Chapter 2

Stability Analysis for Discrete-Time Systems with Uncertainties

2.1 Introduction

In control system design, the study of analyzing the most essential and fundamental problem of stability for TDSs plays a vital role. The stability analysis of discrete-time systems with either time-invariant or time-varying delays have been an interested research area in recent years, since it has a strong background in engineering applications than continuous-time systems [2, 38, 43]. Recently, LKF stability theory aids main focus among researchers to derive the sufficient stability criteria based on the LMIs. For instance, [115] examined the stability problem of discrete-time switched systems with time-delays and the improved robust stability criteria have been proposed in [45] for uncertain discrete-time systems with time-varying delays. However, there are only a few efforts had been taken to investigate stability of DTDSs which intuists to work on such interesting problem.

In some cases, the values of system parameters are inevitable which lead to an undesirable dynamical behavior means that there exist uncertainties in the DSs [23]. They are commonly encountered in many applications such as image processing, pattern recognition, engineering and communication systems, etc. Mostly, uncertainties are chosen
to be norm bounded whereas the fractional uncertainties are incurred here because of more
generality, see [37, 42] and the references therein. Also, norm bounded uncertainties are the
special case of linear fractional uncertainties. In [3], the problem has been investigated for
the delay-interval dependent robust stability criteria of stochastic neural networks with
linear fractional uncertainties. Predominantly, the stability problem for discrete-time
uncertain systems has not yet been tackled fully which has more application in real world
problems.

Besides, a large number of results have been exist in the literature concerning various
types of controller such as state-feedback controller [48], delayed state-feedback controller
[80], adaptive controller [25] and so on. In most cases, the stability analysis of TDS has
been studied via state-feedback controller with delay or without delay separately. However,
the interest in formulating the occurrence of controller dependent on state vector and
delayed state vector randomly (that is, partially delay-dependent controller) renders the
base for the present study. By implementing this controller, the system persuades in random
sense and this randomness obeys mutually uncorrelated Bernoulli-distributed white-noise
sequences. So that there exist two ways of system (controller either with delay or without
delay in the state vector) in single loop and the system dynamics switch between these two
control systems which leads the considered system to be discrete-time Markov chain with
two modes. As well, Markovian system attains essential research among researchers due to
its wide applications in the past decades (see [103, 112] in detail). In [7], it has been stated
that the number of nodes in the Markov chain depends on the number of possible delays in
the network. Hence, this chapter contributes

- a partially delay-dependent controller designed for given discrete-time uncertain
  system;

- the notion of randomness occurring in the controller to formulate the Markovian
discrete-time systems with two modes \((i = 1, 2)\);

- Wirtinger-based integral inequality and convex reciprocal lemma approach used for
  less conservatism.
Motivated by the above discussions, Section 2.2 investigates the problem of the guaranteed cost control for discrete-time system with linear fractional uncertainty via partially delay-dependent controller. Further by constructing suitable LKF, sufficient conditions are derived to ensure the system to be robustly stochastically stable in mean square sense by using the most updated technique Wirtinger-based inequality [65, 81] and convex reciprocal lemma [73]. Also, the appropriate cost function is chosen to guarantee an adequate level of performance. Further, the derived sufficient conditions are expressed in terms of LMIs which can be easily solved by LMI toolbox in Matlab. Finally, as a well-known application, an inverted pendulum model is considered in the numerical examples.

2.2 Robust Guaranteed Cost Analysis for Discrete-Time Systems via Partially Delay-Dependent Controller

2.2.1 Problem Description

Consider the discrete DS with linear fractional uncertainties as

\[
\begin{align*}
    x(k+1) &= A(k)x(k) + A_d(k)x(k - \tau(k)) + Bu(k), \\
    x(s) &= \phi(s), \quad \forall s = -\tau_M, -\tau_M + 1, \ldots, 0,
\end{align*}
\]

(2.1)

where \(x(k) \in \mathbb{R}^n\) is the state vector, \(u(k) \in \mathbb{R}^m\) is the control input. The term \(\tau(k)\) describes the time-varying delay that satisfies

\[0 < \tau_m \leq \tau(k) \leq \tau_M,\]

(2.2)

where \(\tau_m\) and \(\tau_M\) are known positive integers representing the minimum and maximum bounds of the time-varying delay, respectively. The matrices \(A(k) = A + \Delta A(k), A_d(k) = A_d + \Delta A_d(k)\) are bounded matrices containing parameter uncertainties \(\Delta A(k), \Delta A_d(k)\) that satisfy the following linear fractional uncertainty conditions

\[
[\Delta A(k) \; \Delta A_d(k)] = M\Delta(k)(I - J\Delta(k))^{-1}[N_1 \; N_2],
\]

(2.3)
where $J^TJ < I$, $\Delta(k)^T\Delta(k) \leq I$; $A$, $A_d$, $B$, $M$, $N_1$, $N_2$ and $J$ are known constant matrices with appropriate dimensions; $\Delta(k)$ denotes the unknown matrix function with Lebesgue measurable elements and $\psi(s) \in \mathbb{R}^n$ is the initial value.

Now, let define a partially delay-dependent controller as

$$u(k) = [1 - \gamma(k)]K_1x(k) + \gamma(k)K_2x(k - \tau(k)), \quad (2.4)$$

where $K_1$ and $K_2$ are control gain matrices to be determined. The parameter $\gamma(k)$ decides whether the delayed controller added in the system or not and taking the values in a finite set $\mathbb{B} = \{0, 1\}$ with the following transition probability matrix as

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix}. \quad (2.5)$$

The parameters $\alpha$ and $\beta$ are probabilities defined as

$$\begin{cases} 0 \leq \alpha = Pr\{\gamma(k+1) = 1|\gamma(k) = 0\} \leq 1 \\ 0 \leq \beta = Pr\{\gamma(k+1) = 1|\gamma(k) = 1\} \leq 1 \end{cases} \quad (2.6)$$

which are recovery rate and failure rate of the delayed state-feedback controller $u(k) = K_2x(k - \tau(k))$ respectively. Let $\alpha + \beta = 1$, then $\gamma(k)$ will be reduced to the Bernoulli’s type process.

**Remark 2.1.** It is noted that the partially delay-dependent controller (2.4) is more general compared to the state feedback controllers with delay or without delay separately and it spans both the cases. Since there exist two switching modes ($i = 1, 2$) in the controller, the distribution probability described through a Markov process is considered.

Applying the controller (2.4) to the system (2.1), it is easy to see that

$$x(k+1) = [A(k) + (1 - \gamma(k))BK_1]x(k) + [A_d(k) + \gamma(k)BK_2]x(k - \tau(k))$$

which is equivalent to

$$x(k+1) = A(r(k))x(k) + A_d(r(k))x(k - \tau(k)),$$
where \( r(k) \) is a discrete-time homogeneous Markov chain taking values in a finite set \( \mathbb{S} = \{1, 2\} \) whose transition probability is given in (2.5). Let \( r(k) = i \). Thus, one can obtain the Markovian system with two modes \((i - 1, 2)\) as

\[
x(k + 1) = A_i(k)x(k) + A_{di}(k)x(k - \tau(k))
\]

(2.7)

with

\[
\begin{align*}
A_1(k) &= A_1 + \Delta A(k) = A(k) + BK_1, \\
A_{d1}(k) &= A_{d1} + \Delta A(k) = A_d(k), \\
A_2(k) &= A_2 + \Delta A(k) = A(k), \\
A_{d2}(k) &= A_{d2} + \Delta A(k) = A_d(k) + BK_2.
\end{align*}
\]

Remark 2.2. In the literature [21, 28], it is well known that the Markovian system parameters switch simultaneously according to a Markov process, whereas in the resulting system (2.7) all the system parameters are deterministic and only the probability distribution of delayed controller is modeled into a Markov process. So that the system considered here is distinct from normal Markovian jump systems (MJSs).

Given positive-definite matrices \( S, T \), we will consider the performance index associated with the system (2.7) as

\[
J = \sum_{k=0}^{\infty} [x^T(k)Sx(k) + u^T(k)Tu(k)].
\]

(2.8)

From (2.4), it follows that

\[
u(k) = U_{i1}x(k) + U_{i2}x(k - \tau(k))
\]

(2.9)

with \( U_{i1} = K_1, \ U_{i2} = 0, \ U_{21} = 0, \ U_{22} = K_2. \)

2.2.2 Stability Criteria of Discrete-Time Uncertain Systems

In this subsection, sufficient conditions are provided for the stability analysis of the discrete-time closed-loop uncertain system (2.1) with linear fractional uncertainty (2.3) via partially delay-dependent controller (2.4). The following notations are used in the sequel.

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\( e_i = [0_{n \times (i-1)n}, I_n, 0_{n \times (0-i)n}] \),
\( \nu_1 = [e_3 - e_2; \sqrt{3}(e_3 + e_2 - e_3)], \)
\( \nu_2 = [e_4 - e_3; \sqrt{3}(e_4 + e_3 - e_0)], \)
\( \nu_3 = [\nu_1; \nu_2], \)
\( \Gamma = \begin{bmatrix} \tilde{R} & \tilde{Y} \\ \ast & \tilde{R} \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}, \)
\( \tilde{Y} = \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}, \quad \tilde{\pi}_{ij} = \sqrt{\pi_{ij}}, \)
\( \Lambda_s(k) = \Lambda_s + \Delta \Lambda_s(k) \ (s = 1, 2, 3), \)
\( \Lambda_1 = [\tilde{\pi}_{i1} A_i \ 0 \ \tilde{\pi}_{i1} A_d \ 0_{n \times 3n}] , \)
\( \Lambda_2 = [\tilde{\pi}_{i2} A_i \ 0 \ \tilde{\pi}_{i2} A_d \ 0_{n \times 3n}] , \)
\( \Lambda_3 = [(\tau_{M} - \tau_{m})^2(A_i - I) \ 0 \ (\tau_{M} - \tau_{m})^2 A_d \ 0_{n \times 3n}], \)
\( \Delta \Lambda_1(k) = [\tilde{\pi}_{i1} \Delta A(k) \ 0 \ \tilde{\pi}_{i1} \Delta A_d(k) \ 0_{n \times 3n}], \)
\( \Delta \Lambda_2(k) = [\tilde{\pi}_{i2} \Delta A(k) \ 0 \ \tilde{\pi}_{i2} \Delta A_d(k) \ 0_{n \times 3n}], \)
\( \Delta \Lambda_3(k) = [(\tau_{M} - \tau_{m})^2 \Delta A(k) \ 0 \ (\tau_{M} - \tau_{m})^2 \Delta A_d(k) \ 0_{n \times 3n}], \)
\( Z_1 = \begin{bmatrix} 0_{6n \times n} & 0_{6n \times n} \\ \tilde{\pi}_{i1} M & \tilde{\pi}_{i1} M \\ \tilde{\pi}_{i2} M & \tilde{\pi}_{i2} M \\ (\tau_{M} - \tau_{m})^2 M & (\tau_{M} - \tau_{m})^2 M \end{bmatrix} , \)
\( Y_1 = [N_1 \ 0 \ 0 \ 0_{n \times 6n}; 0 \ 0 \ N_2 \ 0_{n \times 6n}], \)
\( \Delta = \Delta_1(I - J_1 \Delta_1)^{-1}, \)
\( \Delta_1 = \begin{bmatrix} \Delta(k) & 0 \\ 0 & \Delta(k) \end{bmatrix}, \quad J_1 = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \)
\( U_i = [U_{i1} \ 0 \ U_{i2} \ 0_{n \times 6n}], \)
\( \xi^T(k) = [x^T(k) \ x^T(k - \tau_{M}) \ x^T(k - \tau(k)) \ x^T(k - \tau_{m}) \ \mu_1^T(k) \ \mu_2^T(k)], \)
\( \mu_1^T(k) = \Lambda(k, \tau(k), \tau_{M}), \mu_2^T(k) = \Lambda(k, \tau_{m}, \tau(k)). \)
Theorem 2.1. Given positive scalars \( \tau_{ij}, \tau_M, \) and \( \tau_m, \) the uncertain system (2.7) is robustly stochastically stable under partially delay-dependent controller (2.4), there exists positive definite matrices \( X_i(i = 1, 2), \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{R}, \tilde{Y}, \tilde{S}, \tilde{T} \) and a scalar \( \varepsilon > 0 \) such that the following LMI holds

\[
\begin{bmatrix}
\tilde{\Sigma} & \tilde{Y}^T_1 & \tilde{Z}_1 & \tilde{U}_i^T & \tilde{X}_i & \tilde{X}_i \\
* & -I & \varepsilon^{-1} J_1 & 0 & 0 & 0 \\
* & * & -\varepsilon^{-1} I & 0 & 0 & 0 \\
* & * & * & -\tilde{T} & 0 & 0 \\
* & * & * & -\tilde{S} & 0 & 0 \\
* & * & * & * & -\tilde{Q}_1 & 0 \\
* & * & * & * & * & -\tilde{Q}_2
\end{bmatrix} < 0 \quad (2.10)
\]

with

\[
\tilde{\Sigma} = \begin{bmatrix}
\tilde{\Omega}_{6 \times 6} & \tilde{\Lambda}_1^T & \tilde{\Lambda}_2^T & \tilde{\Lambda}_3^T \\
* & -X_1 & 0 & 0 \\
* & * & -X_2 & 0 \\
* & * & * & -\tilde{R}
\end{bmatrix} , \quad (2.11)
\]

where

\[
\tilde{\Omega}_{i,1} = -X_i, \quad \tilde{\Omega}_{2,2} = \tilde{Q}_1 - 4\tilde{R} - \tilde{Q}_2 - (\tilde{R} + \tilde{R}^T),
\]

\[
\tilde{\Omega}_{2,3} = 2\tilde{R} + 4\tilde{Y}, \quad \tilde{\Omega}_{2,4} = 2\tilde{Y}, \quad \tilde{\Omega}_{2,5} = 3\tilde{R}, \quad \tilde{\Omega}_{2,6} = 3\tilde{Y},
\]

\[
\tilde{\Omega}_{3,3} = -8\tilde{R} - 4\tilde{Y}, \quad \tilde{\Omega}_{3,4} = -4\tilde{Y} - 2\tilde{R}, \quad \tilde{\Omega}_{3,5} = 3\tilde{R} + 3\tilde{Y},
\]

\[
\tilde{\Omega}_{4,4} = -4\tilde{Y} + 3\tilde{R}, \quad \tilde{\Omega}_{4,5} = -3\tilde{Y} - 4\tilde{R} + \tilde{Q}_2 - (\tilde{R} + \tilde{R}^T),
\]

\[
\tilde{\Omega}_{4,6} = 3\tilde{R}, \quad \tilde{\Omega}_{5,5} = -3\tilde{R}, \quad \tilde{\Omega}_{5,6} = -3\tilde{Y}, \quad \tilde{\Omega}_{6,6} = -3\tilde{R},
\]

\[
\tilde{\Lambda}_1 = [\pi_{i1} X_i A_{i_1} 0 \pi_{i1} \tilde{R} A_{di} 0_{n \times 3n}],
\]

\[
\tilde{\Lambda}_2 = [\pi_{i2} X_i A_{i_1} 0 \pi_{i2} \tilde{R} A_{di} 0_{n \times 3n}],
\]

\[
\tilde{\Lambda}_3 = [(\tau_M - \tau_m)^2 X_i (A_i - I) 0 (\tau_M - \tau_m)^2 \tilde{R} A_{di} 0_{n \times 3n}],
\]

\[
Z_i = I(Z_i I), \quad \tilde{Y}_1 = [X_i N_1 0 0_{n \times 6n}; 0 0 \tilde{R} N_2 0_{n \times 6n}],
\]

\[
\tilde{U}_i = [X_i U_{i_1} 0 \tilde{R} U_{i_2} 0_{n \times 6n}],
\]

\[
\tilde{X}_i = [X_i 0_{n \times 8n}].
\]
The controller gain matrices are given as $K_1 = L_1X_1^{-1}$ and $K_2 = L_2R$.

**Proof.** Let $\eta(k) = x(k+1) - x(k)$ and construct the LKF as

$$V(k) = V_1(k) + V_2(k) + V_3(k), \quad (2.12)$$

where

$$V_1(k) = x^T(k)P_1x(k),$$

$$V_2(k) = \sum_{i=k-\tau_M}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-\tau_M}^{k-\eta_x-1} x^T(i)Q_2x(i) + \sum_{i=k-\tau_m}^{k-1} x^T(i)Q_3x(i),$$

$$V_3(k) = (\tau_M - \tau_m) \sum_{j=-\tau_M}^{-\tau_m-1} \sum_{i=k+j}^{k-1} \eta^T(k)R\eta(k)$$

with $P_i(i = 1, 2), Q_i(i = 1, 2, 3)$ and $R$ are positive-definite matrices.

Now, calculating the forward differences of $V(k)$ and taking the expectation on both sides, one can have

$$\mathbb{E}\{\Delta V_1(k)\} = \mathbb{E}\left\{x^T(k+1) \sum_{j \in \mathbb{S}} \pi_{ij} P_j x(k+1) - x^T(k)P_i x(k)\right\}$$

$$= \mathbb{E}\left\{x^T(k+1)\pi_{11} P_1 x(k+1) + x^T(k+1)\pi_{12} P_2 x(k+1) - x^T(k)P_1 x(k)\right\}$$

$$= \mathbb{E}\left\{\xi^T(k) \left[ A_1^T(k) P_1 A_1(k) + A_2^T(k) P_2 A_2(k) \right] \xi(k) - x^T(k)P_i x(k)\right\}. \quad (2.13)$$

$$\mathbb{E}\{\Delta V_2(k)\} = \sum_{i=k-\tau_M}^{k} x^T(i)Q_1x(i) - \sum_{i=k-\tau_M}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-\tau_m+1}^{k-1} x^T(i)Q_2x(i)$$

$$- \sum_{i=k-\tau_m}^{k-1} x^T(i)Q_2x(i) + \sum_{i=k-\tau_m}^{k} x^T(i)Q_3x(i) - \sum_{i=k-\tau_m}^{k-1} x^T(i)Q_3x(i)$$

$$= \mathbb{E}\left\{x^T(k) \left[ Q_1 + Q_3 \right] x(k) + x^T(k-\tau_m) \left[ Q_2 - Q_3 \right] x(k-\tau_m)$$

$$- x^T(k-\tau_M) \left[ Q_1 + Q_2 \right] x(k-\tau_M)\right\}. \quad (2.14)$$

$$\mathbb{E}\{\Delta V_3(k)\} = \mathbb{E}\left\{(\tau_M - \tau_m) \sum_{j=-\tau_M}^{-\tau_m-1} \left\{\eta^T(k)R\eta(k) - \eta^T(k+j)R\eta(k+j)\right\}\right\}$$

$$= \mathbb{E}\left\{(\tau_M - \tau_m)^2 \eta^T(k)R\eta(k) - (\tau_M - \tau_m) \sum_{j=k-\tau_M}^{k-\eta_x-1} \eta^T(j)R\eta(j)\right\}. \quad (2.15)$$
From (2.15), it follows that
\[-(\tau_M - \tau_m) \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j)R\eta(j) = -(\tau_M - \tau_m) \left\{ \sum_{j=k-\tau_M}^{k-\tau(k)-1} \eta^T(j)R\eta(j) - \sum_{j=k-\tau(k)}^{k-\tau_m-1} \eta^T(j)R\eta(j) \right\}.

From Lemma 1.6, it is obvious that
\[-(\tau_M - \tau_m) \sum_{j=k-\tau_M}^{k-\tau(k)-1} \eta^T(j)R\eta(j) \leq \frac{-(\tau_M - \tau_m)}{\tau_M - \tau(k)} \xi^T(k) \begin{bmatrix} x(k - \tau(k)) - x(k - \tau_M) \\ x(k - \tau(k)) + x(k - \tau_M) - \mu_1(k) \\ x(k - \tau(k)) - x(k - \tau_M) - \mu_1(k) \end{bmatrix}^T \times \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} e_3 - e_2 \\ e_3 + e_2 - e_5 \end{bmatrix} \leq \frac{-(\tau_M - \tau_m)}{\tau_M - \tau(k)} \xi^T(k) \begin{bmatrix} e_3 - e_2 \\ \sqrt{3(e_3 + e_2 - e_5)} \end{bmatrix}^T \times \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} e_3 - e_2 \\ \sqrt{3(e_3 + e_2 - e_5)} \end{bmatrix} \xi(k) \leq \frac{-(\tau_M - \tau_m)}{\tau_M - \tau(k)} \xi^T(k) \begin{bmatrix} e_3 - e_2 \\ e_3 + e_2 - e_5 \end{bmatrix}^T \times \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} e_3 - e_2 \\ e_3 + e_2 - e_5 \end{bmatrix} \xi(k).

Likewise,
\[-(\tau_M - \tau_m) \sum_{j=k-\tau(k)}^{k-\tau_m-1} \eta^T(j)R\eta(j) \leq \frac{(\tau_M - \tau_m)}{\tau(k) - \tau_m} \xi^T(k)\nu_1^T \tilde{R}\nu_1 \xi(k). \tag{2.17}
\]

Therefore, from (2.16) and (2.17) one can get
\[-(\tau_M - \tau_m) \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j)R\eta(j) \leq -\xi^T(k) \left\{ \frac{(\tau_M - \tau_m)}{\tau_M - \tau(k)} \nu_1^T \tilde{R}\nu_1 + \frac{(\tau_M - \tau_m)}{\tau(k) - \tau_m} \nu_2^T \tilde{R}\nu_2 \right\} \xi(k).
\]

By applying convex reciprocal Lemma 1.3 to the above, if there exists \( \tilde{Y} \) such that \( \Gamma > 0 \) holds, then
\[-(\tau_M - \tau_m) \sum_{j=k-\tau_M}^{k-\tau_m-1} \eta^T(j)R\eta(j) \leq -\xi^T(k)\nu_3 \Gamma \nu_3^T \xi(k). \tag{2.18}
\]

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Substitute (2.18) in (2.15), one can achieve
\[
\mathbb{E}\{\Delta V_3(k)\} \leq \mathbb{E}\left\{\xi^T(k)[A_{Y}^T(k)R\Lambda_3(k) - \nu_3^r\Gamma\nu_3]\xi(k)\right\}. \tag{2.19}
\]

Combining the equations (2.13)-(2.19), it can easily seen that
\[
\mathbb{E}\{\Delta V(k)\} \leq \mathbb{E}\left\{\xi^T(k)\begin{bmatrix} e_1^T [Q_1 + Q_3 - P_1 + S]e_1 - e_2^T [Q_1 + Q_2]e_2 \\
+ e_4^T [Q_2 - Q_3]e_4 - \nu_3^r\Gamma\nu_3 + A_1^T(k)P_1\Lambda_1(k) + A_2^T(k)P_2\Lambda_2(k) \\
+ A_3^T(k)R\Lambda_3(k) + U_i^T TU_i\end{bmatrix}\xi(k) - x^T(k)Sx(k) - u^T(k)Tu(k)\right\}
= \mathbb{E}\left\{\xi^T(k)\tilde{\Omega}(k)\xi(k) + \xi^T(k)U_i^T TU_i\xi(k) \right. \\
\left. - x^T(k)Sx(k) - u^T(k)Tu(k)\right\}. \tag{2.20}
\]

Using Schur complement Lemma 1.2,
\[
\tilde{\Omega}(k) = \begin{bmatrix}
\Omega & \Lambda_1^T & \Lambda_2^T & \Lambda_3^T \\
* & -P_1^{-1} & 0 & 0 \\
* & * & -P_2^{-1} & 0 \\
* & * & * & -R^{-1}
\end{bmatrix} + \begin{bmatrix}
0 & \Delta\Lambda_1^T(k) & \Delta\Lambda_2^T(k) & \Delta\Lambda_3^T(k) \\
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0
\end{bmatrix}
= \bar{\Omega} + \Delta\tilde{\Omega}(k), \tag{2.21}
\]
where \( \Omega = e_1^T [Q_1 + Q_3 - P_1 + S]e_1 - e_2^T [Q_1 + Q_2]e_2 + e_4^T [Q_2 - Q_3]e_4 - \nu_3^r\Gamma\nu_3 \). Then, expanding the terms of \( \Delta\tilde{\Omega}(k) \) by applying linear fractional uncertainty (2.3) achieve the following equation
\[
\Delta\tilde{\Omega}(k) = Z_1\Delta Y_1 + [Z_1\Delta Y_1]^T.
\]

Substitute the above equation in (2.20), it is easy to see that
\[
\mathbb{E}\{\Delta V(k)\} \leq \mathbb{E}\left\{\xi^T(k)\left[\tilde{\Omega}(k) + Z_1\Delta Y_1 + [Z_1\Delta Y_1]^T + U_i^T TU_i\right]\xi(k) - x^T(k)Sx(k) - u^T(k)Tu(k)\right\}
\]
and then the uncertainty type Lemma 1.5 can guarantee that
\[
\mathbb{E}\{\Delta V(k)\} \leq \mathbb{E}\left\{\xi^T(k)\tilde{\Omega}(k)\xi(k) - x^T(k)Sx(k) - u^T(k)Tu(k)\right\}, \tag{2.22}
\]
where
\[
\tilde{\Omega} = \begin{bmatrix}
\tilde{\Omega} & \epsilon^{-1}Y_1^T & Z_1 & U_i^T \\
* & -I & \epsilon^{-1}J_1 & 0 \\
* & * & -\epsilon^{-1}I & 0 \\
* & * & * & -\epsilon^{-1}I
\end{bmatrix}. \tag{2.23}
\]
Let $X_i = P_i^{-1}, \bar{R} = R^{-1}, S = S^{-1}, \bar{T} = T^{-1}, \bar{Q}_1 = Q_1^{-1}, \bar{Q}_3 = Q_3^{-1}, \bar{R}(\cdot)\bar{R} = (\cdot)$. Also as for nonlinear term $-\bar{R}(\cdot)\bar{R}$, it is obtained that $-\bar{R}(\cdot)\bar{R} \leq -\bar{R} - \bar{R}^T + (\cdot)^{-1}$. Then, pre- and post-multiply both sides of (2.23) with $\text{diag}\{X_i, \bar{R}_{i,i}, I_{2n}, e_{2n,2n}, J_{3n}\}$ and its transpose respectively. Using Schur complement lemma one can obtain the LMI condition (2.10) and furthermore, one can have

$$\mathbb{E}\{\Delta V(k)\} \leq 0$$

which implies that the function $V(k)$ is decreasing and gives

$$\mathbb{E}\{V(k)\} \leq \mathbb{E}\{V(0)\} \leq \lambda_{\text{max}}(\Sigma)\|x(0)\|^2.$$ 

From (2.12), it can be verify that

$$\delta_1 \|x(k)\|^2 < V(k) < \delta_2 \|x(k)\|^2,$$

(2.24)

where $\delta_1 = \lambda_{\text{min}}(P_i)$, $\delta_2 = \lambda_{\text{max}}(P_i) + \tau_M \lambda_{\text{max}}(Q_1) + (\tau_M - \tau_m)\lambda_{\text{max}}(Q_2) + \tau_m \lambda_{\text{max}}(Q_3) + (\tau_M - \tau_m)^2\lambda_{\text{max}}(R)$. Therefore, it follows from the Lyapunov functional stability theory and Definition 1.1 that the system (2.7) is robustly stochastically stable in the mean square with the controller gain matrices $K_1 = L_1 X_1^{-1}$ and $K_2 = L_2 \bar{R}$. To find the guaranteed cost value, (2.22) affords that

$$x^T(k)Sx(k) + u^T(k)Tu(k) \leq \mathbb{E}\{V(k) - V(k + 1)\}.$$ 

Summing both sides from 0 to $n$ yields

$$\sum_{k=0}^{n} [x^T(k)Sx(k) + u^T(k)Tu(k)] \leq \mathbb{E}\{V(0) - V(n)\}.$$ 

Letting $n \to \infty$, noting that $V(n) \to 0$, so that

$$\sum_{k=0}^{\infty} [x^T(k)Sx(k) + u^T(k)Tu(k)] \leq \delta_2 \|x(0)\|^2 = J^*.$$ 

This completes the proof. \qed

**Remark 2.3.** It is easy to verify that the above result holds for the system (2.7) without linear fractional uncertainties by assigning the value of $M$, $N_1$, and $N_2$ to be zero. Also, the proposed result holds good to the system (2.7) via state-feedback controllers with or without delay individually by letting $\gamma(k) = 1$ and $\gamma(k) = 0$, respectively.
Consider a system (2.7) without uncertainties as

$$x(k + 1) = A_i x(k) + A_{d1} x(k - \tau(k))$$  \hspace{1cm} (2.25)

with

$$A_1 = A + BK_1, \quad A_{d1} = A_d,$$

$$A_2(k) = A, \quad A_{d2}(k) = A_d + BK_2,$$

we have the following corollary to prove the system (2.25) is robustly stochastically stable.

**Corollary 2.1.** Given positive scalars $\pi_{ij}$, $\tau_M$ and $\tau_m$, the system (2.25) is robustly stochastically stable under the partially delay-dependent controller (2.4), there exists positive definite matrices $X_i (i = 1, 2)$, $Q_1$, $Q_2$, $Q_3$, $R$, $\hat{Y}$, $\hat{S}$, $\hat{T}$ and a scalar $\epsilon > 0$ such that the following LMI holds

$$\tilde{\Sigma} = \begin{bmatrix}
\bar{\Omega}_{6 \times 6} & \bar{U}^T & \bar{X}_i & \bar{X}_i & \bar{X}_i \\
* & -\bar{T} & 0 & 0 & 0 \\
* & * & -\bar{S} & 0 & 0 \\
* & * & * & -\bar{Q}_1 & 0 \\
* & * & * & * & -\bar{Q}_3 \\
\end{bmatrix} < 0$$  \hspace{1cm} (2.26)

with the controller gain matrices are given as $K_1 = L_1X_1^{-1}$ and $K_2 = L_2R$. The terms are defined as in Theorem 2.1.

### 2.2.3 Numerical Examples with Application

The following examples show the effectiveness of the obtained theoretical results.

**Example 2.1.** Firstly, consider the uncertain system (2.7) with the following parameters

$$A = \begin{bmatrix}
-0.5 & 0.2 \\
0.1 & 0.4 \\
\end{bmatrix}, \quad A_d = \begin{bmatrix}
0.3 & -0.1 \\
0.5 & 0.2 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0.4 & -0.2 \\
0.5 & 0.4 \\
\end{bmatrix},$$
\[
M = \begin{bmatrix}
0.2 & 0.3 \\
0.8 & 0.2 
\end{bmatrix},
N_1 = \begin{bmatrix}
0.3 & 0.4 \\
0.2 & 0.1 
\end{bmatrix},
N_2 = \begin{bmatrix}
0.1 & 0.5 \\
1 & 0.1 
\end{bmatrix},
\]
\[
J = \begin{bmatrix}
0.2 & 0.4 \\
0.2 & -0.3 
\end{bmatrix},
\Delta(k) = \begin{bmatrix}
sin(k) & 0 \\
0 & sin(k) 
\end{bmatrix},
\]
\[
\alpha = 0.7, \beta = 0.2, \pi_{11} = 0.3, \pi_{12} = 0.7, \pi_{21} = 0.2, \pi_{22} = 0.8.
\]

The delay is assumed to be \( \tau(k) = 3 + \sin(k) \in [2, 4] \), then the corresponding lower and upper bounds are \( \tau_m = 2 \) and \( \tau_M = 4 \). According to Theorem 2.1 and using the LMI toolbox in Matlab to solve the LMI (2.10), the feasibility is archived and get the set of solutions as

\[
X_1 = \begin{bmatrix}
0.0172 & 0.0049 \\
0.0049 & 0.0073 
\end{bmatrix},
X_2 = \begin{bmatrix}
0.0108 & 0.0090 \\
0.0090 & 0.0062 
\end{bmatrix},
\]
\[
L_1 = \begin{bmatrix}
0.0088 & -0.0203 \\
-0.0569 & -0.0283 
\end{bmatrix},
L_2 = \begin{bmatrix}
-0.0364 & -0.0134 \\
-0.0107 & -0.1109 
\end{bmatrix},
\]
\[
\bar{S} = \begin{bmatrix}
2.5992 & 0.1630 \\
0.1630 & 3.5584 
\end{bmatrix},
\hat{S} = \begin{bmatrix}
2.5401 & 0.0301 \\
0.0301 & 2.5506 
\end{bmatrix},
\]
\[
R = \begin{bmatrix}
23.7186 & -1.0517 \\
-1.0517 & 4.8169 
\end{bmatrix} \text{ and } \epsilon = 0.1065.
\]

Thus, the controller gain matrix can be acquired as

\[
K_1 = \begin{bmatrix}
0.5809 & -0.2380 \\
-3.2767 & -0.1248 
\end{bmatrix},
K_2 = \begin{bmatrix}
-0.8484 & -0.0262 \\
-0.1379 & -0.5230 
\end{bmatrix}.
\]

Figure 2.1 depicts the random behavior of the stochastic variable \( \gamma(k) \). Figure 2.2 and Figure 2.3 represent the states of the uncertain system (2.7) with and without controller (2.4), respectively with the initial conditions \( x(0) = [0.1 \ -0.2] \). Figure 2.2 shows that the system is unstable without controller and it is clear from Figure 2.3 that the proposed controller stabilizes the system (2.7). Thus, the simulations affirm the proposed theoretical results.
Figure 2.1: Random nature of the stochastic variable $\gamma(k)$ in Example 2.1

Figure 2.2: State responses of the uncertain system (2.7) without the controller (2.4) in Example 2.1

Figure 2.3: State trajectories of the uncertain system (2.7) with the partially delay-dependent controller (2.4) in Example 2.1
Example 2.2. Consider the inverted pendulum model [24, 45] described in Figure 2.4

\[
\dot{y}(t) = \begin{bmatrix}
0 & 1 \\
\frac{3(M+m)g}{l(4M+m)} & 0
\end{bmatrix} y(t) + \begin{bmatrix}
0 \\
\frac{3}{l(4M+m)}
\end{bmatrix} u(t).
\] (2.27)

By choosing \( M = 8 \text{kg}, \ m = 2.0 \text{kg}, \ l = 0.5 \text{m}, \ g = 9.8 \text{m/s}^2 \), the continuous-time system (2.27) can be transformed as the following discrete-time system

\[
x(k + 1) = \begin{bmatrix}
1.0078 & 0.0301 \\
0.5202 & 1.0078
\end{bmatrix} x(k) + \begin{bmatrix}
-0.0001 \\
0.0053
\end{bmatrix} u(k),
\] (2.28)

where the matrices \( A, \ B \) are given and assume \( A_d = 0 \).

Also, the uncertain parameters are chosen as

\[
M = \begin{bmatrix}
0.01 & 0.1 \\
-0.1 & 0.5
\end{bmatrix}, \quad \mathcal{N}_1 = \begin{bmatrix}
0.1 & 0.2 \\
0.05 & 0
\end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix}
0.2 & -0.4 \\
0.2 & 0.1
\end{bmatrix}.
\]

The closed-loop system may be stable with the controller \( u(k) = K x(t - \tau(k)) \), when the control gain is imposed as \( K = [245.2412 \ 58.9918] \). Figure 2.5 depicts the state of the considered inverted pendulum system (2.28) without control and Figure 2.6 shows that the state responses converge to zero with initial values of the state as \( x(0) = [0.1 \ -1] \) and time delay \( \tau(k) \) is assumed as \( \tau(k) = 3 + \pi \sin(k) \in [2, 4] \).
Figure 2.5: State trajectories of the inverted pendulum model (2.28) without control in Example 2.2

Figure 2.6: State responses of the inverted pendulum model (2.28) with the control in Example 2.2
2.3 Conclusions & Future Directions

The stochastic stability problem of guaranteed performance analysis for a class of uncertain discrete-time systems with linear fractional uncertainties has been investigated. The delay has been taken as time-varying rather than constant and also the adequate level of performance has been guaranteed by choosing desirable cost function. The sufficient criteria have been achieved by constructing appropriate LKFs and using most updated techniques such as Wirtinger-based inequality and convex reciprocal lemma. The partially delay-dependent controller has been adopted to control the given system where the state-feedback controller and delayed state-feedback controller occur randomly. The effectiveness of the obtained results have been illustrated through the numerical examples. Further, it is worth to extend the proposed stability results to neural network problems which plays a crucial role in artificial intelligence.