Chapter 2

Literature review

Let \((a_n)\) be a real-valued sequence and \(c_i, i = 1, 2, \ldots, n\) denote any real constants. The \(k^{th}\) order linear homogeneous recurrence relation with constant coefficients given by

\[
a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k}, \quad n, k \in \mathbb{N} \text{ and } k \leq n,
\]

occur in various branches of Science and Social Science. There are numerous techniques of solving this equation. One technique which is stated in [32] and which we shall use in this thesis is listed below:

Consider the characteristic equation corresponding to (2.1),

\[
\lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - c_3\lambda^{n-3} - \cdots - c_k\lambda^{n-k} = 0
\]

and let \(\alpha_i, i = 1, 2, \ldots, n\) be the distinct roots of this characteristic equation. Then the solution of (2.1) is given by

\[
a_n = \sum_{i=1}^{n} C_i\alpha_i^n, \quad \text{where } C_i, \quad i = 1, 2, \ldots, n, \quad \text{are any constants.}
\]

For example, if \(k = 2, c_1 = 1, c_2 = 1, a_0 = 0\) and \(a_1 = 1\), then (2.1) reduces to the Fibonacci sequence defined by (1.2).

Another concept which we shall use in this thesis is of Generating function.
Generating functions are powerful tools used for solving linear recursion relations and identities relating to them. The function \( g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \) generates the terms of the recurrence relation defined by (2.1) and hence it is called the generating function of the sequence \( a_n \). Thus we have the following definition.

**Definition 2.0.1.** Let \((a_n)\) be a sequence of real (or complex) numbers. If there exists a function \( g : X \rightarrow \mathbb{R} \) such that

\[
g(x) = \sum_{i=0}^{\infty} a_i x^i
\]

then \( g(x) \) is called the generating function of the sequence \((a_n)\).

In 1718, the French mathematician Abraham De Moivre (1667-1754) used the generating function to generate the terms of the Fibonacci sequence (1.2) [31]. He proved that the function \( f(x) = \frac{1}{1-x(1+x)} \) generates the terms of the Fibonacci sequence. The generating function of Lucas sequence is given by \( g(x) = \frac{2-x}{1-x(1+x)} \).

Since (1.2) is a linear homogeneous recurrence relation of second degree, it can be solved using the characteristic equation

\[
\lambda^2 - \lambda - 1 = 0
\]

If the distinct roots of (2.4) are \( \phi_1 \) and \( \phi_2 \), then the \( n^{th} \) term of (1.2) is given by

\[
F_n = \frac{\phi_1^n}{\phi_1 - \phi_2} + \frac{\phi_2^n}{\phi_2 - \phi_1}
\]

Note that \( \phi_1 = \frac{1+\sqrt{5}}{2} \) and \( \phi_2 = \frac{1-\sqrt{5}}{2} \). Hence (2.5) reduces to

\[
F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]
Similarly, we have

\[ L_n = \frac{(2\phi_1 - a)\phi_1^n}{\phi_1 - \phi_2} + \frac{(2\phi_2 - a)\phi_2^n}{\phi_2 - \phi_1} = \phi_1^n + \phi_2^n. \]  

(2.7)

In 1843, the French mathematician Jacques-Phillipe-Marie Binet [31] discovered this formula which is one of the techniques of finding the \( n^{th} \) term of (1.2). It is called the Binet’s formula.

Some of the identities obtained in the thesis are in terms of falling factorial power \( n^k \) (read as \( n \) to the \( k \) falling) [3]. We define it below. For \( n \in \mathbb{N} \cup \{0\} \),

\[ n^k = \begin{cases} 
  n(n-1) \cdots (n-(k-1)), & \text{if } k \in \mathbb{N}, k \leq n; \\
  0, & \text{if } k > n; \\
  1, & \text{if } k = 0; \\
  \frac{1}{(n+1)(n+2) \cdots (n-k)}, & \text{if } k \text{ is a negative integer.} 
\end{cases} \]  

(2.8)

The factorial of negative integers \( k \) is defined by [3]:

\[ (-k)! = (-k)(-k+1)(-k+2) \cdots (-1). \]  

(2.9)

For negative integer \( n \) and integer \( k \) ([19], [32]),

\[ n^k = \begin{cases} 
  k!(-1)^{n-k} \frac{(-1)^{n-k}}{(n-k)!}, & \text{if } k \leq n; \\
  (-1)^k(-n + k - 1)^k, & \text{if } k \geq 0; \\
  0, & \text{otherwise.} 
\end{cases} \]  

(2.10)

We now state the identity related to \( n^{th} \) term of (1.2) and (1.3) respectively.

\[ F_n = \sum_{r=0}^{\frac{n-1}{2}} \frac{(n - 1 - r)^{\xi}}{r!}, \forall n \geq 1, \]  

(2.11)
\[ L_n = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-r} \frac{(n-r)^2}{r!}, \forall n \geq 1. \] (2.12)

2.1 Identities of Fibonacci sequence

In this section, we state some of the interesting properties of Fibonacci sequence.

(1) If 1 is added to the sum of \( n+1 \) terms of Fibonacci sequence with initial term \( F_0 \), the resultant sum is \( (n+2) \)th term. i.e.
\[ 1 + \sum_{i=0}^{n} F_i = F_{n+2}. \]

(2) The sum of the first \( n \) terms with odd suffices with initial value \( F_1 \), is the \( (2n) \)th term which is the term with even suffix. i.e.
\[ \sum_{i=1}^{n} F_{2i-1} = F_{2n}. \]

On the other hand, if 1 is added to the sum of the first \( n+1 \) terms with even suffices with initial term \( F_0 \), the sum is \( (2n+1) \)th term. i.e.
\[ 1 + \sum_{i=0}^{n} F_{2i} = F_{2n+1}. \]

(4) The sum of the squares of the first \( (n+1) \) terms with initial term \( F_0 \) of (1.2), is the product of \( n \)th term and \( (n+1) \)th term of (1.2), i.e.
\[ \sum_{i=0}^{n} F_i^2 = F_n F_{n+1}. \]

(5) Sum of the squares of \( n \)th term and \( (n+1) \)th term is \( (2n+1) \)th term. i.e.
\[ F_n^2 + F_{n+1}^2 = F_{2n+1}. \]
(6) The difference of the product of $(n + 1)^{th}$ term and $(n - 1)^{th}$ term, and the square of $n^{th}$ term of the Fibonacci sequence is $(-1)^n$. i.e.

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$ 

**Remark:** The above identities can be proved using the generating function, Binet’s formula or by Mathematical induction on $n$.

Rewriting (1.2) as $F_{n-1} = F_{n+1} - F_n$ and using $F_0 = 0$ and $F_1 = 1$, we can obtain the following sequence.

$$F_0 = 0, F_1 = 1, F_{-1} = 1, F_{-2} = -1, F_{-3} = 2, F_{-4} = -3, \ldots.$$ 

Note that $F_{-n} = (-1)^{n+1}F_n$.

In [31], some of the identities of Fibonacci sequence stated above are proven for Lucas sequence.

Following identities show the relation between Fibonacci and Lucas sequences.

(1) The sum of $(n + 1)^{th}$ and $(n - 1)^{th}$ Fibonacci numbers is the $n^{th}$ Lucas number. i.e

$$L_n = F_{n+1} + F_{n-1}.$$ 

(2) If $(n - 2)^{th}$ Fibonacci number is subtracted from $(n + 2)^{th}$ Fibonacci number then the resultant value is the $n^{th}$ Lucas number. i.e.

$$L_n = F_{n+2} - F_{n-2}.$$ 

(3) The product of $F_{n+1}$ and $L_n$ is $F_{2n+1} - 1$, if $n$ is odd and $F_{2n+1} + 1$, if $n$ is even, i.e.

$$F_{n+1}L_n = \begin{cases} 
F_{2n+1} - 1, & \text{if } n \text{ is odd;} \\
F_{2n+1} + 1, & \text{if } n \text{ is even.}
\end{cases}$$

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2.2 Generalized Fibonacci sequence

In [31], the generalized Fibonacci sequence $G_n$ with initial conditions $G_1 = a$ and $G_2 = b$ is defined by

$$G_{n+1} = G_n + G_{n-1}$$

(2.13)

The terms generated by this sequence are

$G_1 = a, G_2 = b, G_3 = a + b, G_4 = a + 2b, G_5 = 2a + 3b$ etc. It is interesting to see that the coefficients of the terms are the terms of classical Fibonacci sequence (1.2).

Thus, we have

$$G_{n+1} = a F_{n-1} + F_n b, \forall n \geq 1.$$ 

(2.14)

(2.14) can be proved using induction on $n$, see [31]. With $a=1$ and $b=1$, the sequence (2.14) reduces to $G_{n+1} = F_{n-1} + F_n$ which is Fibonacci sequence (1.2) and if $a=2$ and $b=1$, it reduces to Lucas sequence (1.3).

We state below some of the properties of (2.14).

1. **Sum of the first $n$ terms:**

$$\sum_{r=1}^{n} G_r = aF_n + bF_{n+1} - b, \forall n \geq 1.$$ 

(2.15)

(2) **Binet’s Formula:**

$$G_n = c \phi_1^{n-2} - d \phi_2^{n-2},$$

where $c = a + b \phi_1$ and $d = a + b \phi_2$.

(2.16)

In [8], the authors consider the set of all sequences $(A_n)$ satisfying the following equation

$$A_{n+2} = aA_{n+1} + bA_n$$

(2.17)

with initial terms, $A_0$ and $A_1$ and later list various cases of this sequence by giving the choices for $a, b, A_0$ and $A_1$ including the generalized Fibonacci and Lucas sequences.
which are defined below respectively by:

\[ F_{n+2} = aF_{n+1} + bF_n, \text{ with } F_0 = 0 \text{ and } F_1 = 1, \]  
(2.18)

\[ L_{n+2} = aL_{n+1} + bL_n, \text{ with } L_0 = 2 \text{ and } L_1 = a, \]  
(2.19)

where \( a \) and \( b \) are fixed real constants. The authors in this paper have studied various properties of generalized Fibonacci sequence and Lucas sequence using the Difference operator.

First few terms of the sequence (2.18) are \( F_0 = 0, \ F_1 = 1, \ F_2 = a, \ F_3 = a^2 + b, \ F_4 = a^3 + 2ab, \ F_5 = a^4 + 3a^2b + b^2, \ F_6 = a^5 + 4a^3b + 3ab^2. \)

For \( 0 \leq n \leq 4 \), terms of the sequence (2.19) are \( L_0 = 2, \ L_1 = a, \ L_2 = a^2 + 2b, \ L_3 = a^3 + 3ab, \ L_4 = a^4 + 4a^2b + 2b^2. \)

The terms of (2.18) can also be obtained by adding the anti-diagonal terms of the following Pascal type triangle.

\[
\begin{array}{cccc}
1 \\
& a & b \\
& a^2 & 2ab & b^2 \\
& a^3 & 3a^2b & 3ab^2 & b^3 \\
\ldots
\end{array}
\]

Rewriting equation (2.18), we get

\[ F_{n-1} = \frac{1}{b}(F_{n+1} - aF_n), \text{ with } F_0 = 0 \text{ and } F_1 = 1. \]  
(2.20)

For \(-2 \leq n \leq 0\), we obtain the terms \( F_{-1} = \frac{1}{b}, \ F_{-2} = \frac{-a}{b^2}, \ F_{-3} = \frac{a^2+b}{b^3}. \)
We list below some of the properties of (2.18) that fascinated us ([8] and [25]).

1. The \( n \)th number \( F_n \) is given by

\[
F_n = \begin{cases} 
\frac{\phi_1^n}{\phi_1-\phi_2} + \frac{\phi_2^n}{\phi_2-\phi_1}, & a^2 + 4b \neq 0; \\
n\phi^{n-1}, & a^2 + 4b = 0, \quad \phi_1 = \phi_2 = \phi,
\end{cases}
\]

where \( \phi_1 = \frac{a + \sqrt{a^2 + 4b}}{2} \) and \( \phi_2 = \frac{a - \sqrt{a^2 + 4b}}{2} \), for all \( a, b \in \mathbb{R} \setminus \{0\} \), are roots of the equation \( \lambda^2 - a\lambda - b = 0 \).

2. The generating function for Fibonacci sequence (2.18) is given by

\[
G(x) = \frac{1}{1 - x(a + bx)}.
\]

3. The \( n \)th number \( F_n \) is also given by

\[
F_n = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} a^{n-1-2r} b^r, \quad \forall n \geq 1.
\]

With \( a = k \) and \( b = 1 \), the result can be seen in [25].

4. For all \( n \geq 0 \),

\[
\sum_{r=0}^{n} F_r = \frac{bF_n + F_{n+1} - 1}{a + b - 1},
\]

provided \( a + b \neq 1 \).

Another form of the extended Fibonacci sequence defined in [25] and [21], the \( k \)-Fibonacci sequence can be obtained from (2.18) by substituting \( a = k \) and \( b = 1 \). Various identities related to this sequence are included in this paper.
In Matrix form, the Fibonacci sequence (see [8]) is represented by

$$\begin{bmatrix}
F_n \\
F_{n+1}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
b & a
\end{bmatrix} \begin{bmatrix}
F_{n-1} \\
F_n
\end{bmatrix}.$$ 

Let $A = \begin{bmatrix}
0 & 1 \\
b & a
\end{bmatrix} = \begin{bmatrix}
F_0 & F_1 \\
bF_1 & F_2
\end{bmatrix}$, then $A^n = \begin{bmatrix}
b F_{n-1} & F_n \\
bF_n & F_{n+1}
\end{bmatrix}$.

Following identities can be proved by using the above matrix representation.

(5) **(Honsberger identity)**

For any $m, n \in \mathbb{Z}$,

$$F_{n+m-1} = b F_{n-1} F_{m-1} + F_n F_m.$$ (2.25)

With $m = n$ identity (2.25) reduces to

(a) $F_{2n-1} = b F_{n-1}^2 + F_n^2$.

With $m = n + 1$ identity (2.25) reduces to

(b) $F_{2n} = b F_{n-1} F_n + F_n F_{n+1}$.

Also using (2.18) and (5b), we can obtain, $F_{2n} = a F_n^2 + 2b F_{n-1} F_n$.

(6) **(General bilinear identity)**

For all $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ with $m_1 + n_2 = m_2 + n_1$,

$$\begin{vmatrix}
F_{m_1} & F_{n_1} \\
F_{m_2} & F_{n_2}
\end{vmatrix} = (-b)^s \begin{vmatrix}
F_{m_1-s} & F_{n_1-s} \\
F_{m_2-s} & F_{n_2-s}
\end{vmatrix}.$$ (2.26)

(7) **(d’Ocagne identity)**

For all $m, n \in \mathbb{Z}$,

$$\begin{vmatrix}
F_m & F_n \\
F_{m+1} & F_{n+1}
\end{vmatrix} = (-b)^n F_{m-n}.$$ (2.27)
For all \( n, r \in \mathbb{Z} \),

\[
\left| \begin{array}{cc}
F_n & F_{n+r} \\
F_{n-r} & F_n
\end{array} \right| = -(-b)^n F_r F_{-r} = (-b)^{n-r} F_r^2.
\] (2.28)

Putting \( r = 1 \), in (2.28) we get the following identity.

(9) (Cassini identity)

\[
\left| \begin{array}{cc}
F_n & F_{n-1} \\
F_{n+1} & F_n
\end{array} \right| = (-b)^{n-1}, \forall n \in \mathbb{Z}.
\] (2.29)

### 2.3 Incomplete Fibonacci and Lucas sequences

Filipponi introduced the incomplete Fibonacci numbers \( F^l_n \), incomplete Lucas numbers \( L^l_n \) as well as studied their various identities in [22]. Various identities related to the incomplete \( k \)-Fibonacci and \( k \)-Lucas numbers are studied in [12]. The author defines the incomplete \( k \)-Fibonacci and \( k \)-Lucas numbers respectively by

\[
F^l_{k,n} = \sum_{i=0}^{l} \frac{(n - 1 - i)^i}{i!} k^{n-1-2i}, \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \forall n \geq 1,
\] (2.30)

\[
L^l_{k,n} = \sum_{i=0}^{l} \frac{n}{n-i} \frac{(n - i)^i}{i!} k^{n-2i}, \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor, \quad \forall n \geq 1,
\] (2.31)

where \( k \) is a positive real number.

He also studied various identities of these sequences.

We list below the properties of (2.30) and (2.31), (see [12]).

1. For all \( n \geq 2 \),

\[
F^l_{k,n+2} = k F^l_{k,n+1} + F^l_{k,n}, \quad 0 \leq l \leq \left\lfloor \frac{n-2}{2} \right\rfloor.
\] (2.32)
Using (2.30), equation (2.32) can be rewritten as

\[ F_{k,n+2}^l = k F_{k,n+1}^l + F_{k,n}^l - \frac{(n - 1 - l)l!}{l!} k^{n-1-2l}. \]  

(2.33)

(2) For all \( s \geq 0 \),

\[ \sum_{i=0}^{s} \frac{s^i}{i!} F_{k,n+i}^l k^i = F_{k,n+s+1}^{l+s} \quad 0 \leq l \leq \left\lfloor \frac{n-s-1}{2} \right\rfloor. \]  

(2.34)

(3) For all \( s \geq 1 \),

\[ \sum_{i=0}^{s-1} F_{k,n+i}^l k^{s-1-i} = F_{k,n+s+1}^{l+1} - k^s F_{k,n+1}^{l+1}. \]  

(2.35)

(4) For all \( n \geq 2 \),

\[ L_{k,n+2}^{l+1} = k L_{k,n+1}^{l+1} + L_{k,n}^l, \quad 0 \leq l \leq \left\lfloor \frac{n-2}{2} \right\rfloor. \]  

(2.36)

Using (2.31), equation (2.36) can be rewritten as

\[ L_{k,n+2}^l = k L_{k,n+1}^l + L_{k,n}^l - \frac{(n - 1 - l)l!}{l!} k^{n-1-2l}. \]  

(2.37)

(5) For all \( n \geq 1, \ s \geq 0 \),

\[ \sum_{i=0}^{s} \frac{s^i}{i!} L_{k,n+i}^l k^i = L_{k,n+s+1}^{l+s} \quad 0 \leq l \leq \left\lfloor \frac{n-s-1}{2} \right\rfloor. \]  

(2.38)

(6) For all \( s \geq 1 \),

\[ \sum_{i=0}^{s-1} L_{k,n+i}^l k^{s-1-i} = L_{k,n+s+1}^{l+1} - k^s L_{k,n+1}^{l+1}. \]  

(2.39)

(7) For all \( n \geq 2 \),

\[ L_{k,n}^l = F_{k,n-1}^{l-1} + F_{k,n+1}^l, \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \]  

(2.40)
2.4 Fibonacci polynomials

Fibonacci polynomials are the natural extensions of Fibonacci sequence. In [25] and [31], these polynomials are studied in one variable, where as Hongquan Yu and Chuan-guang Liang derives identities involving partial derivatives of bivariate Fibonacci and Lucas polynomials in [11]. In [25], Fibonacci polynomials are defined by

$$F_{n+1}(x) = \begin{cases} 1, & \text{when } n = 0, \\ x, & \text{when } n = 1, \\ xF_n(x) + F_{n-1}(x), & \text{when } n \geq 2, \end{cases}$$

with $F_0(x) = 0$, where $F_n(x)$ is the $n^{th}$ Fibonacci polynomial.

In [15], Lucas polynomials in $x$ are defined by

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \quad \forall n \geq 1,$$

with $L_0(x) = 2$ and $L_1(x) = x$, where $L_n(x)$ is the $n^{th}$ Lucas polynomial.

In [11], the bivariate Fibonacci and Lucas polynomials are respectively defined by

$$F_{n+1}(x, y) = xF_n(x, y) + yF_{n-1}(x, y), \quad \forall n \geq 1,$$

with $F_0(x, y) = 0$ and $F_1(x, y) = 1$, where $F_n(x, y)$ is the $n^{th}$ Fibonacci polynomial.

$$L_{n+1}(x, y) = xL_n(x, y) + yL_{n-1}(x, y), \quad \forall n \geq 1,$$

with $L_0(x, y) = 2$ and $L_1(x, y) = x$, where $L_n(x, y)$ is the $n^{th}$ Lucas polynomial.

Various properties related to the polynomials (2.43) and (2.44) are obtained in [31].

For simplicity, let $F_n$ denote $F_n(x, y)$ and $L_n$ denote $L_n(x, y)$. 
The $n^{th}$ term of (2.43) and (2.44) respectively as defined in [11] are given below

$$F_n = \sum_{i=0}^{\left\lfloor \frac{n-i}{2} \right\rfloor} \frac{(n-1-i)^i}{i!} x^{n-2i-1} y^i, \ \forall n \geq 1. \quad (2.45)$$

$$L_n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{(n-i)} \frac{(n-i)^i}{i!} x^{n-2i} y^i, \ \forall n \geq 1. \quad (2.46)$$

Identities relating Fibonacci and Lucas polynomials (2.43) and (2.44) are

1. $L_n = F_{n+1} + yF_{n-1}$.
2. $L_n = 2F_{n+1} - xF_n$.

Following identities involving partial derivatives of $F_n$ and $L_n$ discussed in [11].

Let $F_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}(F_n)$ and $L_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}(L_n)$, $k, j \geq 0$.

we list the identities below:

1. $L_n^{(k,j)} = yF_{n-1}^{(k,j)} + jF_{n-1}^{(k,j-1)} + F_{n+1}^{(k,j)}$.
2. $F_n^{(k,j)} = xF_{n-1}^{(k,j)} + yF_{n-2}^{(k,j)} + kF_{n-1}^{(k-1,j)} + jF_{n-2}^{(k,j-1)}$.
3. $L_n^{(k,j)} = xL_{n-1}^{(k,j)} + yL_{n-2}^{(k,j)} + kL_{n-1}^{(k-1,j)} + jL_{n-2}^{(k,j-1)}$.
4. $nF_n^{(k,j)} = L_n^{(k+1,j)}$.
5. $nF_{n-1}^{(k,j)} = L_n^{(k,j+1)}$.

### 2.5 Incomplete Fibonacci and Lucas polynomials

The incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials and their identities are introduced in [14], whereas in [13], the incomplete Tribonacci polynomials and their identities are studied.
Definition 2.5.1. The incomplete $h(x)$-Fibonacci polynomials is defined by
\[
F_{h,n}^l(x) = \sum_{i=0}^{l} \frac{(n-1-i)^i}{i!} h^{n-1-2i}(x), \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor.
\] (2.47)

Definition 2.5.2. The incomplete $h(x)$-Lucas polynomials is defined by
\[
L_{h,n}^l(x) = \sum_{i=0}^{l} \frac{n}{n-i} \frac{(n-i)^i}{i!} h^{n-2i}(x), \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor.
\] (2.48)

Identities similar to incomplete $k$-Fibonacci and $k$-Lucas sequences are obtained for the polynomials $F_{h,n}^l(x)$ and $L_{h,n}^l(x)$ in [14].

In [13], the Tribonacci numbers are defined by
\[
t_{n+2} = t_{n+1} + t_n + t_{n-1}, \quad \forall n \geq 1,
\] (2.49)

with $t_0 = 0$, $t_1 = 1$ and $t_2 = 1$.

In [13], Jose L.R. introduces the incomplete Tribonacci numbers and incomplete Tribonacci polynomials. These are respectively defined by
\[
t_{h,n}^l = \sum_{i=0}^{l} \sum_{j=0}^{i} \frac{i^i}{j!} \frac{(n-i-j-1)^i}{i!}, \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor.
\] (2.50)

\[
T_{h,n}^l(x) = \sum_{i=0}^{l} \sum_{j=0}^{i} \frac{i^i}{j!} \frac{(n-i-j-1)^i}{i!} x^{2n-2-3(i+j)}, \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor.
\] (2.51)

Various identities relating to (2.50) and (2.51) are discussed in [13].
2.6 Functional equations

A functional equation is an equation whose solutions are the functions [24]. Stability problems of functional equations have been extensively studied. (see [9], ([23]), ([26]) and references therein). The importance of the topic lies in the fact that stability of functional equation is associated with notions of Controlled Chaos [30] and Shadowing [33]. In [26], the author discusses the stability problem in Banach space for Fibonacci functional equation defined by $f(x) = f(x-1) + f(x-2)$, whereas in [27], he discusses the stability of the generalized functional equation defined by

$$f(x) = pf(x-1) - qf(x-2), \forall p, q \in \mathbb{R},$$

(2.52)

in Banach space. In [20], the problem is discussed in Modular Functional space. In [4], $k$-Fibonacci functional equation is discussed whereas in [18] and [10] solution and stability of Tribonacci functional equation $f(x) = f(x-1) + f(x-2) + f(x-3)$ in non-Archimedean Banach spaces and 2-normed spaces have been discussed respectively. Stability of Tribonacci and $k$-Tribonacci functional equations in Modular spaces are discussed in [17]. In [28], authors investigate the solution of generalized linear Tribonacci functional equation in terms of Fibonacci numbers.