2.1 INTRODUCTION:

Shock phenomena such as a global shock resulting from a supernova explosion or stellar pulsation passing outward through a stellar envelope or an impulsive flare in Sun’s atmosphere or a man-made explosion in the Earth atmosphere, have enormous importance in space sciences and in astrophysics. Similar phenomena also occur in laboratory, for example, when a piston is driven into a shock tube, when an aircraft moves supersonically through the atmosphere, in the blast wave produced by a strong explosion, or when rapidly flowing gas runs into a constriction in a flow-field or runs across a wall.

Parker [81] has indicated that the hydrodynamic blast wave theory can usefully describe the large scale regime to which the flow due to a sudden expansion of the solar corona asymptotically converges. Using similarity assumptions, he has derived a number of numerical solutions for his idealized adiabatic “solar wind” model. Ranga Rao and Ramana [84] and Sedov [93] have indicated that a limiting case of a self-similar flow-field with a power-law shock is the flow-field formed with an exponential shock. Singh and Vishwakarma [97] and Vishwakarma and Nath [122] have obtained solutions for the problem of unsteady self-similar motion of a gas displaced by a piston according to an exponential law.

The study of high speed flow of a mixture of gas and dust particles is of great interest in several branches of engineering and science (Higashino and Suzuki [38], Nath [69], Vishwakarma and Nath [122]). Miura and Glass [64] have studied an analytical solution for a planer dusty gas flow with constant velocities of the shock. As they considered zero volumetric extension for the dust particles mixed into the perfect gas, the dust virtually has a mass fraction but no volume fraction. Their results reflect the influence of the additional inertia of the dust upon the shock wave propagation. Vishwakarma [119] has obtained a non-similarity solution for
the flow-field behind the strong shock propagation at non-constant velocity in a mixture of perfect gas and small solid particles. He has considered the non-zero volume fraction of solid particles in the mixture, and his results reflect the influence of both the decrease of mixture’s compressibility and the increase of mixture’s inertia on the shock propagation (Pai et al. [80], Steiner and Hirschler [107], Vishwakarma and Nath [128], Vishwakarma and Pandey [130]).

At high temperatures that prevail in the problems associated with shock waves, a gas is ionized and the medium behaves like a medium of very high electrical conductivity, then the electromagnetic effects may also be significant. A complete analysis of such a problem should therefore consist of the study of gas dynamic flow and the electromagnetic field simultaneously. The study of propagation of cylindrical shock waves in a conducting gas in the presence of an azimuthal or axial magnetic field is relevant to the experiments on pinch effect, exploding wires and so-forth. This problem both in ideal and non-ideal gases was under taken by many authors, for example, Christer and Helliwell [11], Cole and Greifinger [12], Nath and Sahu [72], Nath et al. [74], Pai [77], Sakurai [92], Singh et al. [99], Vishwakarma and Yadav [139]. Vishwakarma and Singh [134] have investigated the propagation of diverging shock waves in a low conducting and uniform or non-uniform gas as a result of time dependent energy input [21, 22]. Also, Vishwakarma and Srivastava [136] have presented the propagation of a strong cylindrical shock wave in a weakly conducting dusty gas, in presence of a variable axial magnetic field. Recently, Vishwakarma et al. [142] have studied the propagation of diverging cylindrical shock waves in a weakly conducting dusty gas with variable density under the influence of a spatially variable axial magnetic induction.

In all of the above works and the works of other authors as known to us, no one considered the medium as the mixture of non-conducting dust particles and
conducting perfect gas under the influence of a magnetic field. In the present work, we generalize the work of Vishwakarma and Nath [122] to study the propagation of shock waves in the mixture of a perfectly conducting gas and non-conducting dust particles driven out by a cylindrical piston in the presence of an azimuthal magnetic field. It is assumed that the motion of the piston obeys the exponential law (see Ranga Rao and Ramana [84], Vishwakarma and Nath [122, 123])

\[ r_p = A \exp(\lambda t), \quad \lambda > 0, \quad (1.1) \]

where \( r_p \) is the radius of the piston, \( A \) and \( \lambda \) are dimensional constants and \( t \) is the time. The law of piston motion (1.1) implies a boundary condition on the gas speed at a piston, which is required for the determination of the problem. It is also assumed that the shock propagation follows the exponential law

\[ R = B \exp(\lambda t), \quad (1.2) \]

where \( R \) is the radius of the shock and \( B \) is a dimensional constant which is to be determined.

The generalization of shock boundary conditions is made for the shock propagation in a mixture of a perfectly conducting gas and non-conducting dust particles in presence of a magnetic field. The analysis of the flow-field in the region between the piston and the shock are presented for both the cases of isothermal and adiabatic flows. The assumption of isothermal flow is physically realistic, when radiation heat transfer effects are implicitly present. As the shock propagates, the temperature behind the shock increases and becomes very large so that there is intense heat exchange. This causes the temperature gradient to approach zero throughout the flow-field, i.e. \( \frac{\partial T}{\partial r} \to 0 \). Due to this reason, the temperature in the flow-field depends only on time \( t \) and not on the distance \( r \) from the point of explosion, i.e. \( T = T(t) \), and the flow is isothermal as described by Laumbach and
Probstein [55], Sachdev and Ashraf [91], Sedov [93], and Zhuravskaya and Levin [148]. A detailed mathematical theory of one-dimensional isothermal blast waves in a magnetic field was investigated by Lerche [57, 58]. With this assumption, we present the solutions in sections 2 and 4. In section 5, we obtain the solutions for the flow taken to be adiabatic. The effects of variation of mass concentration of solid particles \( k_p \), the ratio of density of solid particles to the initial density of the perfect gas in the mixture \( G_1 \) and the parameter for strength of initial magnetic field \( M_A^{-2} \) on the flow-field behind the shock are obtained. Also, a comparison between the solutions in the cases of isothermal and adiabatic flows is made.

2.2 FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS: ISOTHERMAL FLOW

The fundamental equations for one-dimensional cylindrically symmetric, unsteady and isothermal flow of a mixture of a perfectly conducting gas and non-conducting small solid particles in the presence of an azimuthal magnetic field may be written as (c.f. Pai et al. [80], Whitham [144])

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{u \rho}{r} = 0 ,
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{(1-Z)}{\rho} \left[ \mu h \frac{\partial h}{\partial r} + \frac{h^2 u}{r} \right] = 0 ,
\]

\[
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial r} + h \frac{\partial u}{\partial r} = 0 ,
\]

\[
\frac{\partial T}{\partial r} = 0 ,
\]

where \( \rho \) is the density of the mixture, \( u \) is the flow velocity, \( p \) is the pressure of the mixture, \( h \) is the azimuthal magnetic field, \( Z \) the volume fraction of solid particles
in the mixture, $\mu$ is the magnetic permeability and $r$ and $t$ are the space and time coordinates.

The medium is a mixture of a perfectly conducting gas and non-conducting small solid particles (dust particles). Therefore, the magnetic force (Lorentz force) in the medium acts in the gas and not in the solid particles. Since $Z$ is the volume of solid particles in the unit volume of the mixture, the magnetic force per unit volume of the mixture is $(1 - Z)\mu \vec{H} \times (\nabla \times \vec{H})$ (c.f. Whitham [144]). If the flow is cylindrically symmetric and only the azimuthal (transverse) component $'h'$ of magnetic field is non-zero, the above magnetic force reduces to $(1 - Z)(\mu h \frac{\partial h}{\partial r} + \frac{\mu h^2}{r})$. This is the reason why the factor $(1 - Z)$ is introduced in the last term of the equation (2.2).

For each species in the mixture of a perfectly conducting gas and small solid particles (a pseudo-fluid), we have one equation of state. For the perfectly conducting gas, we may use the perfect gas law which is

$$p_g = R \bar{\rho}_g T, \quad (2.5)$$

where $p$ and $\bar{\rho}_g$ are the partial pressure and partial density of the gas in the mixture, $T$ is the temperature of the gas (and of the solid particles as the equilibrium flow condition is maintained) and $R$ is the gas constant.

The specific volume of solid particles is assumed to remain unchanged by variations in temperature and pressure. Therefore the equation of state of the solid particles in the mixture is, simply,

$$\rho_{sp} = \text{constant}, \quad (2.6)$$
where $\rho_{sp}$ is the species density of the solid particles. The equation of the state of the mixtures is given as (Pai [79])

$$p = \frac{1-k_p}{1-z} \rho RT, \quad (2.7)$$

where $Z = \frac{V_{sp}}{V}$ is the volume fraction and $k_p = \frac{m_{sp}}{M}$ is the mass fraction of the solid particles in the mixture, $m_{sp}$ and $V_{sp}$ being the total mass and the volumetric extension of the solid particles and $V$ and $M$ the total volume and total mass of the mixture. The relation between $k_p$ and $Z$ is given by

$$k_p = \frac{z\rho_{sp}}{\rho}. \quad (2.8)$$

In the equilibrium flow, $k_p$ is a constant in the whole flow-field. Therefore, from equation (2.8)

$$\frac{z}{\rho} = \text{constant}. \quad (2.9)$$

Also, we have the relation (Pai [14])

$$Z = \frac{k_p}{G(1-k_p)+k_p}, \quad (2.10)$$

where $G = \frac{\rho_{sp}}{\rho_g}$ is the ratio of the density of the solid particle to the species density of the gas. The internal energy per unit mass of the mixture may be written as

$$e = [k_pC_{sp} + (1 - k_p)C_v]T = C_{vm}T, \quad (2.11)$$

where $C_{sp}$ is the specific heat of the solid particles,$C_v$ the specific heat of the gas at constant volume and $C_{vm}$ the specific heat of the mixture at constant volume. The specific heat of the mixture at constant pressure is

$$C_{pm} = k_pC_{sp} + (1 - k_p)C_p, \quad (2.12)$$
where $C_p$ is the specific heat of the gas at constant pressure.

The ratio of the specific heats of the mixture is given by (Marble [60], Pai et al. [80])

$$\Gamma = \frac{C_{pm}}{C_{vm}} = \gamma \left( \frac{1 + \delta \beta'}{1 + \delta \beta'} \right), \quad (2.13)$$

where $\gamma = \frac{C_p}{C_v}$, $\delta = \frac{k_p}{1-k_p}$ and $\beta' = \frac{C_{sp}}{C_v}$. Now,

$$C_{pm} - C_{vm} = (1 - k_p)(C_p - C_v) = (1 - k_p)R. \quad (2.14)$$

The internal energy per unit mass of the mixture is, therefore, given by

$$e = \frac{p(1-Z)}{\rho(\Gamma-1)}. \quad (2.15)$$

We consider that a cylindrical shock is propagating into a mixture of perfectly conducting gas and non-conducting small solid particles of constant density at rest in presence of a variable azimuthal magnetic field.

The azimuthal magnetic field in undisturbed dusty gas is assumed to vary as

$$h = \frac{A}{r^l}, \quad (2.16)$$

where $A$ and $l$ are constants.

The flow variables immediately ahead of the shock front are

$$u = u_1 = 0, \quad (2.17)$$

$$\rho = \rho_1 = \text{constant}, \quad (2.18)$$
\[ h = h_1 = AR^{-l}, \]  
\[ p = p_1 = \frac{(1-Z_1)(1-l)}{2l} \frac{\mu A^2}{R^{2l}} = \text{constant}, \quad (0 < l \leq 1) \]  

where subscript ‘1’ refers to the value in front of the shock and \( R \) is the shock radius.

For isentropic change of state of the gas-particle mixture, we have from first law of thermodynamics and the equation of state of the mixture, the speed of sound ‘\( a \)’ as

\[ a^2 = \frac{\gamma p}{\rho (1-Z)}, \]  

### 2.3 SHOCK CONDITIONS:

The jump conditions across the shock front relate the fluid properties behind the shock (downstream of the shock) to the fluid properties ahead of the shock (upstream of the shock). These conditions are derived from the principles of conservation of mass, magnetic flux, momentum and energy under the assumption that the shock front is a discontinuity surface with no thickness. In a frame of reference moving with the shock front, the principle of conservation of mass across the shock front gives

\[ \rho_1 U = \rho_2 (U - u_2) = m, \quad \text{say}, \]  

where \( U = \frac{dr_2}{dt} \) is the shock velocity, \( m \) is the mass flux per unit time per unit area across the shock and subscript ‘2’ refers to the value just behind the shock.

The magnetic flux per unit time per unit area entering the shock front from upstream reason is \((1 - Z_1)h_1 U\), since the volume \((1 - Z_1)U\) of the perfectly conducting gas and volume \(Z_1 U\) of non-conducting small solid particle are entering the shock front across unit area in unit time. Similarly, the magnetic flux entering
the downstream region across unit area of the shock front in unit time is \((1 - Z_2)h_2(U - u_2)\). Therefore, by principle of conservation of magnetic flux, we have

\[
(1 - Z_1)h_1U = (1 - Z_2)h_2(U - u_2). \tag{3.2}
\]

As assumed by Pai [79], the solid particles in the mixture are very small spheres, uniformly distributed in the mixture and occupying total volume \(Z_1\) per unit volume of the undisturbed mixture, where \(Z_1\) is also small in comparison with unity. Thus the area of contact by solid particles with unit area of the shock surface may be neglected. The magnetic pressures are therefore, \(\frac{1}{2}\mu h_1^2\) and \(\frac{1}{2}\mu h_2^2\) just ahead and just behind the shock, respectively. Therefore the net force on unit area of the shock front from upstream region to downstream region is \((p_1 + \frac{1}{2}\mu h_1^2) - (p_2 + \frac{1}{2}\mu h_2^2)\). Also, the momentum gain per unit time across unit area from upstream region to downstream region is \(m\{(U - u_2) - U\}\). Hence

\[
\left( p_1 + \frac{1}{2}\mu h_1^2 \right) - \left( p_2 + \frac{1}{2}\mu h_2^2 \right) = m(U - u_2) - mU.
\]

Using equation (3.1), above relation becomes

\[
p_1 + \frac{1}{2}\mu h_1^2 + \rho_1 U^2 = p_2 + \frac{1}{2}\mu h_2^2 + \rho_2(U - u_2)^2. \tag{3.3}
\]

The rate at which the pressures (fluid pressure + magnetic pressure) do work on the mass flux \(m\) is \(\left( p_1 + \frac{1}{2}\mu h_1^2 \right) U - \left( p_2 + \frac{1}{2}\mu h_2^2 \right) (U - u_2)\). This is equal to the total rate of gain of energy of \(m\). Now the total energy per unit mass of the mixture just ahead of the shock is due to three causes: Kinetic, magnetic and internal. These give the respective contributions \(\frac{1}{2}U^2\), \((1 - Z_1)\frac{\mu h_1^2}{2\rho_1}\) and \(e_1\). Thus the total change of energy per unit mass in crossing the shock front is
\[
\left\{ \frac{1}{2} (U - u_2)^2 + (1 - Z_2) \frac{\mu h_2^2}{2 \rho_2} + e_2 \right\} - \left\{ \frac{1}{2} U^2 + (1 - Z_1) \frac{\mu h_1^2}{2 \rho_1} + e_1 \right\}
\]

and so the total rate of change applied to mass flux \( m \) is \( m \) times the above. Hence, by conservation of energy

\[
\left( p_1 + \frac{1}{2} \mu h_1^2 \right) U - \left( p_2 + \frac{1}{2} \mu h_2^2 \right) (U - u_2) = m \left\{ \frac{1}{2} (U - u_2)^2 + (1 - Z_2) \frac{\mu h_2^2}{2 \rho_2} + e_2 \right\} - m \left\{ \frac{1}{2} U^2 + (1 - Z_1) \frac{\mu h_1^2}{2 \rho_1} + e_1 \right\}.
\]

Using equation (3.1), above relation can be written as

\[
\frac{1}{2} U^2 + e_1 + \frac{p_1}{\rho_1} + (2 - Z_1) \frac{\mu h_1^2}{2 \rho_1} = \frac{1}{2} (U - u_2)^2 + e_2 + \frac{p_2}{\rho_2} + (2 - Z_2) \frac{\mu h_2^2}{2 \rho_2}.
\]

(3.4)

Further, from equation (2.8), we have

\[
\frac{Z_1}{\rho_1} = \frac{Z_2}{\rho_2}.
\]

(3.5)

Using the strong shock approximations \( p_1 << p_2 \) \( (p_1 \approx 0, e_1 \approx 0) \), we have from above relations

\[
u_2 = (1 - \beta) U ,
\]

(3.6a)

\[
\rho_2 = \frac{\rho_1}{\beta} ,
\]

(3.6b)

\[
h_2 = \frac{(1 - Z_1)}{(\beta - Z_1)} h_1 ,
\]

(3.6c)

\[
p_2 = \left[ (1 - \beta) + \frac{M_\alpha^2}{2} \left\{ 1 - \left( \frac{1 - Z_1}{\beta - Z_1} \right)^2 \right\} \right] \rho_1 U^2 ,
\]

(3.6d)

\[
Z_2 = \frac{Z_1}{\beta} ,
\]

(3.6e)
where \( M_A = \left( \frac{\rho_1 U^2}{\mu h_1^2} \right)^{\frac{1}{2}} \) is the Alfven-Mach number.

The quantity \( \beta (0 < \beta < 1) \) is obtained by the relation

\[
\beta^3 (\Gamma + 1) + \beta^2 (1 - \Gamma - 4Z_1 - 2\Gamma Z_1 - \Gamma M_A^{-2}) + \beta (\Gamma Z_1^2 + 5Z_1^2 + 2\Gamma Z_1 - 2Z_1) + M_A^{-2} (\Gamma - 2 + 2Z_1 + \Gamma Z_1) - \left( (\Gamma Z_1^2 - Z_1^2 + 2Z_1^3) + M_A^{-2} (Z_1 \Gamma - 2Z_1 + 2Z_1^2) \right) = 0. \tag{3.7}
\]

The expression for the initial volume fraction of the solid particles \( Z_1 \) is given by,

\[
Z_1 = \frac{k_p}{(1-k_p)G_1 + k_p}, \tag{3.8}
\]

where \( G_1 = \frac{\rho_{sp}}{\rho_{g1}} \) is the ratio of the species density of solid particles to the initial species density of the gas in the mixture.

Equation (2.4) together with equation (2.7) gives

\[
\frac{p}{p_2} = \frac{(1-Z_2)}{(1-Z)} \frac{\rho}{\rho_2}. \tag{3.9}
\]

2.4 SIMILARITY SOLUTIONS:

The similarity transformations for the problem under consideration are taken as

\[
u = UV(\eta), \quad \rho = \rho_1 D(\eta), \quad p = \rho_1 U^2 P(\eta),
\]

\[
Z = Z_1 D(\eta), \quad \mu^2 h = \rho_1^2 UH(\eta), \tag{4.1}
\]
where $V$, $D$, $P$ and $H$ are the functions of the non-dimensional variable (similarity variable) $\eta = \frac{r}{R}$ only. The variable $\eta$ assumes the value ‘1’ at the shock front and $\eta_p$ on the piston.

Equations (1.1), (1.2), (2.1) and (4.1) yield a relation between $A$ and $B$ in the form

$$A = B\eta_p .$$

Equation (3.9) with the aid of equations (4.1) yields a relation between $P$ and $D$ in the form

$$P(\eta) = \frac{(\beta-Z_1) \left[ (1-\beta) + \frac{M}{\infty} \left\{ 1-\left(\frac{1-Z_1}{\beta-Z_1}\right)^2 \right\} \right] D(\eta)}{(1-Z_1D(\eta))} .$$

Using the similarity transformations (4.1), the conservation equations (2.1)-(2.4) can be transformed into a system of ordinary differential equations with respect to $\eta$, as

$$
(V - \eta) \frac{dD}{d\eta} + D \frac{dV}{d\eta} = -\frac{dV}{\eta} ,
$$

$$
(V - \eta) \frac{dH}{d\eta} + H \frac{dV}{d\eta} = -H ,
$$

$$
(V - \eta)D \frac{dV}{d\eta} + \frac{P}{D(1-Z_1D)} \frac{dD}{d\eta} + (1 - Z_1D)H \frac{dH}{d\eta} = - \left[ \frac{(1-Z_1D)H^2 + VD\eta}{\eta} \right] .
$$

Equations (4.4) - (4.6) can be simplified to

$$
\frac{dV}{d\eta} = \frac{PV + 2(1-Z_1D)^2\eta H^2 -(1-Z_1D)^2 V H^2 -(V - \eta)(1-Z_1D)V D\eta}{\eta[(1-Z_1D)(V-\eta)^2 D-P -(1-Z_1D)^2 H^2]} ,
$$

$$
\frac{dD}{d\eta} = -\frac{D(1-Z_1D)[(V-\eta)VD-2(1-Z_1D)H^2-VD\eta]}{\eta[(1-Z_1D)(V-\eta)^2 D-P -(1-Z_1D)^2 H^2]} ,
$$
By use of equations (4.1), the shock conditions (3.6) take the form

\[ V(1) = (1 - \beta) , \quad (4.10a) \]

\[ D(1) = \frac{1}{\beta} , \quad (4.10b) \]

\[ H(1) = \left( \frac{1-Z_1}{\beta-Z_1} \right) M_A^{-1} , \quad (4.10c) \]

\[ P(1) = \left[ (1 - \beta) + \frac{M_A^2}{2} \left\{ 1 - \left( \frac{1-Z_1}{\beta-Z_1} \right)^2 \right\} \right] . \quad (4.10d) \]

At the inner boundary (surface) of the flow-field behind the shock, the condition is that the velocity of the surface is equal to the velocity of fluid on the surface. This kinematic condition from (4.1), at the inner boundary surface can be written as

\[ V(\eta_p) = \eta_p , \quad (4.11) \]

where \( \eta_p \) is the value of \( \eta \) at the inner surface.

Now, the differential equations (4.7-4.9) can be integrated, numerically, with the boundary conditions (4.10) to obtain the solutions for the isothermal flow behind the shock surface.

2.5 ADIABATIC FLOW:

In this part, we introduced self-similar solution for adiabatic flow behind a strong shock driven by a cylindrical piston moving according to the exponential law (1.1) in the mixture of a perfectly conducting gas and non-conducting small solid particles in presence of an azimuthal magnetic field.
For adiabatic flow, equation (2.4) is replaced by

\[
\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial r} - \frac{p}{\rho^2} \left( \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} \right) = 0. \tag{5.1}
\]

The equations (2.1)-(2.3) and (5.1) can be transformed, using equations (4.1), into

\[
D(V - \eta) \frac{dV}{d\eta} + \frac{d\rho}{d\eta} + (1 - Z_1 D) H \frac{dH}{d\eta} = \frac{[(1-Z_1 D)H^2 + VD\eta]}{\eta}, \tag{5.2}
\]

\[
(V - \eta) \frac{dD}{d\eta} + D \frac{dV}{d\eta} = -\frac{dV}{\eta}, \tag{5.3}
\]

\[
(V - \eta) \frac{dH}{d\eta} + H \frac{dV}{d\eta} = -H, \tag{5.4}
\]

\[
(V - \eta)(1 - Z_1 D) \frac{d\rho}{d\eta} - \frac{(V - \eta)}{D} P \frac{dD}{d\eta} = -2P(1 - Z_1 D), \tag{5.5}
\]

Solving equations (5.2)-(5.5) for \(\frac{dV}{d\eta}, \frac{dD}{d\eta}, \frac{d\rho}{d\eta}\) and \(\frac{dH}{d\eta}\), we have

\[
\frac{dV}{d\eta} = \frac{PV\Gamma + 2P\eta(1-Z_1 D) + 2\eta H^2(1-Z_1 D)^2 - VH^2(1-Z_1 D)^2 - VD\eta(V - \eta)(1-Z_1 D)}{\eta[D(V - \eta)^2(1-Z_1 D) - P\Gamma - H^2(1-Z_1 D)^2]}, \tag{5.6}
\]

\[
\frac{dD}{d\eta} = \frac{D(1-Z_1 D)[VD(V - \eta)^2 - 2H^2(V - \eta)(1-Z_1 D) + 2P\eta - (V - \eta)VD\eta]}{\eta(V - \eta)[D(V - \eta)^2(1-Z_1 D) - P\Gamma - H^2(1-Z_1 D)^2]}, \tag{5.7}
\]

\[
\frac{dH}{d\eta} = \frac{H[\eta D(1-Z_1 D)(V - \eta)^2 + (V - \eta)P\Gamma + 2P\eta(1-Z_1 D)(V - \eta)(VD\eta + (1-Z_1 D)H^2)(1-Z_1 D)]}{\eta(V - \eta)[D(V - \eta)^2(1-Z_1 D) - P\Gamma - H^2(1-Z_1 D)^2]}, \tag{5.8}
\]

\[
\frac{d\rho}{d\eta} = \frac{P[(V\Gamma + 2(1-Z_1 D)\eta)D(V - \eta)^2 - \Gamma(V - \eta)(VD\eta + 2(1-Z_1 D)H^2) - 2\eta H^2(1-Z_1 D)^2]}{\eta(V - \eta)[D(V - \eta)^2(1-Z_1 D) - P\Gamma - H^2(1-Z_1 D)^2]} \tag{5.9}
\]

The transformed shock conditions (4.10) and the kinematic conditions (4.11) at the piston will be same as in the case of isothermal flow. The ordinary differential equations (5.6)-(5.10) with boundary conditions (4.10) can now be numerically integrated to obtain the solution for the adiabatic flow behind the shock surface.
2.6 RESULTS AND DISCUSSION:

The flow variables $V$ the reduced fluid velocity, $D$ the reduced density of the mixture, $P$ the reduced fluid pressure and $H$ the reduced magnetic field against the similarity variable $\eta$ are obtained by numerical integration of the differential equations (4.7)-(4.9) for isothermal flow and equations (5.6)-(5.9) for adiabatic flow with the boundary conditions (4.10) by fourth order Runge-Kutta algorithm using MATHEMATICA software. For the purpose of numerical calculations the values of constant parameters are taken as (Pai et al. [80], Miura and Glass [64], Vishwakarma [119]) ; $\gamma = \frac{5}{3}$; $\beta' = 1$; $k_p = 0, 0.2, 0.3$; $G_1 = 1, 100$ and $M_{A^{-2}} = 0, 0.075, 0.1$. The value $k_p = 0$ corresponds to the dust-free case (perfect gas) and $M_{A^{-2}} = 0$ to a non-magnetic case. Figures (1)-(4) and figures (5)-(8) show that the variation of the flow variables $V$, $D$, $P$ and $H$ with respect to $\eta$ in isothermal and adiabatic flows respectively. Table 1 shows the values of density ratio across the shock front $\beta \left( = \frac{\rho_1}{\rho_2} \right)$ and the position of piston $\eta_p$ at various values of $k_p$, $G_1$ and $M_{A^{-2}}$.

Effects of an increase in the value of $G_1$ the ratio of density of dust particles and initial density of gas in the mixture are

I. to decrease the value of $\beta$ (i.e. to increase the shock strength) (see table 1);

II. to decrease the distance of piston from the shock front (see table 1);

III. to increase the flow variables $V$, $D$ and $H$ at any point in the flow-field behind the shock (see figures 1,2,4 for isothermal flow and figures 5, 6, 8 for isothermal flow); and

IV. to increase the pressure $P$ at any point in the flow-field behind the shock when $M_{A^{-2}} = 0$, in both, the isothermal and adiabatic flows. In isothermal
flow at $M_A^{-2} = 0.075$, the pressure $P$ increases; whereas at $M_A^{-2} = 0.1$, it increases near the shock front and decreases near the piston (see figure 3). In adiabatic flow, an increase in $G_1$ when $k_p = 0.2, 0.3$ and $M_A^{-2} = 0.075, 0.1$ exhibits somewhat different behavior (see figure 7).

Due to increase in $G_1$ (at constant $k_p$), there is high decrease in $Z_1$, i.e. the volume fraction of solid particles in the mixture becomes comparatively very small. This effect induces comparatively more compression of the mixture in the region between the shock and piston, which displays the above effects. Thus in the dusty gas flow an increase in $G_1$ increases the shock strength and at higher values of $G_1 (= 100)$ the behavior of the dusty gas becomes closure to that of the dust-free gas.

It is found that an increase in the value of $k_p$

I. increases the density ratio across the shock $\beta \left( = \frac{\rho_1}{\rho_2} \right)$ when $G_1 = 1$, but in case of $G_1 = 100$ the density ratio decreases;

II. increases the distance of piston from the shock front when $G_1 = 1$, and decreases it when $G_1 = 100$ (see table 1);

III. decreases the flow variables $V$, $D$ and $H$ at any point in the flow-field behind the shock when $G_1 = 1$ and increases these when $G_1 = 100$ (see figures 1, 2, 4 for isothermal flow and figures 5, 6, 8 for adiabatic flow); and

IV. decreases the pressure $P$ at any point in the flow-field when $G_1 = 1$ and increases it when $G_1 = 100$, in both, the isothermal and adiabatic flows in non-magnetic case. In the magnetic cases the pressure $P$ decreases when the flow is isothermal; whereas in adiabatic flow, the pressure $P$ decreases in all cases expect for $M_A^{-2} = 0.1$ at $G_1 = 1$ where it decreases
near the shock front and increases near the piston (see figure 3 for isothermal flow and figure 7 for adiabatic flow).

In the mixture with $G_1 = 1$, small solid particles of density equal to that of the perfect gas occupy a significant part of the volume which lowers the compressibility of the medium. Also, the compressibility of the mixture is reduced by an increase in $k_p$ which causes an increase in the distance between the shock front and the piston, a decrease in the shock strength, and the above nature of the flow variables. In the mixture with $G_1 = 100$, small solid particles of density equal to 100 times that of the perfect gas occupy a very small portion of the volume, and therefore compressibility is not lowered much; the inertia of the medium is increased significantly due to dust load. An increase in $k_p$, from 0.1 to 0.3 in the mixture for $G_1 = 100$, means that the perfect gas constituting 90% of the total mass and occupying 99.889% of the total volume now constitutes 70% of the total mass and occupies 99.573% of the total volume. Due to this reason, the density of the perfect gas in mixture is highly decreased which overcomes the effect of incompressibility of the mixture and finally causes a small decrease in the distance between piston and shock front, an increase in the shock strength, and the above behavior of flow variables.

An increase in the value of the parameter for strength of the magnetic field $M_A^{-2}$ shows the following effects in the case of isothermal flow:

I. The density ratio $\beta$ increases (i.e. shock strength decreases) (see table 1).

II. $\eta_p$ decreases except for $G_1 = 1$, $Z_1 = 0.3$ (see table 1). In general, the presence of magnetic field has decaying effect on shock wave (i.e. the
effect of decreasing $\eta_p$), but this effect is slowed down when volume fraction of solid particles $Z_1$ is increased. This fact results in the above behavior of $\eta_p$ at $G_1 = 1, Z_1 = 0.3$.

III. The fluid velocity $V$ decreases in all cases expect for $k_p = 0.3$ at $G_1 = 1$ where it decreases near the piston and increases near the shock front.

IV. The density $D$ decreases in the dust-free case ($k_p = 0$) and in dusty case when $G_1 = 100$; whereas at $G_1 = 1$, it is increased.

V. The pressure $P$ decreases in the flow-field behind the shock and the magnetic field $H$ increases.

VI.

Following results are obtained by increasing $M_A^{-2}$ in the case of adiabatic flow:

I. The density ratio $\beta$ increases (i.e. shock strength decreases).

II. $\eta_p$ decreases except for $G_1 = 1, Z_1 = 0.3$ (see table 1), which is same as given in (II) (see above in isothermal flow).

III. The flow variables $V, D$ and $P$ decrease at any point in the flow-field behind the shock.

IV. The magnetic field $H$ decreases in the dust-free case ($k_p = 0$) and in dusty case when $G_1 = 100$; whereas at $G_1 = 1$ it increases.

2.7 **COMPARISON BETWEEN ISOTHERMAL AND ADIABATIC FLOWS:**

I. From table 1 it is clear that $\lambda_p$ (position of the piston surface) in adiabatic flow is greater than that in the isothermal flow. Physically, it means that the gas is less compressed in the isothermal flow in comparison to that in adiabatic flow. Thus the strength of the shock is higher in the adiabatic flow than that in the isothermal flow.
II. In adiabatic flow, the pressure $P$ decreases near the piston and tends to zero in dust-free case ($k_p = 0$) and in dusty case when $G_1 = 100$ (see figure 3); whereas in isothermal flow, $P$ increases from shock to piston and its gradient is almost constant in all the cases (see figure 7).

III. In adiabatic flow, figure 6 shows that there is unbounded density distribution near the piston. This is quite expected and may be explained as follows. First of all Sedov [93] (see also Ranga Rao and Ramana [84] and Vishwakarma and Nath [122]) pointed out that a limiting case, as $n \to \infty$ of a self-similar flow-field with a power law shock

$$R \sim t^{n+1},$$

is the flow-field formed with an exponential shock described by (1.2). For such flow with a power law shock, in the adiabatic case, it can be seen from the asymptotic form of the adiabatic integral (see appendix)

$$\left(\frac{D}{(1-Z_1 D)}\right)^{\frac{n}{2}} \Gamma D^{(n+1)} = \frac{p \eta^{(n+1)}}{M} (V - n - 1) \left(\frac{n}{n+1}\right)^{\frac{n}{n+1}},$$

that the density tends to infinity for $n > 0$ as the piston is approached, provided $(1 - Z_1 D)$ does not tend to zero. The density distribution exhibits this nature in the cases of non-magnetic dust-free gas and non-magnetic dusty gas with higher values $G_1 (= 100)$ (see curves 1, 4, 5 in figure 7) and in the remaining cases this behavior of density is removed. In the later cases, this is perhaps due to the fact that the expression $(1 - Z_1 D)$ in (6.1) tends to zero as the piston is approached.

The above phenomenon of density distribution can be physically interpreted as follows. In the case of a non-magnetic dust-free gas or in the case of a non-magnetic dusty gas with higher value of $G_1$ the path of the piston converges with the path of the particle immediately ahead, thus compressing the gas to infinite density, but in the remaining cases, the path of the piston
is almost parallel to the path of particle immediately ahead, the above behavior of the density distribution is removed. It may therefore be concluded that the magnetic field has tendency to remove the nature of unbounded density distribution near the piston.

In the case of isothermal flow, the density is finite at the piston for all values of $k_p$ and $G_1$. This seems to be necessary because with an unbounded density near the piston the temperature there approaches zero violating the basic assumption of zero temperature gradient throughout the flow. Therefore, it may be observed that the assumption of zero temperature gradient brings a profound change in the density distribution as compared to that of the adiabatic flow.

IV. In adiabatic flow, the reduced magnetic field $H$ increases as we move from the shock to the piston (see figure 8); whereas in isothermal flow, when $k_p = 0.3$, $G_1 = 1$, $H$ at first increases, becomes maximum and then decreases near the piston (see curves 3, 8 in figure 4).

2.8 CONCLUSION:

In this work, we have studied the self-similar solution for the flow behind a strong exponential cylindrical shock wave propagating in a mixture of perfectly conducting gas and non-conducting small solid particles in presence of an azimuthal magnetic field. The shock is driven by a piston moving with time according to an exponential law. On the basis of this work, one may draw the following conclusions:

I. An increase in the mass concentration of solid particles ($k_p$) decreases the shock strength at lower values of $G_1$, and increases it at its higher values. Also for $G_1 = 1$, it decreases the velocity, density and magnetic field at any
point in the flow-field behind shock; whereas for $G_1 = 100$, it increases the velocity, density and magnetic field.

II. An increase in the value of the ratio of the density of solid particles and the initial density of the perfect gas in the mixture ($G_1$) increases the shock strength. Also, it increases the velocity, density and magnetic field at any point in the flow-field behind the shock.

III. The presence of magnetic field decreases the velocity, pressure and density at any point in the flow-field behind the shock when the flow is adiabatic; whereas in isothermal flow, the pressure decreases but velocity and density show somewhat different behavior when $G_1 = 1$.

IV. The presence of magnetic field or the presence of dust particles with lower values of $G_1$ removes the nature of unbounded density distribution near the piston in the adiabatic flow.

V. The presence of magnetic field decreases the shock strength, in general. This effect of magnetic field on shock strength, in both cases (isothermal and adiabatic flows) decreases significantly by increasing $k_p$ at $G_1 = 1$; whereas at $G_1 = 100$ the effect of magnetic field on the shock strength is almost not influenced by increasing $k_p$.

VI. The shock strength is higher in adiabatic flow than that in isothermal flow.

VII. The pressure $P$ increases from shock to piston in isothermal flow, whereas it decreases to zero in some cases of adiabatic flow. Also, the tendency of maxima formation in the magnetic field distribution in some cases of the isothermal flow is removed in comparison to the adiabatic flow.

Appendix: From the fundamental equations (2.1) to (2.3) and (5.1) with the similarity transformations
\[
\eta = \frac{r}{R}, \quad \text{(A.1)}
\]

\[
u = \frac{r}{t} V(\eta), \quad \rho = \rho_1 D(\eta), \quad p = \frac{r^2}{t^2} \rho_1 P(\eta),
\]

\[
Z = Z_1 D(\eta), \quad \mu \frac{h}{\eta} = \frac{r}{t} \rho_1^2 H(\eta), \quad \text{(A.2)}
\]

where the variable \( \eta \) assumes the value ‘1’ at the shock front and \( \eta_p \) on the piston surface, such that the piston radius \( r_p = \eta_p R \), \( R (\sim t^{n+1}) \) being the shock radius, we obtain

\[
(V - n - 1) \frac{dD}{d\eta} + D \frac{dV}{d\eta} + 2 \frac{dV}{\eta} = 0 , \quad \text{(A.3)}
\]

\[
(V - n - 1) \frac{dV}{d\eta} + \frac{1}{D} \frac{dP}{d\eta} + \frac{(1-Z_1 D) H}{D} \frac{dH}{d\eta} + \frac{(V^2-V)}{\eta} + \frac{2(P+(1-Z_1 D)H^2)}{D\eta} = 0 , \quad \text{(A.4)}
\]

\[
(V - n - 1) \frac{dH}{d\eta} + H \frac{dV}{d\eta} + \frac{(2V-1)H}{\eta} = 0 , \quad \text{(A.5)}
\]

\[
\frac{2(V-1)}{\eta (V-n-1)} - \frac{r}{D (1-Z_1 D)} \frac{dD}{d\eta} + \frac{1}{P} \frac{dP}{d\eta} = 0 . \quad \text{(A.6)}
\]

The boundary conditions for the strong shock in the mixture at \( \eta = 1 \) are given by

\[
V(1) = (1 - \beta)(n + 1) , \quad \text{(A.7a)}
\]

\[
D(1) = \frac{1}{\beta} , \quad \text{(A.7b)}
\]

\[
P(1) = \left[ (1 - \beta) + \frac{M_A^{-2}}{2} \left\{ 1 - \left( \frac{1-Z_1}{\beta-Z_1} \right)^2 \right\} \right] (n + 1)^2 , \quad \text{(A.7c)}
\]

\[
H(1) = \frac{(1-Z_1)}{(\beta-Z_1)} (n + 1) M_A^{-1} , \quad \text{(A.7d)}
\]
In addition to the shock conditions (A.7), there is the kinematic condition \( V(\eta_p) = (n + 1) \) at the piston surface. From equations (A.3) and (A.6), one can get the relation

\[
\left( \frac{D}{(1 - z_1 D)} \right)^\Gamma D^{\frac{n}{(n+1)}} = \frac{p \eta^{\frac{n+1}{2}}}{M} (V - n - 1)^{-\frac{n}{n+1}}, \tag{A.8}
\]

where \( M \) is a constant to be determined from the shock conditions (A.7).
Table 1. Variation of density ratio $\beta = \frac{\rho_1}{\rho_2}$ across the shock front and the position of the piston surface $\eta_p$ for different values of $k_p, Z_1, G_1$ and $M_A^{-2}$ with $\gamma = \frac{5}{3}$.

| $k_p$ | $Z_1$ | $M_A^{-2} = 0$ | $M_A^{-2} = 0.075$ | $M_A^{-2} = 0.1$ | $Z_1$ | $M_A^{-2} = 0$ | $M_A^{-2} = 0.075$ | $M_A^{-2} = 0.1$
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$\eta_p$ (Isentropic Flow)

| $k_p$ | $Z_1$ | $M_A^{-2} = 0$ | $M_A^{-2} = 0.075$ | $M_A^{-2} = 0.1$ | $Z_1$ | $M_A^{-2} = 0$ | $M_A^{-2} = 0.075$ | $M_A^{-2} = 0.1$
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$\eta_p$ (Adiabatic Flow)

| $k_p$ | $Z_1$ | $M_A^{-2} = 0$ | $M_A^{-2} = 0.075$ | $M_A^{-2} = 0.1$ | $Z_1$ | $M_A^{-2} = 0$ | $M_A^{-2} = 0.075$ | $M_A^{-2} = 0.1$
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Figure 1: Velocity distribution in the region behind the shock front for isothermal flow.
Figure 2: Density distribution in the region behind the shock front for isothermal flow.
Figure 3: Pressure distribution in the region behind the shock front for isothermal flow.
Figure 4: Magnetic field distribution in the region behind the shock front for isothermal flow.
Figure 5: Velocity distribution in the region behind the shock front for adiabatic flow.
Figure 6: Density distribution in the region behind the shock front for adiabatic flow.
Figure 7: Pressure distribution in the region behind the shock front for adiabatic flow.
Figure 8: Magnetic field distribution in the region behind the shock front for adiabatic flow.