y)$

(ii) $\mu(x) \geq \min \left\{ \inf_{a \in x \circ y} \mu(a), \mu(y) \right\}$.

Theorem 8.3.2

A fuzzy set $\mu$ in a hyper TM-algebra $H$ is a fuzzy hyper TM-ideal of $H$ if and only if the $t$ cut $\mu_t$ of $\mu$ is a hyper TM-ideal of $H$ whenever $\mu_t \neq \phi$ for $t \in [0,1]$.

Proof

It can be proved in the similar way of Theorem 5.3.8.
Theorem 8.3.3

Let $A$ be a subset of a hyper TM-algebra $H$. If $I$ is a hyper TM-ideal of $H$ such that $A \preccurlyeq I$ then $A \subseteq I$.

Proof

Since $A \preccurlyeq I$, for every $a \in A$ there exists $0 \in I$ such that $a \preccurlyeq 0$. As $a \circ 0 = \{a\}$ and I is an ideal, $a \preccurlyeq 0$ and $0 \in I$ imply that $a \in I$ and hence $A \subseteq I$.

Definition 8.3.4

A fuzzy set $\mu$ in a hyper TM-algebra $H$ is called a fuzzy weakly positive implicative hyper TM-ideal of $H$ if it satisfies

(i) $x \preccurlyeq y \Rightarrow \mu(x) \geq \mu(y)$

(ii) $\inf_{a \in (x \circ z) \circ z} \mu(a) \geq \min \left\{ \inf_{b \in (x \circ y) \circ z} \mu(b), \inf_{c \in y \circ z} \mu(c) \right\}$ for all $x, y, z \in H$.

Theorem 8.3.5

Every fuzzy weakly positive implicative hyper TM-ideal of a hyper TM-algebra $H$ is a fuzzy hyper TM-ideal of $H$.

Proof

From the definition, (i) $x \preccurlyeq y \Rightarrow \mu(x) \geq \mu(y)$ is satisfied. Let $x, y \in H$.

Taking $z = 0$ in $\inf_{a \in (x \circ z) \circ z} \mu(a) \geq \min \left\{ \inf_{b \in (x \circ y) \circ z} \mu(b), \inf_{c \in y \circ z} \mu(c) \right\}$ and using $x \circ 0 = \{x\}$
and $A \circ 0 = A$.

$$
\mu(x) = \inf_{a \in (x \circ 0) \circ 0} \mu(a) \geq \min \left\{ \inf_{b \in (x \circ y) \circ 0} \mu(b), \inf_{c \in y \circ 0} \mu(c) \right\}
$$

$$
= \min \left\{ \inf_{b \in (x \circ y)} \mu(b), \inf_{c \in y} \mu(c) \right\}
$$

$$
= \min \left\{ \inf_{b \in xy} \mu(b), \mu(y) \right\}
$$

Hence, $\mu$ is a fuzzy hyper TM-ideal of $H$.

**Theorem 8.3.6**

If $\mu$ is a fuzzy weakly positive implicative hyper TM-ideal of a hyper TM-algebra $H$, then the $t$ cut $\mu_t$ of $\mu$ is a weakly positive implicative hyper TM-ideal of $H$, where $t \in \text{Im}(\mu)$.

**Proof**

Let $\mu$ be a fuzzy weakly positive implicative hyper TM-ideal of $H$.

Therefore,

$$
x \ll y \Rightarrow \mu(x) \geq \mu(y) \text{ and } \inf_{a \in (x \circ y) \circ z} \mu(a) \geq \min \left\{ \inf_{b \in (x \circ y) \circ z} \mu(b), \inf_{c \in y \circ z} \mu(c) \right\}
$$

for all $x, y, z \in H$.

$$
\mu_t = \{ x \in H / \mu(x) \geq t \}
$$

Both the results $0 \ll x$ and $x \ll y \Rightarrow \mu(x) \geq \mu(y)$ induce the inequality

$\mu(0) \geq \mu(x)$ for all $x \in H$. Therefore, $0 \in \mu_t$ for all $t \in \text{Im}(\mu)$. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll \mu_t$ and $y \circ z \subseteq \mu_t$, where $t \in \text{Im}(\mu)$. Then, for every $b \in (x \circ y) \circ z$, there exists $b_1 \in \mu_t$ such that $b \ll b_1$, and therefore $\mu(b) \geq \mu(b_1)$.

Hence $\mu(b) \geq t$ for all $b \in (x \circ y) \circ z$. 
It follows from \( \inf_{a \in (x \circ z) \circ z} \mu(a) \geq \min \{ \inf_{b \in (x \circ y) \circ z} \mu(b), \inf_{c \in y \circ z} \mu(c) \} \) that, for every \( a \in (x \circ z) \circ z \),

\[
\mu(a) \geq \inf_{c \in (x \circ z) \circ z} \mu(c) \geq \min \{ \inf_{a \in (x \circ y) \circ z} \mu(a), \inf_{d \in y \circ z} \mu(d) \} \leq t.
\]

This implies \( a \in \mu_t \). That is \((x \circ z) \circ z \subseteq \mu_t \). Hence, \( \mu_t \) is a weakly positive implicative hyper TM-ideal of \( H \).

Now the converse of the above theorem is stated below.

**Theorem 8.3.7**

Let \( \mu \) be a fuzzy set in a hyper TM-algebra \( H \) such that \( \mu_t, t \in Im(\mu) \) is a weakly positive implicative hyper TM-ideal of \( H \). Then \( \mu \) is a fuzzy weakly positive implicative hyper TM-ideal of \( H \).

**Proof**

Assume that \( \mu_t, t \in Im(\mu) \) is a weakly positive implicative hyper TM-ideal of \( H \). Then \( \mu_t \) is a hyper TM-ideal of \( H \). It follows from Theorem 8.3.2 that \( \mu \) is a fuzzy hyper TM-ideal of \( H \), and so \( x \ll y \Rightarrow \mu(x) \geq \mu(y) \). Now, let \( t = \min \{ \inf_{b \in (x \circ y) \circ z} \mu(b), \inf_{c \in y \circ z} \mu(c) \} \). Then \( \mu(b^1) \geq \inf_{b \in (x \circ y) \circ z} \mu(b) \geq t \) and \( \mu(c^1) \geq \inf_{c \in y \circ z} \mu(c) \geq t \) for all \( b^1 \in (x \circ y) \circ z \) and \( c^1 \in y \circ z \). Hence \( b^1, c^1 \in \mu_t \).

This implies that \((x \circ y) \circ z \subseteq \mu_t \) and \( y \circ z \subseteq \mu(t) \). It follows from \( A \subseteq B \) implies \( A \ll B \) and \((x \circ y) \circ z \ll I \) and \( y \circ z \ll I \) imply \((x \circ z) \circ z \subseteq I \) that \((x \circ z) \circ z \subseteq \mu_t \) so that \( \mu(d) \geq t \) for all \( d \in (x \circ z) \circ z \). Consequently,

\[
\inf_{a \in (x \circ z) \circ z} \mu(a) \geq t = \min \left\{ \inf_{b \in (x \circ y) \circ z} \mu(b), \inf_{c \in y \circ z} \mu(c) \right\}
\]

for all \( x, y, z \in H \).
Thus $\mu$ is a fuzzy weakly positive implicative hyper TM-ideal of $H$.

8.4 ($\in, \in \lor q$)-FUZZY HYPER TM-SUBALGEBRAS

Definition 8.4.1

A fuzzy set $\mu$ in a hyper TM-algebra $H$ is called a fuzzy hyper TM-subalgebra of $H$ if

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \text{ for all } x, y \in H.$$ 

Definition 8.4.2

A fuzzy set $\mu$ in a hyper TM-algebra $H$ is called an $(\alpha, \beta)$-fuzzy hyper TM-subalgebra of $H$, where $\alpha, \beta \in \{\in, q, \in \lor q, \in \land q\}$ and $\alpha \neq \in \land q$, if for all $x, y \in H$ and $t_1, t_2 \in (0, 1]$,

$$x_{t_1 \alpha \mu}, y_{t_2 \alpha \mu} \Rightarrow z_{\min\{t_1, t_2\} \beta \mu} \text{ for all } z \in x \circ y.$$ 

Note 8.4.3

To consider the notion of fuzzy hyper TM-algebra of type $(\in \land q, \beta)$ where $\beta \in \{\in, q, \in \lor q, \in \land q\}$, a fuzzy set $\mu$ is taken in $H$ satisfying $\mu(x) > 0.5$ for some $x \in H$.

Theorem 8.4.4

A fuzzy set $\mu$ in a hyper TM-algebra $H$ is an $(\in, \in \lor q)$-fuzzy hyper TM-
subalgebra of $H$ if and only if it satisfies

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\} \text{ for all } x, y \in H$$

**Theorem 8.4.5**

Let $\mu$ be a fuzzy set in a hyper TM-algebra $H$. Then $\mu$ is an $(\in, \in \lor q)$-fuzzy hyper TM-subalgebra of $H$ if and only if its non-empty $t$-level set $U(\mu; t)$ is a hyper TM-subalgebra of $H$ for all $t \in (0, 0.5]$.

**Proof**

Let $\mu$ be an $(\in, \in \lor q)$-fuzzy hyper TM-subalgebra of $H$. Let $t \in (0, 0.5]$ and $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. For any $z \in x \circ y$, $\mu(z) \geq \inf_{u \in x \circ y} \mu(u) \geq \min\{\mu(x), \mu(y), 0.5\} \geq t$ by Theorem 8.4.4. Thus $z \in U(\mu; t)$, and so $x \circ y \subseteq U(\mu; t)$. Hence $U(\mu; t)$ is a hyper TM-subalgebra of $H$ for all $t \in (0, 0.5]$.

Conversely, suppose that the non-empty $t$-level set $U(\mu; t)$ of $\mu$ is a hyper TM-subalgebra of $H$ for all $t \in (0, 0.5]$. For any $x, y \in H$, let $t_0 = \min\{\mu(x), \mu(y), 0.5\}$. Then, $t_0 \in (0, 0.5]$, $\mu(x) \geq \min\{\mu(x), \mu(y), 0.5\} = t_0$ and $\mu(y) \geq \min\{\mu(x), \mu(y), 0.5\} = t_0$. It follows that $x, y \in U(\mu; t_0)$ so that $x \circ y \subseteq U(\mu; t_0)$ since $U(\mu : t_0)$ is a hyper TM-subalgebra of $H$. Therefore $z \in U(\mu; t_0)$, that is $\mu(z) \geq t_0$ for all $z \in x \circ y$. Thus $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$, which implies from Theorem 8.4.4, that $\mu$ is an $(\in, \in \lor q)$-fuzzy hyper TM-subalgebra of $H$.

For any fuzzy set $\mu$ of a hyper TM-algebra $H$ and any $t \in (0, 1]$, consider two subsets $Q(\mu; t) = \{x \in H / x_t \in q\mu\}$ and $[\mu]_t = \{x \in H / x_t \in \lor q\mu\}$. It is clear that $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$. 

Theorem 8.4.6

If $\mu$ is an $(\in, \in \lor q)$-fuzzy hyper TM-subalgebra of a hyper TM-algebra $H$, then the set $Q(\mu; t)$ is a hyper TM-subalgebra of $H$ for all $t \in (0.5, 1]$. 

Proof

For any $t \in (0.5, 1]$, let $x, y \in Q(\mu; t)$. Then $\mu(x) + t > 1$ and $\mu(y) + t > 1$. Using Theorem 8.4.4, $\inf_{z \in x \circ y} \mu(z) \geq \min \{\mu(x), \mu(y), 0.5\}.$

If $\min \{\mu(x), \mu(y), 0.5\} \geq 0.5$, then $\inf_{z \in x \circ y} \mu(z) \geq 0.5$ and so $\mu(z) \geq 0.5 > 1 - t$ for all $z \in x \circ y$. Thus $z_{t\mu}$. That is, $z \in Q(\mu; t)$ for all $z \in x \circ y$. If $\min \{\mu(x), \mu(y)\} < 0.5$, then $\inf_{z \in x \circ y} \mu(z) \geq \min \{\mu(x), \mu(y)\}$ and so $\mu(z) \geq \min \{\mu(x), \mu(y)\} > 1 - t$ for all $z \in x \circ y$. This shows that $z_{t\mu}$. That is, $z \in Q(\mu; t)$ for all $z \in x \circ y$. Consequently, $Q(\mu; t)$ is a hyper TM-subalgebra of $H$ for all $t \in (0.5, 1]$. 

Corollary 8.4.7

If $\mu$ is an $(\in, \in)$-fuzzy hyper TM-subalgebra of a hyper TM-algebra $H$, then the set $Q(\mu; t)$ is a hyper TM-subalgebra of $H$ for all $t \in (0.5, 1]$. 

Theorem 8.4.8

For any set $\mu$ in a hyper TM-algebra of $H$, the following are equivalent.

(i) $\mu$ is an $(\in, \in \lor q)$-fuzzy hyper TM-subalgebra of $H$.

(ii) $[\mu]_t$ is a hyper TM-subalgebra of $H$ for all $t \in (0, 1]$. 

Suppose that $\mu$ is an $(\in, \in \lor q)$-fuzzy hyper TM-subalgebra of $H$. Let $x, y \in [\mu]_t$ for $t \in (0, 1]$. Then $\mu(x) \geq t$ or $\mu(x) + t > 1$, and $\mu(y) \geq t$ or $\mu(y) + t > 1$.

The following four cases are considered.

Case (i) $\mu(x) \geq t$ and $\mu(y) \geq t$

Case (ii) $\mu(x) \geq t$ and $\mu(y) + t > 1$

Case (iii) $\mu(x) + t > 1$ and $\mu(y) \geq t$

Case (iv) $\mu(x) + t > 1$ and $\mu(y) + t > 1$

For the first case, Theorem 8.4.4 implies that

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$$

$$\geq \min\{t, 0.5\}$$

$$= \begin{cases} 0.5 & \text{if } t > 0.5 \\ t & \text{if } t \leq 0.5 \end{cases}$$

Hence if $t > 0.5$, then $\mu(z) \geq 0.5$ for all $z \in x \circ y$, which implies that $\mu(z) + t \geq 0.5 + t > 1$. That is, $z \in Q(\mu; t)$ for all $z \in x \circ y$. If $t \leq 0.5$, then $\mu(z) \geq t$ for all $z \in x \circ y$, and so $z \in U(\mu; t)$ for all $z \in x \circ y$. Therefore $x \circ y \subseteq U(\mu; t) \cup Q(\mu; t) = [\mu]_t$.

For case (ii), assume that $t > 0.5$. Then $1 - t < 0.5$. If $\min\{\mu(y), 0.5\} \leq \mu(x)$, then $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(y), 0.5\}$ and so $\mu(z) \geq \min\{\mu(y), 0.5\} > 1 - t$ for all $z \in x \circ y$. This shows that $z \in Q(\mu; t)$, that is $z \in Q(\mu; t)$ for all $z \in x \circ y$. If $\min\{\mu(y), 0.5\} > \mu(x)$, then $\inf_{z \in x \circ y} \mu(z) \geq \mu(x)$ which implies that $\mu(z) \geq \mu(x) \geq t$.
for all $z \in x \circ y$. Thus $z \in U(\mu; t)$ for all $z \in x \circ y$. Therefore, $z \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$ for all $z \in x \circ y$, that is $x \circ y \subseteq [\mu]_t$ for $t > 0.5$.

Suppose that $t \leq 0.5$. Then $1 - t \geq 0.5$. If $\min\{\mu(x), 0.5\} \leq \mu(y)$, then

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), 0.5\}$$

and so $\mu(z) \geq \min\{\mu(x), 0.5\} > t$ for all $z \in x \circ y$. If $\min\{\mu(x), 0.5\} > \mu(y)$, then $\inf_{z \in x \circ y} \mu(z) \geq \mu(y)$ and so $\mu(z) \geq \mu(y) > 1 - t$ for all $z \in x \circ y$. Hence $z \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$ for all $z \in x \circ y$ whenever $t \leq 0.5$, that is, $x \circ y \subseteq [\mu]_t$ for $t \leq 0.5$.

Similarly the result (iii) follows.

Now consider the final case.

If $t > 0.5$, then $1 - t < 0.5$.

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$$

= \begin{cases} 0.5 > 1 - t, & \text{if } \min\{\mu(x), \mu(y)\} \geq 0.5 \\ \min\{\mu(x), \mu(y)\} > 1 - t & \text{if } \min\{\mu(x), \mu(y)\} < 0.5 \end{cases}

and so $z \in Q(\mu; t) \subseteq [\mu]_t$, for all $z \in x \circ y$.

If $t \leq 0.5$, then $1 - t \geq 0.5$. Thus

$$\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$$

= \begin{cases} 0.5 \geq t, & \text{if } \min\{\mu(x), \mu(y)\} \geq 0.5 \\ \min\{\mu(x), \mu(y)\} > 1 - t & \text{if } \min\{\mu(x), \mu(y)\} < 0.5 \end{cases}

which implies that $z \in U(\mu; t) \cup Q(\mu; t) = [\mu]_t$ for all $z \in x \circ y$. Consequently, $[\mu]_t$ is a hyper TM-subalgebra of $H$ for all $t \in (0.1]$. 

Conversely, let $\mu$ be a fuzzy set in $H$ such that $[\mu]_t$ is a hyper TM-subalgebra of $H$ for all $t \in (0,1]$. Assume that the result

$$\inf_{z \in xy} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$$

for all $x, y \in H$ in Theorem 8.4.4 is false. Then there exists $a, b \in H$ such that

$$\inf_{z \in ab} \mu(z)t < \min\{\mu(a), \mu(b), 0.5\}.$$ 

It follows that $\mu(w) < \min\{\mu(a), \mu(b), 0.5\}$ for some $w \in a \circ b$ so that there exists $t_w \in (0,1]$ such that $\mu(w) < t_w \leq \min\{\mu(a), \mu(b), 0.5\}$. Then, $t_w \in (0,0.5]$ and $a, b \in U(\mu; t_w) \subseteq [\mu]_{t_w}$. Since $[\mu]_{t_w}$ is a hyper TM-subalgebra of $H$, $a \circ b \subseteq \mu$ and so $w \in [\mu]_{t_w}$ for all $w \in a \circ b$. This is a contradiction since $w \not\in U(\mu; t_w)$ and $\mu(w) + t_w < 2t_w \leq 1$, that is, $w \not\in Q(\mu; t_w)$. Therefore

$$\inf_{z \in xy} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$$

for all $x, y \in H$.

Hence $\mu$ is an $(\in, \in \lor q)$-fuzzy hyper TM-subalgebra of $H$.

**Corollary 8.4.9**

If $\mu$ is an $(\in, \in)$-fuzzy hyper TM-subalgebra of a hyper TM-algebra of $H$, then the set $[\mu]_t$ is a hyper TM-subalgebra of $H$ for all $t \in (0,1]$.

### 8.5 Homomorphisms on Hyper TM-Algebras

**Definition 8.5.1**

A mapping $f : H_1 \rightarrow H_2$ of hyper TM-algebras is called a hyper homomorphism if
(i) $f(0) = 0$

(ii) $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in H_1$.

**Proposition 8.5.2**

Let $f : H_1 \rightarrow H_2$ be a hyper homomorphism of hyper TM-algebras. If $A \ll B$ in $H_1$, then $f(A) \ll f(B)$ for all $A, B \subseteq H_1$.

**Proof**

Let $A, B \subseteq H_1$ such that $A \ll B$. Then for all $a \in A$ there exists $b \in B$ such that $a \ll b$, that is, $0 \in a \circ b$. Hence, $0 = f(0) \in f(a \circ b) = f(a) \circ f(b)$. This implies $f(a) \ll f(b)$ in $H$.

**Proposition 8.5.3**

Let $f : H_1 \rightarrow H_2$ be a hyper homomorphism of hyper TM-algebras. If $x \ll y$ in $H_1$, then $f(x) \ll f(y)$ in $H_2$.

**Proof**

Let $x, y \in H_1$ be such that $x \ll y$. Then $0 \in x \circ y$. Therefore, $0 = f(0) \in f(x \circ y) = f(x) \circ f(y)$. This implies $f(x) \ll f(y)$ in $H_2$.

**Definition 8.5.4**

Let $f : H_1 \rightarrow H_2$ be a hyper homomorphism of hyper TM-algebras. Then $\ker(f) = \{x \in H_1 / f(x) = 0\}$ is called the kernel of $f$. 

**Theorem 8.5.5**

Let $f : H_1 \to H_2$ be a homomorphism of hyper TM-algebras. If $I$ is a weakly positive implicative hyper TM-ideal of $H_2$, then $f^{-1}(I)$ is a weakly positive hyper TM-ideal of $H_1$.

**Proof**

Let $I$ be a weakly positive implicative hyper TM-ideal of $H_2$. Clearly $0 \in f^{-1}(I)$. Let $x, y \in H_1$ be such that $(x \circ y) \circ z \ll f^{-1}(I)$ and $y \circ z \subseteq f^{-1}(I)$.

Then,

\[ f((x \circ y) \circ z) \ll I \text{ and } f(y \circ z) \subseteq I. \]

\[ \Rightarrow (f(x \circ y) \circ f(z)) \ll I \text{ and } (f(y) \circ f(z)) \subseteq I. \]

\[ \Rightarrow (f(x \circ f(y)) \circ f(z)) \ll I \text{ and } (f(y) \circ f(z)) \subseteq I. \]

Since $I$ is a weakly positive implicative hyper TM-ideal of $H_2$

\[ (f(x) \circ f(z)) \circ f(z) \subseteq I \]

\[ \Rightarrow (f(x \circ z)) \circ f(z) \subseteq I \]

\[ \Rightarrow (f(x \circ z) \circ z) \subseteq I \]

\[ \Rightarrow (x \circ z) \circ z \subseteq f^{-1}(I). \]

Hence $f^{-1}(I)$ is a weakly positive implicative hyper TM-ideal of $H_1$.

**8.6 CONCLUSION**

In the present work the hyper structure in TM-algebras is introduced. The concepts of positive weakly hyper TM-ideals and fuzzy hyper TM-subalgebras of type $(\alpha, \beta)$ are introduced and their properties are duly characterized. It has been
observed that the TM-algebras satisfy various conditions stated in the BCK / BCI algebras.