Chapter 1

Introduction

Leonardo Pisa popularly known as Fibonacci was famous for his book on *Liber Abaci* published around 1202. In this book, he states a recurrence relation which starts with 0 and 1 and the subsequent terms are obtained by adding the preceding two terms. Thus, we have

\[0, 1, 1, 2, 3, 5, \ldots\] (1.1)

Equation (1.1) was later named as the Fibonacci sequence. This name was given by the French mathematician Edouard Lucas in 1876. These numbers can be mathematically expressed in terms of the recurrence relation ([7], [31] and [29]) by

\[F_{n+1} = F_n + F_{n-1}, \forall n \geq 1, \text{ with } F_0 = 0 \text{ and } F_1 = 1,\] (1.2)

where \(F_n\) is the \(n^{th}\) Fibonacci number.

Edouard Lucas used the same recurrence relation of Fibonacci sequence with different seed values to generate a new sequence which is now called Lucas sequence. Lucas sequence is given by the recurrence relation

\[L_{n+1} = L_n + L_{n-1}, \forall n \geq 1, \text{ with } L_0 = 2 \text{ and } L_1 = 1,\] (1.3)
where $L_n$ is the $n^{th}$ Lucas number. Identities similar to the identities of Fibonacci sequence can be also obtained for the equation (1.3) (see [31], [29]).

Fibonacci is the family surname in Italian and it means "son of the simpleton (Bonaccio)". He was born around the year 1170. Fibonacci studied Indo-Arabic numeration system and computation techniques from his school teacher. Although the Fibonacci sequence was described earlier in Indian mathematics, Fibonacci was the first person to introduce it to the world through his book on Liber Abaci. He also included arithmetic, elementary Algebra, Indo-numeration system, elementary algorithms and some examples of business problems. However today he is known to the world mostly for the Fibonacci sequence. People appreciated his work in Indo-Arabic system. Leonardo Fibonacci used this sequence to win a competition sponsored by Emperor Frederick II in 1225. The contest question was: Start with a pair of rabbits. Every month, every pair of rabbits who are over a month old gives birth to a new pair of rabbits. After \(n\) months, how many pairs of rabbits are there? He found that solution for this problem was the Fibonacci sequence.

Kepler studied the Fibonacci sequence independently and also its properties [31]. One of the recurrence property he discovered is about the ratios of the consecutive terms of the Fibonacci sequence, that is

\[
\frac{1}{1} = 1, \quad \frac{2}{1} = 2, \quad \frac{3}{2} = 1.5, \quad \frac{5}{3} = 1.666..., \quad \frac{8}{5} = 1.6, \quad \frac{13}{8} = 1.625, \quad \frac{21}{13} = 1.61538\cdots
\]

He showed that this ratio approaches a number 1.618 (approx.) which is denoted by $\phi$ and is commonly known as the Golden ratio, named after the Greek sculptor Phidias, who used it in his artwork. This ratio has many applications. The rectangle in which the sides are in the ratio $\phi : 1$, is considered to be most pleasing to the human eye. There are many more applications in which this golden ratio appears. Like Fibonacci numbers, Tribonacci numbers also play an important role in problems of combinatorics ([16]) and also in the evaluation of determinants of circulant matrices ([6]).
It is this sequence which created interest within us to explore its extensions and learn various extended identities as (1.2) has wide varieties of interesting mathematical properties and various applications ranging from Nature to Technology.

The thesis is designed as follows:

**Chapter 1** is Introduction and **Chapter 2** deals with an overview of literature work.

In **Chapter 3** we have introduced $B$-Tribonacci and $B$-Tri Lucas sequences, incomplete $B$-Tribonacci and $B$-Tri Lucas sequences. We also study various identities related to these sequences.

**Chapter 4** deals with the $q^{th}$ order linear recurrence relation as an extension of the ideas introduced in Chapter 3. Here $q \geq 2$ and $q \in \mathbb{N}$. In this chapter $B$-$q$ bonacci, $B$-$q$ Lucas, incomplete $B$-$q$ bonacci and incomplete $B$-$q$ Lucas sequences are introduced.

In **Chapter 5**, the generalized bivariate $B$-Tribonacci, $B$-Tri Lucas, $B$-$q$ bonacci, $B$-$q$ Lucas, incomplete $B$-Tribonacci, incomplete $B$-Tri Lucas, incomplete $B$-$q$ bonacci and incomplete $B$-$q$ Lucas polynomials are introduced. The results discussed in Chapter 3 and Chapter 4 are extended to these polynomials. In this chapter, the identities involving partial derivatives of these polynomials are included.

In **Chapter 6**, the Fibonacci functional equation is extended to the generalized linear Tribonacci functional equation and proven that its solution is associated with generalized Tribonacci sequence. Its stability in the class of functions $f : \mathbb{R} \to X$, where $X$ is a real (or complex) Banach space is obtained. These results are further extended to the generalized linear $q$-bonacci functional equation.

At the end a brief summary of the work done is included. Few Python programming codes which are used to verify the identities are given in Appendix. This is followed by a list of publications. The thesis ends with a bibliography.