Chapter 5

Generalized Bivariate $B$-$q$ bonacci and $B$-$q$ Lucas

Polynomials

This Chapter includes the content of published papers (P2), (P3) and (E1).
Chapter 5

Generalized Bivariate $B$-$q$ bonacci and $B$-$q$ Lucas Polynomials

5.1 Introduction

It is known that one way of studying the extensions of Fibonacci sequence is the study of polynomials associated with it. In this Chapter, we generalize and extend bivariate Fibonacci polynomials defined by (2.43). The coefficients $x$ and $y$ of $F_n$ and $F_{n-1}$ in (2.43) is generalized to non-zero polynomials $h(x)$ and $g(y)$ with real coefficients respectively. Thus, we rewrite (2.43) and (2.44) respectively as

$$ (fB)_{h,g,n+1}(x, y) = h(x) (fB)_{h,g,n}(x, y) + g(y) (fB)_{h,g,n-1}(x, y), \quad (5.1) $$

with $(fB)_{h,g,0}(x, y) = 0$, $(fB)_{h,g,1}(x, y) = 1$.

and

$$ (fL)_{h,g,n+1}(x, y) = h(x) (fL)_{h,g,n}(x, y) + g(y) (fL)_{h,g,n-1}(x, y), \quad (5.2) $$

with $(fL)_{h,g,0}(x, y) = 2$, $(fL)_{h,g,1}(x, y) = x$.

We call (5.1) and (5.2), generalized bivariate $B$-Fibonacci polynomials and generalized bivariate $B$-Lucas polynomials respectively. With $g(y) = 1$, identities of (5.1) and (5.2) can be seen in [14] and [2]. In this Chapter, we extend and generalized (5.1)
and (5.2). This extension is such that the $n^{th}$ polynomial is constructed by adding the preceding three terms having the coefficients as the terms of the binomial expansion of $(h(x) + g(y))^2$. We call them, generalized bivariate $B$-Tribonacci polynomials and generalized bivariate $B$-Tri Lucas polynomials respectively. We also extend and generalized incomplete Fibonacci and Lucas polynomials defined by (2.47) and (2.48) respectively. Further they are extended to $q^{th}$ order polynomials.

In Section 2, we introduce and obtain various identities relating generalized bivariate $B$-Tribonacci polynomials. Section 3 deals with $B$-Tri Lucas polynomials and their identities. In Section 4 and Section 5, we introduce incomplete generalized bivariate $B$-Tribonacci polynomials and incomplete generalized bivariate $B$-Tri Lucas polynomials respectively. Section 6 deals with a generalized bivariate $B$-$q$ bonacci polynomials. In Section 7, we study generalized bivariate $B$-$q$ Lucas polynomials. In Section 8 and Section 9, we study incomplete generalized bivariate $B$-$q$ bonacci polynomials and incomplete generalized bivariate $B$-$q$ Lucas polynomials respectively.

We also study their various identities. Throughout this Chapter we take $h(x)$ and $g(y)$ to be two non-zero polynomials in $x$ and $y$ with real coefficients respectively.

### 5.2 Generalized bivariate $B$-Tribonacci polynomials

We define now generalized bivariate $B$-Tribonacci polynomials.

**Definition 5.2.1.** The generalized bivariate $B$-Tribonacci polynomials are defined by

$$(tB)_{h,g,n+2}(x, y)$$

$$= h^2(x)(tB)_{h,g,n+1}(x, y) + 2h(x)g(y)(tB)_{h,g,n}(x, y) + g^2(y)(tB)_{h,g,n-1}(x, y), \forall n \in \mathbb{N},$$

$$\text{(5.3)}$$

with $(tB)_{h,g,0}(x, y) = 0, (tB)_{h,g,1}(x, y) = 0$ and $(tB)_{h,g,2}(x, y) = 1,$
where the coefficients of the terms on right hand side of (5.3) are the terms of the binomial expansion of \((h(x) + g(y))^2\) and \(\binom{t}{h,g,n}(x, y)\) is the \(n\)th polynomial.

For \(0 \leq n \leq 6\), the terms of (5.3) are \(\binom{t}{h,g,0}(x, y) = 0\), \(\binom{t}{h,g,1}(x, y) = 0\), \(\binom{t}{h,g,2}(x, y) = 1\), \(\binom{t}{h,g,3}(x, y) = h^2(x)\), \(\binom{t}{h,g,4}(x, y) = h^4(x) + 2h(x)g(y)\), \(\binom{t}{h,g,5}(x, y) = h^6(x) + 4h^3(x)g(y) + g^2(y)\) and \(\binom{t}{h,g,6}(x, y) = h^8(x) + 6h^5(x)g^2(y) + 6h^2(x)g^2(y)\).

In particular, if \(g(y) = 1\), then (5.3) with \(\binom{t}{h,1,n}(x, y)\) written as \(\binom{t}{h,n}(x)\), reduces to (1.1) of (P3), namely

\[
\binom{t}{h,n+2}(x) = h^2(x)\binom{t}{h,n+1}(x) + 2h(x)\binom{t}{h,n}(x) + \binom{t}{h,n-1}(x), \quad \forall n \in \mathbb{N},
\]

with \(\binom{t}{h,0}(x) = 0\), \(\binom{t}{h,1}(x) = 0\) and \(\binom{t}{h,2}(x) = 1\).

For \(0 \leq n \leq 6\), the terms of (5.4) are \(\binom{t}{h,0}(x) = 0\), \(\binom{t}{h,1}(x) = 0\), \(\binom{t}{h,2}(x) = 1\), \(\binom{t}{h,3}(x) = h^2(x)\), \(\binom{t}{h,4}(x) = h^4(x) + 2h(x)\), \(\binom{t}{h,5}(x) = h^6(x) + 4h^3(x) + 1\) and \(\binom{t}{h,6}(x) = h^8(x) + 6h^5(x) + 6h^2(x)\).

If \(h(x) = 1\), then (5.4) reduce to \(B\)-Trionacci sequence (3.4) with \(a = 1\) and \(b = 1\), namely,

\[
\binom{t}{h,1,n+2} = \binom{t}{h,1,n+1} + 2\binom{t}{h,1,n} + \binom{t}{h,1,n-1}, \quad \forall n \geq 1,
\]

with \(\binom{t}{h,1,0} = 0\), \(\binom{t}{h,1,1} = 0\) and \(\binom{t}{h,1,2} = 1\).

First few terms of (5.5) are \(\binom{t}{h,1,0} = 0\), \(\binom{t}{h,1,1} = 0\), \(\binom{t}{h,1,2} = 1\), \(\binom{t}{h,1,3} = 1\), \(\binom{t}{h,1,4} = 3\), \(\binom{t}{h,1,5} = 6\), \(\binom{t}{h,1,6} = 13\), \(\binom{t}{h,1,7} = 28\) and \(\binom{t}{h,1,8} = 60\).

For simplicity, we use \(\binom{t}{h,g,n}(x, y) = \binom{t}{h,g,n}(x)\), \(\binom{t}{h,n}(x) = \binom{t}{h,n}(x)\), \(h(x) = h\) and \(g(y) = g\).

The following table shows the coefficients of \(\binom{t}{h,n}\) defined by (5.4) arranged in ascending order and also the terms of the sequence \(\binom{t}{h,1,n}\) defined by (5.5).
Table 5.1: Coefficients of \((tB)_{h,n}\) and terms of \((tB)_{1,n}\).

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In Table 5.1, the sum of the \(n^{th}\) row is the \(n^{th}\) term of the sequence \((tB)_{1,n}\). Also, for \(n \geq 2\), sum of the elements in the anti-diagonal of corresponding \((2n-3)\times(2n-3)\) matrix is \(2^{2(n-2)}\).

We state below theorems on the \(n^{th}\) term \((tB)_{h,g,n}\) defined by (5.3). These theorems can be proved using the procedure similar to that used to prove theorems in Section 2 of Chapter 3 and hence omitted.

**Theorem 5.2.2.** The \(n^{th}\) term of (5.3) is given by

\[
(tB)_{h,g,n} = \frac{(\alpha - \beta)\gamma^n - (\alpha - \gamma)\beta^n + (\beta - \gamma)\alpha^n}{(\alpha - \beta)(\beta - \gamma)(\alpha - \gamma)},
\]

(5.6)

where \(\alpha, \beta\) and \(\gamma\) are the distinct roots of the characteristics equation

\[\lambda^3 - h^2\lambda^2 - 2hg\lambda - g^2 = 0\]

corresponding to (5.3).

Equation (5.6) is called the Binet type identity for (5.3).

**Theorem 5.2.3.** The \(n^{th}\) term \((tB)_{h,g,n}\) of (5.3) is given by

\[
(tB)_{h,g,n} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n - 4 - 2r)^r}{r!} h^{2n-4-3r} g^r, \ \forall n \geq 2.
\]

(5.7)
Theorem 5.2.4. The sum of the first \( n + 1 \) terms of (5.3) is

\[
\sum_{r=0}^{n} (tB)_{h,g,r} = \frac{(tB)_{h,g,n+1} + (g^2 + 2hg)(tB)_{h,g,n} + g^2(tB)_{h,g,n-1} - 1}{(h + g)^2 - 1},
\]

provided \( h + g \neq \pm 1 \).

Theorem 5.2.5. The generating function for (5.3) is given by

\[
(tG_{(B)})_{h,g}(z) = \frac{1}{1 - z(h + gz)^2}.
\]

The next two theorems are related to the recurrence properties of \((tB)_{h,g,n}\).

Theorem 5.2.6. For all \( s \geq 1 \),

\[
\sum_{i=0}^{2s} \frac{(2s)_i}{i!} (tB)_{h,g,n+i} h^i g^{2s-i} = (tB)_{h,g,n+3s}.
\]

Proof. We prove the theorem by mathematical induction on \( n \). For \( s = 1 \),

L.H.S. of (5.10) = \[
\sum_{i=0}^{2} \frac{(2)_i}{i!} (tB)_{h,g,n+i} h^i g^{2s-i} = (tB)_{h,g,n+3} = \text{R.H.S.}
\]

Therefore (5.10) is true for \( s = 1 \). Assume that the result holds for all \( s \leq m \).

Consider, \[
\sum_{i=0}^{2m+2} \frac{(2m+2)_i}{i!} (tB)_{h,g,n+i} h^i g^{2m+2-i} = \sum_{i=0}^{2m+2} \left( \frac{(2m+2)_i}{(i-2)!} + 2 \frac{(2m+2)_i}{(i-1)!} + \frac{(2m+2)_i}{i!} \right) (tB)_{h,g,n+i} h^i g^{2m+2-i} = \sum_{i=-2}^{2m} \frac{(2m)_i}{i!} (tB)_{h,g,n+i+2} h^{i+2} g^{2m-i}
\]

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\[ +2 \sum_{i=-1}^{2m} \frac{(2m)!}{i!}(tB)_{h,g,n+i+1} h^{i+1} g^{2m-i+1} + \sum_{i=0}^{2m} \frac{(2m)!}{i!} (tB)_{h,g,n+i} h^i g^{2m-i+2} \]
\[ = \sum_{i=0}^{2m} \frac{(2m)!}{i!} h^i g^{2m-i} \left( h^2 (tB)_{h,g,n+i+2} + 2hg (tB)_{h,g,n+i+1} + g^2 (tB)_{h,g,n+i} \right) \]
\[ = h^2 (tB)_{h,g,n+3m+2} + 2hg(tB)_{h,g,n+3m+1} + g^2 (tB)_{h,g,n+3m} \]
\[ = (tB)_{h,g,n+3m+3}. \]

Hence the result is true for \( s = m + 1 \).

Therefore, by mathematical induction on \( s \), the result follows. \( \square \)

**Theorem 5.2.7.** For \( s \geq 1 \),
\[
\sum_{i=0}^{s-1} \left( 2h^{2s-1-2i} g (tB)_{h,g,n+1+i} + h^{2s-2-2i} g^2 (tB)_{h,g,n+i} \right)
\]
\[ = (tB)_{h,g,n+2+s} - h^{2s} (tB)_{h,g,n+2}. \quad (5.11) \]

**Proof.** By induction on \( s \). If \( s = 1 \), then (5.11) reduces to
\[ 2hg (tB)_{h,g,n+1} + g^2 (tB)_{h,g,n} = (tB)_{h,g,n+3} - h^2 (tB)_{h,g,n+2}, \]
which is true from (5.3). Hence (5.11) holds for \( s = 1 \).

Now let the result be true for \( s \leq m \). We prove it for \( s = m + 1 \).
Consider, \( \sum_{i=0}^{m} \left( 2h^{2m+1-2i} g (tB)_{h,g,n+1+i} + h^{2m-2i} g^2 (tB)_{h,g,n+i} \right) \).
\[ = \sum_{i=0}^{m-1} \left( 2h^{2m+1-2i} g (tB)_{h,g,n+1+i} + h^{2m-2i} g^2 (tB)_{h,g,n+i} \right) \]
\[ + \left( 2hg(tB)_{h,g,n+m+1} + g^2 (tB)_{h,g,n+m} \right) \]
\[ = h^2 \left( \sum_{i=0}^{m-1} \left( 2h^{2m-1-2i} g (tB)_{h,g,n+1+i} + h^{2m-2-2i} g^2 (tB)_{h,g,n+i} \right) \right) \]
\[ + \left( 2hg(tB)_{h,g,n+m+1} + g^2(tB)_{h,g,n+m} \right) \]

\[ = h^2 \left( (tB)_{h,g,n+m+2} - h^{2m}(tB)_{h,g,n+2} \right) \]

\[ + 2hg(tB)_{h,g,n+m+1} + g^2(tB)_{h,g,n+m} \]

\[ = h^2(tB)_{h,g,n+m+2} - h^{2m+2}(tB)_{h,g,n+2} + 2hg(tB)_{h,g,n+m+1} + g^2(tB)_{h,g,n+m} \]

\[ = (tB)_{h,g,n+m+3} - h^{2m+2}(tB)_{h,g,n+2}, \text{ from (5.3)}. \]

Hence the theorem is proved. \( \square \)

**Remark 5.2.8.** If \( g(y) = 1 \), then all the identities listed above reduce to corresponding identities of (5.4) which are published in (P3).

### 5.3 Generalized bivariate \( B \)-Tri Lucas polynomials

In this section, we define generalized bivariate \( B \)-Tri Lucas polynomials and study their various identities. We also prove the relation between generalized bivariate \( B \)-Tribonacci polynomials and generalized bivariate \( B \)-Tri Lucas polynomials.

**Definition 5.3.1.** The generalized bivariate \( B \)-Tri Lucas polynomials are defined by

\[ (tL)_{h,g,n+2}(x,y) = h^2(x)(tL)_{h,g,n+1}(x,y) \]

\[ + 2h(x)g(y)(tL)_{h,g,n}(x,y) + g^2(y)(tL)_{h,g,n-1}(x,y), \forall n \in \mathbb{N}, \quad (5.12) \]

with \( (tL)_{h,g,0}(x,y) = 0, (tL)_{h,g,1}(x,y) = 2 \) and \( (tL)_{h,g,2}(x,y) = h^2(x) \),

where the coefficients of the terms on the right hand side are the terms of the binomial expansion of \( \left( h(x) + g(y) \right)^2 \) and \( (tL)_{h,g,n}(x,y) \) is the \( n \)th polynomial.
For $0 \leq n \leq 5$, the terms of (5.12) are $(^tL)_{h,0,0}(x,y) = 0$, $(^tL)_{h,0,1}(x,y) = 2$,

$(^tL)_{h,0,2}(x,y) = h^2(x)$, $(^tL)_{h,0,3}(x,y) = h^4(x) + 4h(x)g(y)$,

$(^tL)_{h,0,4}(x,y) = h^6(x) + 6h^3(x)g(y) + 2g^2(y)$ and

$(^tL)_{h,0,5}(x,y) = h^8(x) + 8h^5(x)g(y) + 11h^2(x)g^2(y)$.

In particular if $g(y) = 1$, then (5.12) with $(^tL)_{h,1,n}(x,y)$ written as $(^tL)_{h,n}(x)$ reduces to (3.1) of (P3), namely,

$$(^tL)_{h,n+2}(x) = h^2(x)(^tL)_{h,n+1}(x) + 2h(x)(^tL)_{h,n}(x) + (^tL)_{h,n-1}(x), \forall n \in \mathbb{N},$$

(5.13)

with $(^tL)_{h,0}(x) = 0$, $(^tL)_{h,1}(x) = 2$ and $(^tL)_{h,2}(x) = h^2(x)$.

For $0 \leq n \leq 5$, the terms of (5.13) are $(^tL)_{h,0}(x) = 0$, $(^tL)_{h,1}(x) = 2$,

$(^tL)_{h,2}(x) = h^2(x)$, $(^tL)_{h,3}(x) = h^4(x) + 4h(x)$, $(^tL)_{h,4}(x) = h^6(x) + 6h^3(x) + 2$ and

$(^tL)_{h,5}(x) = h^8(x) + 8h^5(x) + 11h^2(x)$.

If $h(x) = 1$, then (5.13) reduces to B-Tri Lucas sequence defined by

$$(^tL)_{1,n+2} = (^tL)_{1,n+1} + 2(^tL)_{1,n} + (^tL)_{1,n-1}, \forall n \in \mathbb{N},$$

(5.14)

with $(^tL)_{1,0} = 0$, $(^tL)_{1,1} = 2$ and $(^tL)_{1,2} = 1$.

First few terms of (5.14) are $(^tL)_{1,0} = 0$, $(^tL)_{1,1} = 2$, $(^tL)_{1,2} = 1$, $(^tL)_{1,3} = 5$,

$(^tL)_{1,4} = 9$, $(^tL)_{1,5} = 20$, $(^tL)_{1,6} = 43$ and $(^tL)_{1,7} = 92$.

For simplicity, we use $(^tL)_{h,g,n}(x,y) = (^tL)_{h,g,n}$, $(^tL)_{h,n}(x) = (^tL)_{h,n}$, $h(x) = h$ and $g(y) = g$.

Following table show coefficients of $(^tL)_{h,n}$ arranged in ascending order of $h$ and also terms of sequence $(^tL)_{1,n}$.

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In Table 5.2, the sum of the $n^{th}$ row is the $n^{th}$ term of $(tL)_{1,n}$. Also, for $n \geq 2$, sum of the elements in the anti-diagonal of corresponding $(2n-1) \times (2n-1)$ matrix is $7 \left(2^{2(n-2)} \right)$. 

We state below theorems related to the $n^{th}$ term $(tL)_{h,g,n}$ of $B$-Tri Lucas polynomials. These theorems can be proved using the procedure similar to that of theorems in Section 3 of Chapter 3 and hence omitted.

**Theorem 5.3.2.** The $n^{th}$ term $(tL)_{h,g,n}$ of (5.12) is given by

$$(tL)_{h,g,n} = \frac{(\alpha - \beta)\gamma^n(2\gamma - h^2) - (\alpha - \gamma)\beta^n(2\beta - h^2) + (\beta - \gamma)\alpha^n(2\alpha - h^2)}{(\alpha - \beta)(\beta - \gamma)(\alpha - \gamma)}, \quad (5.15)$$

where $\alpha, \beta$ and $\gamma$ are the distinct roots of the characteristics equation

$$\lambda^3 - h^2\lambda^2 - 2hg\lambda - g^2 = 0$$

corresponding to (5.12).

Equation (5.15) is called Binet type formula for (5.12).

**Theorem 5.3.3.** The $n^{th}$ term $(tL)_{h,g,n}$ of (5.12) is given by

$$(tL)_{h,g,n} = \sum_{r=0}^{\left\lfloor \frac{2n-2}{2} \right\rfloor} \frac{(2n-2) \cdot (2n-2-2r) \cdot \cdots \cdot (2n-2-2r)}{r!} \cdot \frac{r(r-1) \cdot (2n-4-2r) \cdot \cdots \cdot (2n-4-2r)}{r!} \cdot h^{2n-2-3r} g^r, \quad \forall n \geq 2. \quad (5.16)$$
Theorem 5.3.4. The sum of the first \( n + 1 \) terms of \( (tL)_{h,g,n} \) is

\[
\sum_{r=0}^{n} (tL)_{h,g,r} = \frac{(tL)_{h,g,n+1} + (2hg + g^2)(tL)_{h,g,n} + g^2(tL)_{h,g,n-1} + (tL)_{h,g,2} - (tL)_{h,g,1}}{(h + g)^2 - 1},
\]

provided \( h + g \neq \pm 1 \).

Theorem 5.3.5. The generating function for \( (tL)_{h,g,n} \) is given by

\[
(tG(L))_{h,g}(z) = \frac{2 - h^2z}{1-z(h + gz)^2}.
\]

We have the following theorems on recurrence properties of generalized bivariate \( B \)-Tri Lucas polynomials.

Theorem 5.3.6.

\[
(tL)_{h,g,n+1} = (tB)_{h,g,n+2} + 2hg (tB)_{h,g,n} + g^2(tB)_{h,g,n-1}, \forall n \geq 1.
\]

Proof. By induction on \( n \). Since \( (tL)_{h,g,2} = h^2, (tB)_{h,g,3} = h^2, (tB)_{h,g,1} = 0 \) and \( (tB)_{h,g,0} = 0 \), (5.19) holds for \( n = 1 \).

Now assume that it holds for \( n \leq m - 1 \) and consider (5.12),

\[
(tL)_{h,g,m+1} = h^2 (tL)_{h,g,m} + 2hg (tL)_{h,g,m-1} + g^2(tL)_{h,g,m-2}
\]

\[
= h^2 \left( (tB)_{h,g,m+1} + 2hg (tB)_{h,g,m-1} + g^2(tB)_{h,g,m-2} \right)
\]

\[
+ 2hg \left( (tB)_{h,g,m} + 2hg (tB)_{h,g,m-2} + g^2(tB)_{h,g,m-3} \right)
\]

\[
+ g^2 \left( (tB)_{h,g,m-1} + 2hg (tB)_{h,g,m-3} + g^2(tB)_{h,g,m-4} \right), \text{ by assumption.}
\]

\[
= (tB)_{h,g,m+2} + 2hg (tB)_{h,g,m} + g^2(tB)_{h,g,m-1}.
\]

Hence by mathematical induction the result is proved.
Following Corollary can be deduced from (5.3) and (5.19).

**Corollary 5.3.7.**

\[
(t^L)_{h,g,n} = 2 (t^B)_{h,g,n+1} - h^2 (t^B)_{h,g,n}, \quad \forall n \geq 0.
\]  

(5.20)

Using the above Corollary, we can establish the following results.

**Theorem 5.3.8.**

\[
(t^L)_{h,g,n+3s} = \sum_{i=0}^{2s} \frac{(2s)^i}{i!} (t^L)_{h,g,n+i} h^i g^{2s-i}, \quad s \geq 1.
\]  

(5.21)

**Proof.** Since \((t^L)_{h,g,n} = 2 (t^B)_{h,g,n+1} - h^2 (t^B)_{h,g,n},\)

\[
\sum_{i=0}^{2s} \frac{(2s)^i}{i!} (t^L)_{h,g,n+i} h^i g^{2s-i}
\]

\[= \sum_{i=0}^{2s} \frac{(2s)^i}{i!} \left( 2 (t^B)_{h,g,n+1+i} - h^2 (t^B)_{h,g,n+i} \right) h^i g^{2s-i}\]

\[= 2 \sum_{i=0}^{2s} \frac{(2s)^i}{i!} (t^B)_{h,g,n+1+i} h^i g^{2s-i} - h^2 \sum_{i=0}^{2s} \frac{(2s)^i}{i!} (t^B)_{h,g,n+i} h^i g^{2s-i}\]

\[= 2 (t^B)_{h,g,n+1+3s} - h^2 (t^B)_{h,g,n+3s}, \quad \text{from (5.10).}\]

\[= (t^L)_{h,g,n+3s}. \quad \square\]

Using (5.11), (5.20) and the procedure similar to that of Theorem 5.3.8, we get the following result.

**Theorem 5.3.9.**

\[
\sum_{i=0}^{s-1} \left( 2 h^{2s-1-2i} g \left( t^L \right)_{h,g,n+1+i} + h^{2s-2-2i} g^2 \left( t^L \right)_{h,g,n+i} \right) = (t^L)_{h,g,n+2+s} - h^{2s} (t^L)_{h,g,n+2}.
\]  

(5.22)
Following identities involving partial derivatives of the polynomials \((tL)_{h,g,n}\) and \((tL)_{h,g,n}\) are extensions of some identities discussed in [11].

Let \((tB)_{h,g,n}^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} ((tB)_{h,g,n})\), \((tL)_{h,g,n}^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} ((tL)_{h,g,n})\), \(h^{(k,0)} = \frac{d^k}{dx^k} (h(x))\) and \(g^{(0,j)} = \frac{d^j}{dy^j} (g(y))\).

We have the following identities involving \((tB)_{h,g,n}^{(k,j)}\) and \((tL)_{h,g,n}^{(k,j)}\).

**Theorem 5.3.10.**

1. \((tL)_{h,g,n}^{(k,j)} = (tB)_{h,g,n+1}^{(k,j)} + \sum_{r=1}^{2} \frac{2r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k+i}{s!} \frac{j}{r!} (h^{2-r})(s,0) (g^r)^{(0,i)} (tB)_{h,g,n-r}^{(k-s,j-i)}\).

2. \((tB)_{h,g,n}^{(k,j)} = \sum_{r=0}^{2} \frac{2r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k+i}{s!} \frac{j}{r!} (h^{2-r})(s,0) (g^r)^{(0,i)} (tB)_{h,g,n-1-r}^{(k-s,j-i)}\).

3. \((tL)_{h,g,n}^{(k,j)} = \sum_{i=0}^{k} \frac{k}{s!} (h^s)_{h,g,n} + \sum_{i=0}^{k} \frac{k}{s!} (h)^{(s,i)} (tB)_{h,g,n+1}^{(k+,s,j-i)}\).

4. \(2(n-1) \sum_{i=0}^{j} \frac{j}{i!} (g)^{(0,i)} (tB)_{h,g,n+1}^{(k-s,j-i)} = 3 \sum_{i=0}^{j} \frac{j}{i!} (g)^{(0,i)} (tB)_{h,g,n}^{(k+1-j-i)} + \sum_{i=0}^{k} \frac{k}{s!} (h)^{(s,i)} (tB)_{h,g,n+1}^{(k+1-s,j-i)}\).

5. \(2(n-2) \sum_{i=0}^{j} \frac{j}{i!} (g)^{(0,1+i)} (tB)_{h,g,n}^{(k-s,j-i)} = 3 \sum_{i=0}^{j} \frac{j}{i!} (g)^{(0,1+i)} (tB)_{h,g,n}^{(k+1-j-i)} + \sum_{i=0}^{k} \frac{k}{s!} (h)^{(s,0)} (tB)_{h,g,n+1}^{(k-s,j+1)}\).

6. \(\sum_{i=0}^{j} \frac{j}{i!} (g)^{(0,1+i)} (tB)_{h,g,n}^{(k+1-j-i)} = \sum_{i=0}^{k} \frac{k}{s!} (h)^{(1+s,0)} (tB)_{h,g,n+1}^{(k-s,j)}\).

**Proof.**

(1) Equation (5.19) implies \((tL)_{h,g,n} = (tB)_{h,g,n+1} + 2h \cdot (tB)_{h,g,n-1} + g^2 \cdot (tB)_{h,g,n-2}\).

Differentiating both sides \(k\) times with respect to \(x\) and \(j\) times with respect to \(y\) and using Leibnitz theorem for derivatives, we get

\[(tL)_{h,g,n}^{(k,j)} = (tB)_{h,g,n+1}^{(k,j)} + 2 \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k+i}{s!} \frac{j}{r!} h^{(s,i)} (tB)_{h,g,n-1}^{(k-s,j-i)}\]

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\[\sum_{i=0}^{j} \frac{i}{\pi} (g^2)^{(0,i)} (t^B)^{(k,j-i)}_{h,g,n-2}\]

\[= (t^B)^{(k,j)}_{h,g,n+1} + \sum_{r=1}^{2} 2^r \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^s}{s!} \frac{r^j}{j!} (h^{2-r})^{(s,0)} (g^r)^{(0,i)} (t^B)^{(k-s,j-i)}_{h,g,n-r}.\]

(2) Equation (5.3) implies

\[(t^B)_{h,g,n} = h^2 (t^B)_{h,g,n-1} + 2hg (t^B)_{h,g,n-2} + g^2 (t^B)_{h,g,n-3}.\]

Differentiating both sides \(k\) times with respect to \(x\) and \(j\) times with respect to \(y\) and using Leibnitz theorem for derivatives, we get

\[(t^B)^{(k,j)}_{h,g,n} = \sum_{s=0}^{k} \frac{k^s}{s!} (h^2)^{(s,0)} (t^B)^{(k-s,j)}_{h,g,n-1} + 2\sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^s}{s!} \frac{j^i}{i!} h^{(s,0)} g^{(0,i)} (t^B)^{(k-s,j-i)}_{h,g,n-2}\]

\[+ \sum_{i=0}^{j} \frac{j^i}{i!} (g^2)^{(0,i)} (t^B)^{(k,j-i)}_{h,g,n-3}\]

\[= \sum_{r=0}^{2} 2^r \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^s}{s!} \frac{r^j}{j!} (h^{2-r})^{(s,0)} (g^r)^{(0,i)} (t^B)^{(k-s,j-i)}_{h,g,n-1-r}.\]

(3) Equation (5.12) implies

\[(t^L)_{h,g,n} = h^2 (t^L)_{h,g,n-1} + 2hg (t^L)_{h,g,n-2} + g^2 (t^L)_{h,g,n-3}.\]

Hence identity (3) can be proved by a method similar to that used in identity (2) above.

(4) We first prove that \(2(n - 1)h (t^{1,0})_{h,g,n+1}(t^B)^{(1,0)}_{h,g,n+1} = 3g (t^B)^{(1,0)}_{h,g,n} + h (t^B)^{(1,0)}_{h,g,n+1}\), using (5.7). For this purpose, we divide the proof in to three cases depending on \(n\), i.e. \(n = 3k; 3k + 1, 3k + 2\).

Case 1 : Let \(n = 3k\). Consider,

\[3g (t^B)^{(1,0)}_{h,g,n} + h (t^B)^{(1,0)}_{h,g,n+1}\]

\[= 3g \frac{\partial}{\partial x} ((t^B)_{h,g,3k}) + h \frac{\partial}{\partial x} ((t^B)_{h,g,3k+1})\]
\[
\begin{align*}
&= 3g \frac{\partial}{\partial x} \left( \sum_{r=0}^{2k-2} \frac{(6k-4-2r)^2}{r!} h^{6k-4-3r} g^r \right) + h \frac{\partial}{\partial x} \left( \sum_{r=0}^{2k-1} \frac{(6k-2-2r)^2}{r!} h^{6k-2-3r} g^r \right) \\
&= 3 \left( \sum_{r=0}^{2k-2} \frac{(6k-4-2r)^2}{r!} h^{6k-5-3r} h^{(1,0)} g^{r+1} \right) + \left( \sum_{r=0}^{2k-1} \frac{(6k-2-2r)^2}{r!} h^{6k-2-3r} h^{(1,0)} g^r \right) \\
&= (6k - 2) h^{6k-2} h^{(1,0)} + \sum_{r=1}^{2k-1} \left( 3r \frac{(6k-2-2r)^2}{r!} + \frac{(6k-2-2r)^2}{r!} \right) h^{6k-2-3r} h^{(1,0)} g^r \\
&= (6k - 2) h^{6k-2} h^{(1,0)} + \sum_{r=1}^{2k-1} \left( 3r + (6k - 2 - 3r) \right) h^{6k-2-3r} h^{(1,0)} g^r \\
&= (6k - 2) h^{6k-2} h^{(1,0)} + \sum_{r=1}^{2k-1} \frac{(6k-2-2r)^2}{r!} (6k - 2) h^{6k-2} h^{(1,0)} g^r \\
&= \sum_{r=0}^{2k-1} (6k - 2) \frac{(6k-2-2r)^2}{r!} h^{6k-2-3r} h^{(1,0)} g^r \\
&= 2(3k - 1) h^{(1,0)} (^t B)_{h,g,3k+1}.
\end{align*}
\]

Hence the result is true for \( n = 3k \).

Similarly, the result can be proved for \( n = 3k + 1 \) and \( n = 3k + 2 \).

Thus, we have, \( 2(n - 1) h^{(1,0)} (^t B)_{h,g,n+1} = 3g (^t B)^{(1,0)}_{h,g,n} + h (^t B)^{(1,0)}_{h,g,n+1} \).

Now differentiating both sides \( k \) times with respect to \( x \) and \( j \) times with respect to \( y \) and using Leibnitz theorem for derivatives, we get the required result.

(5) We first show that

\[
2(n - 2) g^{(0,1)} (^t B)_{h,g,n} = 3g (^t B)^{(0,1)}_{h,g,n} + h (^t B)^{(0,1)}_{h,g,n+1}.
\]

We consider 3 cases by taking \( n = 3k, 3k + 1, 3k + 2 \).

Case 1: Let \( n = 3k \). Consider,

\[
3g (^t B)^{(0,1)}_{h,g,n} + h (^t B)^{(0,1)}_{h,g,n+1}.
\]
= 3g \frac{\partial}{\partial y} ((t^i B)_{h,g,3k}) + h \frac{\partial}{\partial y} ((t^i B)_{h,g,3k+1}), \text{ from (5.7)}

= 3g \frac{\partial}{\partial y} \left( \sum_{r=0}^{2k-2} \frac{(6k-4-2r)z^r}{r!} h^{6k-4-3r} g^r \right) + h \frac{\partial}{\partial y} \left( \sum_{r=0}^{2k-1} \frac{(6k-2-2r)z^r}{r!} h^{6k-2-3r} g^r \right)

= 3 \left( \sum_{r=1}^{2k-2} \frac{(6k-4-2r)z^r}{(r-1)!} h^{6k-4-3r} g^{r+1} g^{(0,1)} \right) + \left( \sum_{r=1}^{2k-1} \frac{(6k-2-2r)z^r}{(r-1)!} h^{6k-2-3r} g^{r+1} g^{(0,1)} \right)

= 3k(6k - 4) h^{6k-4} g^{(0,1)} + \sum_{r=1}^{2k-2} \frac{(6k-4-2r)z^r}{r!} (6k - 4) h^{6k-4-3r} g^r g^{(0,1)}

= 3(6k - 4) h^{6k-4} g^{(0,1)} + \sum_{r=1}^{2k-2} \frac{(6k-4-2r)z^r}{r!} (6k - 4) h^{6k-4-3r} g^r g^{(0,1)}

= \sum_{r=0}^{2k-2} (6k - 4) \frac{(6k-4-2r)z^r}{r!} h^{6k-4-3r} g^r g^{(0,1)}

= 2(3k - 2) g^{(0,1)} (t^i B)_{h,g,3k}.

Hence the result is true for \( n = 3k \).

Similarly, the result can be proved for \( n = 3k + 1 \) and \( n = 3k + 2 \).

Thus we have, \( 2(n - 2)(t^i B)_{h,g,n} g^{(0,1)} = 3g \ (t^i B)^{(0,1)}_{h,g,n} + h \ (t^i B)^{(0,1)}_{h,g,n+1} \)

Now differentiating above equation both sides \( k \) times with respect to \( x \) and \( j \) times with respect to \( y \) and using Leibnitz theorem for derivatives, we get the required result.

(6) We first show that \( g^{(0,1)} (t^i B)^{(1,0)}_{h,g,n} = h^{(1,0)} (t^i B)^{(0,1)}_{h,g,n+1} \).

We divide the proof in 3 cases, \( n = 3k, 3k + 1, 3k + 2 \).

Putting \( n = 3k \) in (5.3) and differentiate it with respect to \( x \), we get,

\[ (t^i B)^{(1,0)}_{h,g,3k} = \frac{\partial}{\partial x} \left( \sum_{r=0}^{2k-2} \frac{(6k-4-2r)z^r}{r!} h^{6k-4-3r} g^r \right) \]
\[
\sum_{r=0}^{2k-2} \frac{(6k-4-2r)r+1}{r!} h^{6k-5-3r} h^{(1,0)} g^r.
\]

Therefore, \( g^{(0,1)} (tB)^{(1,0)}_{h,g,3k} = \sum_{r=0}^{2k-2} \frac{(6k-4-2r)r+1}{r!} h^{6k-5-3r} h^{(1,0)} g^r g^{(0,1)}. \)

Now consider,

\[
(tB)^{(0,1)}_{h,g,3k+1} = \frac{\partial}{\partial y} (tB)^{(0,1)}_{h,g,3k+1}
\]

\[
= \sum_{r=0}^{2k-1} \frac{(6k-2-2r)r}{r!} h^{6k-2-3r} r g^{r-1} g^{(0,1)}
\]

\[
= \sum_{r=1}^{2k-1} \frac{(6k-2-2r)r}{(r-1)!} h^{6k-2-3r} g^{r-1} g^{(0,1)}
\]

Thus, \( h^{(1,0)} (tB)^{(0,1)}_{h,g,3k+1} = \sum_{r=0}^{2k-2} \frac{(6k-4-2r)r+1}{r!} h^{6k-5-3r} h^{(1,0)} g^r g^{(0,1)}. \)

Therefore, \( (tB)^{(1,0)}_{h,g,n} h^{(1,0)} = (tB)^{(0,1)}_{h,g,n+1} g^{(0,1)}. \)

Differentiating both sides \( k \) times with respect to \( x \) and \( j \) times with respect to \( y \) and using Leibnitz theorem for derivatives, we get

\[
\sum_{i=0}^{j} \frac{j!}{i!} g^{(0,1+i)} (tB)^{(k+1,j-i)}_{h,g,n} = \sum_{s=0}^{k} \frac{k!}{s!} h^{(1+s,0)} (tB)^{(k-s,j+1)}_{h,g,(n+1)}.
\]

With \( h(x) = x \) and \( g(y) = y \), generalized bivariate \( B \)-Tribonacci polynomials and generalized bivariate \( B \)-Tri Lucas polynomials respectively reduce to

\[
(tB)_{n+2}(x,y) = x^2(tB)_{n+1}(x,y) + 2xy(tB)_{n}(x,y) + y^2(tB)_{n-1}(x,y), \forall n \geq 1, \quad (5.23)
\]

with \((tB)_0(x,y) = 0, (tB)_1(x,y) = 0, (tB)_2(x,y) = 1\).

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and

\[(^tL)_{n+2}(x, y) = x^2(^tL)_{n+1}(x, y) + 2xy(^tL)_{n}(x, y) + y^2(^tL)_{n-1}(x, y), \forall n \geq 1, \quad (5.24)\]

with \((^tL)_0(x, y) = 0, (^tL)_1(x, y) = 2\) and \((^tL)_2(x, y) = x^2\).

Following Corollary give the corresponding identities of (5.23) and (5.24).

**Corollary 5.3.11.** For all \(n \geq 2\),

\[\begin{align*}
(1) \quad (^tL)_{n+1}^{(k,j)} &= (^tB)_{n+1}^{(k,j)} + \sum_{r=1}^{2} \frac{n}{r!} \sum_{s=0}^{2-r} \sum_{i=0}^{r} \frac{k^s}{s!} \frac{r}{r!} (x^{2-r})^{s,0}(y^r)^{(0,i)}(^tB)_{n-r}^{(k-s,j-i)} \\
(2) \quad (^tB)_n^{(k,j)} &= \sum_{r=0}^{2} \sum_{s=0}^{r} \sum_{i=0}^{r} \frac{k^s}{s!} \frac{r}{r!} (x^{2-r})^{s,0}(y^r)^{(0,i)}(^tB)_{n-1-r}^{(k-s,j-i)} \\
(3) \quad (^tL)_n^{(k,j)} &= \sum_{r=0}^{2} \sum_{s=0}^{r} \sum_{i=0}^{r} \frac{k^s}{s!} \frac{r}{r!} (x^{2-r})^{s,0}(y^r)^{(0,i)}(^tL)_{n-1-r}^{(k-s,j-i)} \\
(4) \quad 2(n-1)(^tB)_n^{(k,j)} &= 3\sum_{i=0}^{1} \frac{n}{i!} (y^{(0,i)})(^tB)_{n-1}^{(k+1-j-i)} + \sum_{s=0}^{1} \frac{k^s}{s!} (x^{s,0})(^tB)_{n+1}^{(k+1-s,j)} \\
(5) \quad 2(n-2)(^tB)_n^{(k,j)} &= 3\sum_{i=0}^{1} \frac{n}{i!} (y^{(0,i)})(^tB)_{n-1}^{(k+1-j-i)} + \sum_{s=0}^{1} \frac{k^s}{s!} (x^{s,0})(^tB)_{n+1}^{(k-s,j+1)} \\
(6) \quad (^tB)_{n+1}^{(k+1,j)} &= (^tB)_{n+1}^{(k,j+1)}
\end{align*}\]

**Theorem 5.3.12.** (*Convolution property* for \((^tB)_{h,g,n}\))

\[\quad (^tB)_{h,g,n}^{(1,0)} = h^{(1,0)} \sum_{i=0}^{n} \left(2h(^tB)_{h,g,n+1-i} + 2g(^tB)_{h,g,n-i}\right)(^tB)_{h,g,i}. \quad (5.25)\]

*Proof.* Equation (5.9) implies

\[\sum_{n=0}^{\infty} (^tB)_{h,g,n} z^{n-2} = \frac{1}{1 - z(h + g)^2}\]

Differentiating both sides with respect to \(x\) we get,

\[\sum_{n=0}^{\infty} (^tB)_{h,g,n} z^{n-2} = h^{(1,0)} \left(\frac{2hz}{1 - (h + g)^2} + \frac{2gz^2}{1 - (h + g)^2}\right)\]

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\[
\sum_{n=0}^{\infty} (t B)_{h,g,n} z^{n+1} = h^{(1,0)} \left( 2 h \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} (t B)_{h,g,i} (t B)_{h,g,n-i} \right) z^{n} + 2 g \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} (t B)_{h,g,i} (t B)_{h,g,n-i} \right) z^{n+1} \right)
\]

Comparing the coefficients of \(z^{n+1}\),

\[
(t B)_{h,g,n}^{(1,0)} = h^{(1,0)} \left( \sum_{i=0}^{n} \left( 2 h (t B)_{h,g,n+1-i} + 2 g (t B)_{h,g,n-i} \right) (t B)_{h,g,i} \right).
\]

\[\square\]

**Theorem 5.3.13.** *(Convolution property for \((t L)_{h,g,n}\))*

\[
(t L)_{h,g,n}^{(1,0)} = h^{(1,0)} \left[ \sum_{i=0}^{n} \left( 2 h (t L)_{h,g,n+1-i} + 2 g (t L)_{h,g,n-i} \right) (t B)_{h,g,i} - 2 h (t B)_{h,g,n} \right].
\]

**(5.26)**

**Proof.** Equation (5.20) implies

\[
(t L)_{h,g,n} = 2 (t B)_{h,g,n+1} - h^2 (t B)_{h,g,n}
\]

Differentiating both sides with respect to \(x\), we get

\[
(t L)_{h,g,n}^{(1,0)} = 2 (t B)_{h,g,n+1}^{(1,0)} - h^2 (t B)_{h,g,n}^{(1,0)} - 2 h (t B)_{h,g,n}^{(1,0)}
\]

\[
= 2 h^{(1,0)} \sum_{i=0}^{n+1} \left( 2 h (t B)_{h,g,n+2-i} + 2 g (t B)_{h,g,n+1-i} \right) (t B)_{h,g,i}
\]

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5.4 Incomplete generalized bivariate $B$-Tribonacci polynomials

In this section, we define the incomplete generalized bivariate $B$-Tribonacci polynomials and obtain various identities related to these polynomials.

**Definition 5.4.1.** The incomplete generalized bivariate $B$-Tribonacci polynomials are defined by

$$(t^LB)^l_{h,g,n}(x,y) = \sum_{r=0}^{l} \frac{(2n-4-2r)x}{r!} h^{2n-4-3r}(x)g^r(y), \forall 0 \leq l \leq \left\lfloor \frac{2n-4}{3} \right\rfloor \text{ and } n \geq 2.$$  \hspace{1cm} (5.27)

We list below terms of (5.27) for $2 \leq n \leq 5$.

\begin{align*}
(t^B)^0_{h,g,2}(x,y) & = 1, \ (t^B)^0_{h,g,3}(x,y) = h^2(x), \ (t^B)^0_{h,g,4}(x,y) = h^4(x), \\
(t^B)^1_{h,g,4}(x,y) & = h^4(x) + 2h(x)g(y), \ (t^B)^0_{h,g,5}(x,y) = h^6(x), \\
(t^B)^1_{h,g,5}(x,y) & = h^6(x) + 4h^3(x)g(y) \text{ and } (t^B)^2_{h,g,5}(x,y) = h^6(x) + 4h^3(x)g(y) + g^2(y).
\end{align*}

Note that $(t^B)^{\lfloor \frac{2n-4}{3} \rfloor}_{h,g,n}(x,y) = (t^B)^{\lfloor \frac{2n-4}{3} \rfloor}_{h,g,n}(x,y)$. 

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For simplicity, we use \((tB)_{h,g,n}(x, y) = (tB)_{h,g,n}^l, (tB)_{h,g,n}(x, y) = (tB)_{h,g,n}\), 
\(h(x) = h\) and \(g(y) = g\).

Following table shows terms of incomplete generalized bivariate \(B\)-Tribonacci polynomials.

<table>
<thead>
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<th>3</th>
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<tr>
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<td>(h^2)</td>
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</tr>
<tr>
<td>4</td>
<td>(h^4)</td>
<td>(h^4 + 2hg)</td>
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</tr>
<tr>
<td>5</td>
<td>(h^6)</td>
<td>(h^6 + 4h^4g)</td>
<td>(h^6 + 4h^4g + g^2)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(h^8)</td>
<td>(h^8 + 6h^6g)</td>
<td>(h^8 + 6h^6g + 6h^2g^2)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(h^{10})</td>
<td>(h^{10} + 8h^8g)</td>
<td>(h^{10} + 8h^8g + 15h^6g^2 + 15h^4g^3 + 6h^2g^4)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(h^{12})</td>
<td>(h^{12} + 10h^{10}g)</td>
<td>(h^{12} + 10h^{10}g + 28h^8g^2 + 20h^6g^3)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>(h^{14})</td>
<td>(h^{14} + 12h^{12}g)</td>
<td>(h^{14} + 12h^{12}g + 45h^{10}g^2 + 56h^8g^3)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>(h^{16})</td>
<td>(h^{16} + 14h^{14}g)</td>
<td>(h^{16} + 14h^{14}g + 66h^{12}g^2 + 120h^{10}g^3)</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.3: Terms of \((tB)_{h,g,n}^l\) for \(0 \leq l \leq 3\), \(2 \leq n \leq 10\).

With \(g(y) = 1\), the identities of (5.27) can be seen in (P2).

Next, we prove the recurrence properties of polynomials \((tB)_{h,g,n}^l\).

**Theorem 5.4.2.** The recurrence relation of the incomplete generalized bivariate \(B\)-Tribonacci polynomials \((tB)_{h,g,n}^l\) is

\[
(tB)_{h,g,n+3}^{l+2} = h^2(tB)_{h,g,n+2}^{l+2} + 2hg(tB)_{h,g,n+1}^{l+1} + g^2(tB)_{h,g,n}^l, \tag{5.28}
\]

where \(0 \leq l \leq \left\lfloor \frac{2n-6}{3} \right\rfloor\) and \(n \geq 3\).

**Proof.** Consider, \(h^2(tB)_{h,g,n+2}^{l+2} + 2hg(tB)_{h,g,n+1}^{l+1} + g^2(tB)_{h,g,n}^l\)

\[
= \sum_{r=0}^{l+2} \frac{(2n-2r)!}{r!} h^{2n+2-3r}g^r + 2 \sum_{r=0}^{l+1} \frac{(2n-2-2r)!}{r!} h^{2n-1-3r}g^{r+1}
+ \sum_{r=0}^{l} \frac{(2n-4-2r)!}{r!} h^{2n-4-3r}g^{r+2}
\]

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\[= h^{2n+2} + \sum_{r=1}^{l+2} \left[ \frac{(2n-2r)\xi}{r!} + \frac{(2n-2r)^{-1}}{(r-1)!} \right] h^{2n+2-3r} g^r \]

\[+ h^{2n-1} + \sum_{r=1}^{l+1} \left[ \frac{(2n-2-2r)\xi}{r!} + \frac{(2n-2-2r)^{-1}}{(r-1)!} \right] h^{2n-1-3r} g^r \]

\[= h^{2n+2} + \sum_{r=1}^{l+2} \frac{(2n+1-2r)\xi}{r!} h^{2n+2-3r} g^r + \sum_{r=0}^{l+1} \frac{(2n-1-2r)\xi}{r!} h^{2n-1-3r} g^r \]

\[= h^{2n+2} + \sum_{r=1}^{l+1} \frac{(2n+1-2r)\xi}{r!} h^{2n+2-3r} g^r \]

\[= \sum_{r=0}^{l+2} \frac{(2n+2-2r)\xi}{r!} h^{2n+2-3r} g^r \]

\[= \binom{l+2}{n+3}. \quad \Box \]

**Remark 5.4.3.** Using (5.27), equation (5.28) can be rewritten in terms of non-homogeneous recurrence relation as

\[
\binom{l}{B}^{l+2}_{h.g,n+3} = h^2 \binom{l}{B}^{l}_{h,g,n+2} + 2hg \binom{l}{B}^{l+1}_{h,g,n+1} + g^2 \binom{l}{B}^{l+2}_{h,g,n} 
\]

\[- \left[ \frac{(2n-4-2l)\xi}{l!} h^{2n-4-3l} g^{l+2} + \left( 2 \frac{(2n-2-2l)\xi}{l!} + \frac{(2n-2-2l)^{-1}}{(l-1)!} \right) h^{2n-1-3l} g^{l+1} \right]. \quad (5.29)\]

**Theorem 5.4.4.** For \( s \geq 1, \)

\[
\sum_{i=0}^{2s} \frac{(2s)\xi}{i!} \binom{l}{B}^{l+i}_{h.g,n+i} h^i g^{2s-i} = \binom{l}{B}^{l+2s}_{h,g,n+3s}, \quad (5.30)\]

\[0 \leq l \leq \left\lfloor \frac{2n-2s-1}{3} \right\rfloor. \]

**Proof.** We prove (5.30) by mathematical induction on \( s. \)

Let \( s = 1. \) Then L.H.S. of (5.30) = \[ \sum_{i=0}^{2} \frac{2s}{i!} \binom{l}{B}^{l+i}_{h.g,n+i} h^i g^{2s-i} = \binom{l}{B}^{l+2}_{h,g,n+3} = R.H.S. \]

Thus, the theorem is true for \( s = 1. \) Assume that the result is true for all \( s \leq m. \)

Consider, \[ \sum_{i=0}^{2m+2} \frac{(2m+2)\xi}{i!} \binom{l}{B}^{l+i}_{h.g,n+i} h^i g^{2m+2-i} \]

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\[
= \sum_{i=0}^{2m+2} \left( \frac{(2m-2)^{i+2}}{(i+2)} \left( t B \right)_h g^{2m+2-i} \right) + 2 \frac{(2m-1)^{i+1}}{(i+1)!} \left( t B \right)_h g^{2m+2-i} + \left( t B \right)_h g^{2m+2-i} \\
= \sum_{i=0}^{2m} \left( \frac{(2m)^i}{i!} \left( t B \right)_h g^{2m-i} \right) + 2 \frac{(2m)^{i+1}}{(i+1)!} \left( t B \right)_h g^{2m-i+1} + \left( t B \right)_h g^{2m-i+2} \\
= h^2 (t B)^{l+2m+2}_{h,g,n+3m+2} + 2hg (t B)^{l+2m+1}_{h,g,n+3m+1} + g^2 (t B)^{l+2m}_{h,g,n+3m} \\
= (t B)^{l+2m+2}_{h,g,n+3m+3}.
\]

Hence the result is true for \( s = m + 1 \).

Thus, by mathematical induction the theorem is proved.

\[\square\]

**Theorem 5.4.5.** For \( n \geq \left\lceil \frac{3s+6}{2} \right\rceil \) and \( s \geq 1 \),

\[
\sum_{i=0}^{s-1} \left( 2h^{2s-1-2i}g + h^{2s-2-2i}g^2 \right) (t B)^{l+i}_{h,g,n+1+i} + h^{2s-2-2i}g^2 (t B)^{l}_{h,g,n+i} = (t B)^{l+2}_{h,g,n+2+s} - h^{2s} (t B)^{l+2}_{h,g,n+2}.
\]

\[(5.31)\]

**Proof.** By mathematical induction on \( s \).

Note that (5.28) implies, (5.31) holds for \( s = 1 \). Now let the result be true for \( s \leq m \).

We prove it for \( s = m + 1 \). Consider,

\[
\sum_{i=0}^{m} \left( 2h^{2m+1-2i}g + h^{2m-2i}g^2 \right) (t B)^{l+i}_{h,g,n+1+i} + h^{2m-2i}g^2 (t B)^{l}_{h,g,n+i}
\]

\[
= \sum_{i=0}^{m-1} \left( 2h^{2m+1-2i}g + h^{2m-2i}g^2 (t B)^{l+i}_{h,g,n+1+i} \right) + \left( 2hg (t B)^{l+i}_{h,g,n+1+m} + g^2 (t B)^{l}_{h,g,n+m} \right)
\]

\[
= h^2 \left( \sum_{i=0}^{m-1} \left( 2h^{2m+1-2i}g + h^{2m-2i}g^2 (t B)^{l+i}_{h,g,n+1+i} \right) \right)
\]

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\[ + \left( 2 \, hg \, (tB)_{h,g,n+1+m} + g^2 \, (tB)_{h,g,n+m} \right) \]

\[ = h^2 \left( (tB)_{h,g,n+2+m} - h^{2m} \, (tB)_{h,g,n+2} \right) \]

\[ + 2hg \, (tB)_{h,g,n+1+m} + g^2 \, (tB)_{h,g,n+m}, \text{ by induction assumption.} \]

\[ = h^2 (tB)_{h,g,n+2+m} - h^{2m+2} \, (tB)_{h,g,n+2} + 2hg \, (tB)_{h,g,n+1+m} + g^2 \, (tB)_{h,g,n+m} \]

\[ = (tB)_{h,g,n+3+m} - h^{2m+2} \, (tB)_{h,g,n+2}, \text{ from (5.28).} \]

Hence the theorem is proved. 

**Lemma 5.4.6.** For all \( n \geq 2 \),

\[ \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} r \frac{(2n-4-2r)!}{r!} h^{2n-4-3r} \, g^r \]

\[ = \frac{2n-4}{3} \, (tB)_{h,g,n} - \frac{h}{3} \sum_{i=0}^{n} \left( 2h (tB)_{h,g,n+1-i} + 2g (tB)_{h,g,n-i} \right) (tB)_{h,g,i}. \quad (5.32) \]

**Proof.** We use (5.7) to prove the result.

Consider, \((tB)_{h,g,n} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-2r)!}{r!} h^{2n-4-3r} \, g^r \), \( \forall n \geq 2 \).

Differentiating both sides with respect to \( x \), we get

\[ (tB)^{(1,0)}_{h,g,n} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-2r)!}{r!} (2n-4-3r) h^{2n-5-3r} \, g^r \]

Therefore, \((tB)^{(1,0)}_{h,g,n} = (2n-4) \, h^{(1,0)} \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-2r)!}{r!} \, h^{2n-4-3r} \, g^r \)

\[ - 3 \, h^{(1,0)} \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} r \frac{(2n-4-2r)!}{r!} h^{2n-4-3r} \, g^r. \]

Using Convolution property of \((tB)_{h,g,n}\), we get

\[ \left( h^{(1,0)} \sum_{i=0}^{n} \left( 2h (tB)_{h,g,n+1-i} + 2g (tB)_{h,g,n-i} \right) (tB)_{h,g,i} \right) h \]
Thus, 
\[ h \sum_{i=0}^{n} \left( 2h(t^i B)_{h,g,n+1-i} + 2g(t^i B)_{h,g,n-i} \right) (t^i B)_{h,g,i} \]

Therefore, 
\[ \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} r \frac{(2n-4-2r)^2}{r!} h^{2n-4-3r} g^r \]

Hence the lemma is proved. 

Theorem 5.4.7. For all \( n \geq 2 \), 
\[
\sum_{l=0}^{\lfloor \frac{2n-4}{3} \rfloor} (t^l B)_{h,g,n} \]

\[
= \left( \left[ \frac{2n-4}{3} \right] - \frac{2n-7}{3} \right) (t^i B)_{h,g,n} + \frac{h}{3} \sum_{i=0}^{n} \left( 2h(t^i B)_{h,g,n+1-i} + 2g(t^i B)_{h,g,n-i} \right) (t^i B)_{h,g,i}.
\]

\begin{equation}
\tag{5.33}
\end{equation}

Proof. 
\[
\sum_{l=0}^{\lfloor \frac{2n-4}{3} \rfloor} (t^l B)_{h,g,n} = (t^0 B)_{h,g,n} + (t^1 B)_{h,g,n} + \ldots + (t^\lfloor \frac{2n-4}{3} \rfloor B)_{h,g,n}
\]

\[
= \left( \frac{2n-4-2r)^2}{0!} h^{2n-4} + \left( \frac{2n-4)^2}{0!} h^{2n-4} + \frac{(2n-4-2)^2}{1!} h^{2n-4-3} g \right) + \ldots + \left[ (2n-4)^2 \right] h^{2n-4} + \frac{(2n-4)(2n-2)^2}{r!} h^{2n-4-3r} g^r \right] + \ldots + \left[ (2n-4)(2n-2)^2 \right] h^{2n-4-3r} g^r + \ldots
\]

\[
+ \left[ (2n-4-2)^2 \left( \frac{2n-4}{3} \right) \right] \left( \frac{2n-4}{3} \right) ! h^{2n-4-3r} g^{\lfloor \frac{2n-4}{3} \rfloor}
\]

\[
= \left( \left( \frac{2n-4}{3} \right) + 1 \right) \left( \frac{2n-4)!}{0!} h^{2n-4} + \left( \left( \frac{2n-4}{3} \right) \right) \left( \frac{2n-4-2)!}{1!} h^{2n-4-2} g + \ldots + \left( \frac{2n-4-2}{3} \right) \left( \frac{2n-4}{3} \right) ! h^{2n-4-3r} g^r + \ldots
\]

+ \left( \frac{2n-4}{3} + 1 - r \right) \left( \frac{2n-4-2r)!}{r!} h^{2n-4-3r} g^r + \ldots + \left( \frac{2n-4-2}{3} \right) \left( \frac{2n-4}{3} \right) ! h^{2n-4-3r} g^{\lfloor \frac{2n-4}{3} \rfloor}
\]

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\[
\left[\frac{2n-4}{3}\right] + 1 - \frac{(2n-4-2r)l}{r!} h^{2n-4-3r} g^r
\]

\[
= \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \right) \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} r \frac{(2n-4-2r)l}{r!} h^{2n-4-3r} g^r - \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} r \frac{(2n-4-2r)l}{r!} h^{2n-4-3r} g^r
\]

\[
= \left( \left\lfloor \frac{2n-4}{3} \right\rfloor + 1 \right) (tB)_{h,g,n} h^{2n-4-3r} g^r
\]

\[
+ \frac{h}{3} \sum_{i=0}^{n} \left( 2h \ (tB)_{h,g,n+1-i} + 2g(tB)_{h,g,n-i} \right) (tB)_{h,g,i}, \text{ by Lemma 5.4.6.}
\]

\[
= \left( \left\lfloor \frac{2n-4}{3} \right\rfloor - \frac{2n-7}{3} \right) (tB)_{h,g,n} + \frac{h}{3} \sum_{i=0}^{n} \left( 2h \ (tB)_{h,g,n+1-i} + 2g(tB)_{h,g,n-i} \right) (tB)_{h,g,i}. \quad \Box
\]

### 5.5 Incomplete generalized bivariate B-Tri Lucas polynomials

In this section, we introduce the incomplete generalized bivariate B-Tri Lucas polynomials and study some identities related to it. We also study its relation with the incomplete generalized bivariate B-Tri bonacci polynomials.

**Definition 5.5.1.** The incomplete generalized bivariate B-Tri Lucas polynomials are defined by

\[
(tL)^l_{h,g,n}(x, y)
\]

\[
= \sum_{r=0}^{l} \left( \frac{(2n-2)}{(2n-2-2r)} - \frac{(2n-4-2r)^{r-2}}{(r-2)!} \right) h^{2n-2-3r}(x) g^r(y), \quad (5.34)
\]

\[
\forall \ 0 \leq l \leq \left\lfloor \frac{2n-2}{3} \right\rfloor \text{ and } n \geq 2.
\]

Note that \((tL)^{\left\lfloor \frac{2n-2}{3} \right\rfloor}_{h,g,n} (x, y) = (tL)_{h,g,n}(x, y)\).
For simplicity, we use \((tL)^{\left\lfloor \frac{2n-2}{3} \right\rfloor}_{h,g,n}(x,y) = (tL)^{\left\lfloor \frac{2n-2}{3} \right\rfloor}_{h,g,n}, h(x) = h\) and \(g(y) = g\).

Following table shows terms of incomplete generalized bivariate \(B\)-Tri Lucas polynomials.

<table>
<thead>
<tr>
<th>(n \setminus l)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(h^2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(h^4)</td>
<td>(h^4 + 4hg)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(h^6)</td>
<td>(h^6 + 6h^3g + 2g^2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(h^8)</td>
<td>(h^8 + 8h^3g + 11h^2g^2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(h^{10})</td>
<td>(h^{10} + 10h^9g + 24h^3g^2 + 8hg^3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(h^{12})</td>
<td>(h^{12} + 12h^9g + 41h^6g^2 + 36h^3g^4)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4: Terms of \((tL)_{h,g,n}^l\), for \(0 \leq l \leq 3\) and \(0 \leq n \leq 7\).

Following theorems give the relation between incomplete generalized bivariate \(B\)-Trionacci and \(B\)-Tri Lucas polynomials.

**Theorem 5.5.2.**

\[ (tL)_{h,g,n}^l = (tB)_{h,g,n+1}^l + 2hg (tB)_{h,g,n-1}^{l-1} + g^2 (tB)_{h,g,n-2}^{l-2}, \]  
(5.35)

\(2 \leq l \leq \left\lfloor \frac{2n-2}{3} \right\rfloor\).

**Proof.** From (5.27), we have

\[
(tB)_{h,g,n+1}^l + 2hg (tB)_{h,g,n-1}^{l-1} + g^2 (tB)_{h,g,n-2}^{l-2}
\]

\[
= \sum_{r=0}^{l} \frac{(2n-2-2r)x}{r!} h^{2n-2-3r}g^r + 2hg \sum_{r=0}^{l-1} \frac{(2n-6-2r)x}{r!} h^{2n-6-3r}g^r
\]

\[+ g^2 \sum_{r=0}^{l-2} \frac{(2n-8-2r)x}{r!} h^{2n-8-3r}g^r \]

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\[
= \sum_{r=0}^{l} \left( \frac{(2n-2-2r)\xi}{r!} h^{2n-2-3r} g^r \right) + 2 \sum_{r=1}^{l} \left( \frac{(2n-2-2r)^{r-1}}{(r-1)!} h^{2n-2-3r} g^r \right)
\]

\[
= \sum_{r=0}^{l} \left[ \frac{(2n-2-2r)^2}{r!} + 2 \left( \frac{(2n-4-2r)^{r-1}}{(r-1)!} - \frac{(2n-4-2r)^{r-2}}{(r-2)!} \right) \right] h^{2n-2-3r} g^r
\]

\[
= \sum_{r=0}^{l} \left[ \frac{(2n-2-2r)^2}{r!} + 2 \left( \frac{(2n-3-2r)^{r-1}}{(r-1)!} - \frac{(2n-4-2r)^{r-2}}{(r-2)!} \right) \right] h^{2n-2-3r} g^r
\]

\[
= \sum_{r=0}^{l} \left[ \frac{2n-2}{2n-2-2r} \left( \frac{(2n-2-2r)^2}{r!} \right) - \frac{(2n-4-2r)^{r-2}}{(r-2)!} \right] h^{2n-2-3r} g^r
\]

\[
= \left( t^L \right)_{l+h,g,n}.
\]

Hence the theorem is proved. \(\square\)

Using (5.28) and (5.35), following Corollary can be proved.

**Corollary 5.5.3.**

\[
\left( t^L \right)_{l+h,g,n}^l = 2 \left( t^B \right)_{l+h,g,n+1}^l - h^2 \left( t^B \right)_{l+h,g,n}^l,
\]  \(5.36\)

\(0 \leq l \leq \left[ \frac{2n-2}{3} \right].\)

**Theorem 5.5.4.** *The recurrence relation of the incomplete generalized bivariate B-Tri Lucas sequence \(\left( t^L \right)_{l+h,g,n}^l\) is given by*

\[
\left( t^L \right)_{l+h,g,n+3}^{l+2} = h^2 \left( t^L \right)_{l+h,g,n+2}^{l+2} + 2hg \left( t^L \right)_{l+h,g,n+1}^{l+1} + g^2 \left( t^L \right)_{l+h,g,n}^l,
\]  \(5.37\)

\(0 \leq l \leq \left[ \frac{2n-2}{3} \right].\)

**Proof.** Equation (5.35) implies

\[
\left( t^L \right)_{l+h,g,n+3}^{l+2} = \left( t^B \right)_{l+h,g,n+4}^{l+2} + 2hg \left( t^B \right)_{l+h,g,n+2}^{l+1} + g^2 \left( t^B \right)_{l+h,g,n+1}^l
\]

\(117\)
\[ h^2(B)_{h,g,n+3} + 2hg(B)_{h,g,n+2} + g^2(B)_{h,g,n+1} \]
\[ + 2hg(h^2(B)_{h,g,n+1} + 2hg(B)_{h,g,n} + g^2(B)_{h,g,n-1}) \]
\[ + h^2(B)_{h,g,n} + 2hg(B)_{h,g,n-1} + g^2(B)_{h,g,n-2}, \text{ from (5.28)} \]
\[ = h^2(L)_{h,g,n+2} + 2hg(L)_{h,g,n+1} + g^2(L)_{h,g,n}, \text{ from (5.35)}. \]

**Theorem 5.5.5.** For \( s \geq 1 \),

\[ (tL)_{h,g,n+3s}^{l+2s} = \sum_{i=0}^{2s} \left( \frac{(2s)!}{i!} (tL)_{h,g,n+i}^{l+i} \right) h^i g^{2s-i}, \quad 0 \leq l \leq \left\lfloor \frac{2n-2-2s}{3} \right\rfloor. \] \hspace{1cm} (5.38)

**Proof.** Equation (5.36) implies,

\[ \sum_{i=0}^{2s} \left( \frac{(2s)!}{i!} (tL)_{h,g,n+i}^{l+i} \right) h^i g^{2s-i} \]
\[ = \sum_{i=0}^{2s} \left( \frac{(2s)!}{i!} (tL)_{h,g,n+1+i}^{l+i} h^i g^{2s-i} \right) \]
\[ = 2 \sum_{i=0}^{2s} \left( \frac{(2s)!}{i!} (tL)_{h,g,n+i}^{l+i} h^i g^{2s-i} \right) - h^2 (tL)_{h,g,n+i}^{l+i} h^i g^{2s-i}, \text{ from (5.10)}. \]
\[ = 2 \left( (tL)_{h,g,n+1+2s}^{l+2s} - h^2 (tL)_{h,g,n+2s}^{l+2s} \right) \]
\[ = (tL)_{h,g,n+2s}^{l+2s}. \]

Similarly, using (5.36) following theorem can be proved.

**Theorem 5.5.6.** For \( s \geq 1 \),

\[ (tL)_{h,g,n+2s}^{l+2s} - h^2 (tL)_{h,g,n+2s}^{l+2s} = \sum_{i=0}^{s-1} \left( 2 h^{2s-2-2i} (tL)_{h,g,n+i+1}^{l+1} + h^{2s-2-2i} (tL)_{h,g,n+i}^{l+1} \right), \quad (5.39) \]
\[ 0 \leq l \leq \left\lfloor \frac{2n-6}{3} \right\rfloor. \]
Lemma 5.5.7. For all \( n \geq 2 \),
\[
\sum_{r=0}^{\lfloor \frac{2n-2}{2} \rfloor} r \left( \frac{(2n-2)}{(2n-2-2r)} \frac{2n-2-2r}{r!} - \frac{(2n-4-2r)}{(r-2)!} \right) h^{2n-2-3r} g^r
\]
\[
= \frac{(2n-2)}{3} \left( \sum_{i=0}^{n} \left( 2h \left( tL \right)_{g,n+1-i} + 2g \left( tL \right)_{h,n-i} \right) (tB)_{h,g,i} - 2h \left( tB \right)_{h,g,n} \right).
\]

(5.40)

Proof. Equation (5.21) implies

\[
\left( tL \right)_{h,g,n} = \sum_{r=0}^{\lfloor \frac{2n-2}{2} \rfloor} \left( \frac{(2n-2)}{(2n-2-2r)} \frac{2n-2-2r}{r!} - r(r-1) \frac{(2n-4-2r)}{(r-2)!} \right) h^{2n-2-3r} g^r.
\]

Differentiating both sides with respect to \( x \).

\[
\left( tL \right)^{(1,0)}_{h,g,n}
\]

\[
= \sum_{r=0}^{\lfloor \frac{2n-2}{2} \rfloor} (2n-2-3r) \left( \frac{(2n-2)}{(2n-2-2r)} \frac{2n-2-2r}{r!} - \frac{(2n-4-2r)}{(r-2)!} \right) h^{2n-2-3r} g^r.
\]

This implies, \( h(tL)^{(1,0)}_{h,g,n} = (2n-2)h^{(1,0)}_h (tL)_{h,g,n} \)

\[
-3h^{(1,0)} \sum_{r=0}^{\lfloor \frac{2n-2}{2} \rfloor} r \left( \frac{(2n-2)}{(2n-2-2r)} \frac{2n-2-2r}{r!} - \frac{(2n-4-2r)}{(r-2)!} \right) h^{2n-2-3r} g^r.
\]

Thus, \( h^{(1,0)} \sum_{r=0}^{\lfloor \frac{2n-2}{2} \rfloor} r \left( \frac{(2n-2)}{(2n-2-2r)} \frac{2n-2-2r}{r!} - \frac{(2n-4-2r)}{(r-2)!} \right) h^{2n-2-3r} g^r

\[
= \frac{(2n-2)}{3} \left( tL \right)_{h,g,n} h^{(1,0)}_{h,g,n} - \frac{h}{3} \left( tL \right)^{(1,0)}_{h,g,n}.
\]

Therefore, \( \sum_{r=0}^{\lfloor \frac{2n-2}{2} \rfloor} r \left( \frac{(2n-2)}{(2n-2-2r)} \frac{2n-2-2r}{r!} - \frac{(2n-4-2r)}{(r-2)!} \right) h^{2n-2-3r} g^r

\[
= \frac{(2n-2)}{3} \left( tL \right)_{h,g,n} - \frac{h}{3} \left[ \sum_{i=0}^{n} \left( 2h \left( tL \right)_{h,g,n+1-i} + 2g \left( tL \right)_{h,n-i} \right) (tB)_{h,g,i} - 2h \left( tB \right)_{h,g,n} \right], \text{ from (5.26).}
\]

\[\square\]

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Theorem 5.5.8. For all $n \geq 2$,

$$
\begin{align*}
\sum_{l=0}^{2n-2} (\ell L)_{h,g,n}^l & = \left( \left\lfloor \frac{2n-2}{3} \right\rfloor - \frac{2n-5}{3} \right) (\ell L)_{h,g,n} \\
& + \frac{h}{3} \left[ \sum_{i=0}^{n} \left( 2h (\ell L)_{h,g,n+1-i} + 2g (\ell L)_{h,g,n-i} \right) (\ell B)_{h,g,i} - 2h (\ell B)_{h,g,n} \right].
\end{align*}
\tag{5.41}
$$

Proof. \(\sum_{l=0}^{2n-2} (\ell L)_{h,g,n}^l = (\ell L)_{h,g,n}^0 + (\ell L)_{h,g,n}^1 + \cdots + (\ell L)_{h,g,n}^{2n-2} \)

\begin{align*}
&= \left( \frac{(2n-2)(2n-2)!}{(2n-2)!} \right) h^{2n-2} g^0 + \left( \frac{(2n-2)(2n-2)!}{(2n-4)!} \right) h^{2n-2} g^0 + \left( \frac{(2n-2)(2n-4)!}{2!} \right) h^{2n-5} g^1 \\
&+ \left( \frac{(2n-2)(2n-2)!}{(2n-4)!} \right) h^{2n-2} g^0 + \left( \frac{(2n-2)(2n-4)!}{(2n-6)!} \right) h^{2n-5} g^1 + \left( \frac{(2n-2)(2n-6)!}{2!} \right) h^{2n-8} g^2 \\
&+ \cdots \\
&+ \left( \frac{(2n-2)(2n-2)!}{(2n-2)!} \right) h^{2n-2} g^0 + \left( \frac{(2n-2)(2n-4)!}{(2n-4)!} \right) h^{2n-5} g^1 + \left( \frac{(2n-2)(2n-6)!}{2!} \right) h^{2n-8} g^2 \\
&+ \cdots + \left( \frac{(2n-2)(2n-2)!}{(2n-2)!} \right) h^{2n-2} g^0 + \left( \frac{(2n-2)(2n-4)!}{(2n-6)!} \right) h^{2n-5} g^1 + \left( \frac{(2n-2)(2n-6)!}{2!} \right) h^{2n-8} g^2 \\
&+ \cdots + \left( \frac{(2n-2)(2n-2)!}{(2n-2)!} \right) h^{2n-2} g^0 + \left( \frac{(2n-2)(2n-4)!}{(2n-6)!} \right) h^{2n-5} g^1 + \left( \frac{(2n-2)(2n-6)!}{2!} \right) h^{2n-8} g^2 \\
&+ \cdots \\
&= \left( \left\lfloor \frac{2n-2}{3} \right\rfloor + 1 \right) \left( \frac{(2n-2)(2n-2)!}{(2n-2)!} \right) h^{2n-2} g^0 + \left( \left\lfloor \frac{2n-2}{3} \right\rfloor \right) \left( \frac{(2n-2)(2n-4)!}{(2n-4)!} \right) h^{2n-5} g^1 + \cdots \\
&+ \left( \left\lfloor \frac{2n-2}{3} \right\rfloor + 1 - r \right) \left( \frac{(2n-2)(2n-2)!}{(2n-2)!} \right) \left( \frac{(2n-2)(2n-4)!}{(2n-4)!} \right) h^{2n-5} g^1 + \cdots \\
&+ \left( \left\lfloor \frac{2n-2}{3} \right\rfloor + 1 - r \right) \left( \frac{(2n-2)(2n-2)!}{(2n-2)!} \right) \left( \frac{(2n-2)(2n-4)!}{(2n-4)!} \right) h^{2n-5} g^1 + \cdots
\end{align*}

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\[
+ \left( \frac{(2n-2)}{(2n-2)} r! \right) \left( \frac{2n-2}{(2n-2-2r)} \right) \frac{(2n-2-2r)}{r!} - \left( \frac{2n-4-2}{(2n-2-2r)} \right) \frac{(2n-2-2r)}{r!} h^{2n-2-3} \left[ \begin{array}{c} 2n-2 \end{array} \right] g^{\frac{2n-2}{3}}
\]

\[
= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor + \left( \frac{2n-2}{3} \right) + 1 - r} \left( \frac{(2n-2)}{(2n-2-2r)} \right) \left( \frac{2n-2-2r}{r!} \right) - \left( \frac{2n-4-2}{(2n-2-2r)} \right) \frac{(2n-2-2r)}{r!} h^{2n-2-3r} g^r
\]

\[
= \left( \frac{2n-2}{3} \right) + 1 - \frac{(2n-2)}{3} \right) (tL)_{h,g,n}
\]

\[
+ \frac{h}{3} \left[ \sum_{i=0}^{n} \left( 2h \left( ^{t}L \right)_{h,g,n+i+1} + 2g \left( ^{t}L \right)_{h,g,n-i} \right) \right] (tB)_{h,g,i} - 2h \left( ^{t}B \right)_{h,g,n}, \text{ from (5.40)}
\]

\[
= \left( \frac{2n-2}{3} \right) - \frac{2n-5}{3} \right) (tL)_{h,g,n}
\]

\[
+ \frac{h}{3} \left[ \sum_{i=0}^{n} \left( 2h \left( ^{t}L \right)_{h,g,n+i+1} + 2g \left( ^{t}L \right)_{h,g,n-i} \right) \right] (tB)_{h,g,i} - 2h \left( ^{t}B \right)_{h,g,n}
\]

\[
\square
\]

Let \(( ^{t}B \)_{h,g,n})^{(k,j)} = \frac{\partial^{k+j}}{\partial x^{k} \partial y^{j}} \left( ^{t}B \right)_{h,g,n}\) and \(( ^{t}L \)_{h,g,n})^{(k,j)} = \frac{\partial^{k+j}}{\partial x^{k} \partial y^{j}} \left( ^{t}L \right)_{h,g,n}.

We have the following identities involving \(( ^{t}B \)_{h,g,n})^{(k,j)} and \(( ^{t}L \)_{h,g,n})^{(k,j)}.

**Theorem 5.5.9.** For \( n \geq 2, \)

1. \(( ^{t}L \)_{h,g,n})^{(k,j)} = \left( ^{t}B \right)_{h,g,n+1}^{(k,j)}

\[+ \sum_{r=1}^{2n} \frac{2n}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k-s}{s!} \frac{j}{i!} (h^{2-r})^{(s,0)} g^{r} \left( ^{t}B \right)_{h,g,n-r}^{(k-s,j-i)}.\]

2. \(( ^{t}B \)_{h,g,n})^{(k,j)}

\[= \sum_{r=0}^{2n} \frac{2n}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k-s}{s!} \frac{j}{i!} (h^{2-r})^{(s,0)} g^{r} \left( ^{t}B \right)_{h,g,n-1-r}^{(k-s,j-i)}.\]

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(3) \[
\left(\left\langle L^{l}_{h,g,n}\right\rangle^{(k,j)}\right) = \sum_{r=0}^{2} \frac{2r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^s}{s!} \frac{j^i}{i!} \left(h^{2-r}\right)^{(s,0)} \left(g^{r}\right)^{(0,i)} \left(\left\langle L^{l-r}_{h,g,n-1}\right\rangle\right)^{(k-s,j-i)}.
\]

(4) \[
\sum_{s=0}^{k} \frac{k^s}{s!} \left(\left\langle B^{l+1}_{h,g,n+1}\right\rangle\right)^{(k-s,j+1)} f^{(s+1,0)} = \sum_{i=0}^{j} \frac{j^i}{i!} \left(\left\langle B^{l}_{h,g,n}\right\rangle\right)^{(k+1,j-i)} g^{(0,i+1)}.
\]

Proof. (1) Consider,

\[
\left\langle L^{l}_{h,g,n}\right\rangle = \left\langle B^{l}_{h,g,n+1}\right\rangle + 2hg\left\langle B^{l-1}_{h,g,n-1}\right\rangle + g^2\left\langle B^{l-2}_{h,g,n-2}\right\rangle.
\]

Differentiating both sides \(k\) times with respect to \(x\) and \(j\) times with respect to \(y\) and using Leibnitz theorem for derivatives, we get

\[
\left(\left\langle L^{l}_{h,g,n}\right\rangle\right)^{(k,j)} = \left(\left\langle B^{l}_{h,g,n+1}\right\rangle\right)^{(k,j)} + 2 \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^s}{s!} \frac{j^i}{i!} h^{(s,0)} g^{(0,i)} \left(\left\langle B^{l-1}_{h,g,n-1}\right\rangle\right)^{(k-s,j-i)}
\]

\[
+ \sum_{i=0}^{j} \frac{j^i}{i!} \left(g^2\right)^{(0,i)} \left(\left\langle B^{l-2}_{h,g,n-2}\right\rangle\right)^{(k,j-i)}
\]

\[
= \left(\left\langle B^{l}_{h,g,n+1}\right\rangle\right)^{(k,j)}
\]

\[
+ \sum_{r=1}^{2} \frac{2r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^s}{s!} \frac{j^i}{i!} \left(h^{2-r}\right)^{(s,0)} \left(g^{r}\right)^{(0,i)} \left(\left\langle B^{l-r}_{h,g,n-r}\right\rangle\right)^{(k-s,j-i)}.
\]

(2) From (5.28), we have

\[
\left\langle B^{l+2}_{h,g,n}\right\rangle = h^2\left\langle B^{l+2}_{h,g,n-1}\right\rangle + 2hg\left\langle B^{l+1}_{h,g,n-1}\right\rangle + g^2\left\langle B^{l}_{h,g,n-2}\right\rangle.
\]

Differentiating both sides \(k\) times with respect to \(x\) and \(j\) times with respect to \(y\) and using Leibnitz theorem for derivatives, we get

\[
\left(\left\langle B^{l}_{h,g,n}\right\rangle\right)^{(k,j)} = \sum_{s=0}^{k} \frac{k^s}{s!} h^{(s,0)} \left(\left\langle B^{l}_{h,g,n-1}\right\rangle\right)^{(k-s,j)}
\]

\[
+ 2 \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^s}{s!} \frac{j^i}{i!} h^{(s,0)} g^{(0,i)} \left(\left\langle B^{l}_{h,g,n-2}\right\rangle\right)^{(k-s,j-i)}
\]
\[ \sum_{i=0}^{j} \frac{k^2}{i!} (g^2)^{(0,i)} \left( (t^l)_{h,g,n-3} \right)^{(k,j-i)} \]

\[ = \sum_{r=0}^{2} \frac{2r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^2}{r!} \left( h^{2-r} \right)^{(s,0)} (g^r)^{(0,i)} \left( (t^l)_{h,g,n-1-r} \right)^{(k-s,j-i)}. \]

(3) Identity (3) can be proved by differentiating (5.37), \( k \) times with respect to \( x \) and \( j \) times with respect to \( y \) and using Leibnitz theorem for derivatives.

(4) Differentiating (5.27) with respect to \( x \), we get

\[ ((t^l)_{h,g,n})^{(1,0)} = \sum_{r=0}^{l} \frac{(2n - 4 - 2r)^{r+1}}{r!} h^{2n-5-3r} h^{(1,0)} g^r. \]

Also, (5.27) implies,

\[ (t^l)_{h,g,n+1} = \sum_{r=0}^{l+1} \frac{(2n - 2 - 2r)^{r}}{r!} h^{2n-2-3r} g^r. \]

Differentiating both sides with respect to \( y \),

\[ ((t^l)_{h,g,n+1})^{(0,1)} = \sum_{r=0}^{l+1} \frac{(2n - 2 - 2r)^{r}}{r!} h^{2n-2-3r} r g^{-1} g^{(0,1)} \]

\[ = \sum_{r=1}^{l+1} \frac{(2n - 2 - 2r)^{r}}{r!} h^{2n-2-3r} r g^{-1} g^{(0,1)} \]

\[ = \sum_{r=0}^{l} \frac{(2n - 4 - 2r)^{r+1}}{r!} h^{2n-5-3r} g^r g^{(0,1)} \]

Therefore, \( ((t^l)_{h,g,n+1})^{(0,1)} h^{(1,0)} = (t^l)_{h,g,n}^{(1,0)} g^{(0,1)}. \)

Differentiating \( k \) times both sides with respect to \( x \) and \( j \) times with respect to \( y \) and using Leibnitz theorem, we get

\[ \sum_{s=0}^{k} \frac{k^2}{s!} ((t^l)_{h,g,n+1})^{(k-s,j+1)} h^{(s+1,0)} = \sum_{i=0}^{j} \frac{j^2}{i!} ((t^l)_{h,g,n})^{(k+1,j-i)} g^{(0,i+1)}. \]

\[ \square \]


5.6 Generalized bivariate $B$-$q$ bonacci polynomials

In this section, we extend generalized bivariate $B$-Tribonacci polynomials to generalized $B$-$q$ bonacci polynomials and state its identities. These identities are similar to the identities of $B$-$q$ bonacci sequence defined by (4.1), studied in Section 2 of Chapter 4. Hence the proof of these results is omitted.

Definition 5.6.1. Let $n \in \mathbb{N} \cup \{0\}$. The generalized $B$-$q$ bonacci polynomials $(^q \text{B})_{h,g,n}(x, y)$, are defined by

$$(^q \text{B})_{h,g,n+q-1}(x, y) = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r}(x) g^r(y) (^q \text{B})_{h,g,n+q-r}(x, y), \forall n \geq 1,$$

with $(^q \text{B})_{h,g,1}(x, y) = 0$, $i = 0, 1, 2, 3, \cdots q - 2$, and $(^q \text{B})_{h,g,q-1}(x, y) = 1$,

where $(^q \text{B})_{h,g,n}(x, y)$ is $n^{th}$ $B$-$q$ bonacci polynomial.

Few terms of (5.42) are $(^q \text{B})_{h,g,q}(x, y) = h^{q-1}(x)$

$(^q \text{B})_{h,g,q+1}(x, y) = h^2(q-1)(x) + (q - 1)h^{q-2}(x) g(y),$

$(^q \text{B})_{h,g,q+2}(x, y) = h^3(q-1)(x) + \frac{(2(q-1))!}{1!} h^{q-3}(x) g(y) + \frac{(q-1)!^2}{2!} h^{q-3}(x) g(y),$

$(^q \text{B})_{h,g,q+3}(x, y) = h^4(q-1)(x) + \frac{(3(q-1))!}{1!} h^{q-4}(x) g(y) + \frac{(2(q-1))!^2}{2!} h^{q-4}(x) g(y)$

$\quad + \frac{(q-1)!^2}{3!} h^{q-4}(y) g(y),$

$(^q \text{B})_{h,g,q+4}(x, y) = h^5(q-1)(x) + \frac{(4(q-1))!}{1!} h^{q-5}(x) g(y) + \frac{(3(q-1))!^2}{2!} h^{q-5}(x) g(y)$

$\quad + \frac{(2(q-1))!^2}{3!} h^{q-5}(x) g(y) + \frac{(q-1)!^2}{4!} h^{q-5}(x) g(y).$

For simplicity, we write $(^q \text{B})_{h,g,n}(x, y) = (^q \text{B})_{h,g,n}$ and $h(x) = h$ and $g(y) = g$.

We have following results for $(^q \text{B})_{h,g,n}$.

Theorem 5.6.2. The $n^{th}$ term of (5.42) is given by

$$(^q \text{B})_{h,g,n} = \frac{\sum_{k=1}^{q} (-1)^{k+1} \prod_{1 \leq i < j \leq q, i \neq k} (\phi_i - \phi_j) \phi_k^n}{\prod_{1 \leq i < j \leq q} (\phi_i - \phi_j)}, \quad (5.43)$$

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where \( \phi_p, p = 1, 2, \cdots, q \) are \( q \) distinct roots of characteristic equation corresponding to (5.42).

Equation (5.43) is called Binet type formula for (5.42).

**Theorem 5.6.3.** The \( n^{th} \) term \((qB)_{h,g,n}\) of (5.42) is given by

\[
(qB)_{h,g,n} = \sum_{r=0}^{(q-1)(n-(q-1))} \binom{(q-1)(n-(q-1))}{r} h^{(q-1)(n-(q-1))-r} g^r,
\]

for all \( n \geq q - 1 \).

**Theorem 5.6.4.** The sum of the first \( n + 1 \) terms of (5.42) is given by

\[
\sum_{r=0}^{n} (qB)_{h,g,r} = \frac{(qB)_{h,g,n+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} g^r (qB)_{h,g,n-i} - 1}{(h + g)^{q-1} - 1},
\]

provided \( \begin{cases} h + g \neq 1, & \text{if } q \text{ is even;} \\ h + g \neq \pm 1, & \text{if } q \text{ is odd.} \end{cases} \)

**Theorem 5.6.5.** The generating function for (5.42) is given by

\[
(qG(qB))_{h,g}(z) = \frac{1}{1 - z(h + gz)^{q-1}}
\]

The next two theorems are related to the recurrence properties of \((qB)_{h,g,n}\). Proof of these theorems is similar to the proof of Theorem 5.2.6 and Theorem 5.2.7 respectively.

**Theorem 5.6.6.** For all \( s \geq 1 \),

\[
(qB)_{h,g,n+qs} = \sum_{i=0}^{(q-1)s} \binom{(q-1)s}{i} \frac{((q-1)s)^i}{i!} (qB)_{h,g,n+i} h^i g^{(q-1)s-i}.
\]
Theorem 5.6.7. For all $s \geq 1$ and $q \geq 2$,

$$(qB)_{h,g,n+(q-1)+s} - h^{(q-1)s}(qB)_{h,g,n+q-1} = \sum_{i=0}^{s} \sum_{j=1}^{q-1} \frac{(q-1)^{i}}{j!} g^{j} h^{(q-1)(s-i)-j}(qB)_{h,g,n+(q-1)+i-j}.$$  

(5.48)

We prove below the results related to first order partial derivative of $(qB)_{h,g,n}$ with respect to $x$ and $y$.

Theorem 5.6.8. For all $n \geq 0$,

1. $qq \frac{\partial}{\partial x}[(qB)_{h,g,n}] + h \frac{\partial}{\partial x}[(qB)_{h,g,n+1}] = (q-1)(n - (q-2))(qB)_{h,g,n+1} h^{(1,0)}$.
2. $g^{(0,1)} \frac{\partial}{\partial x}[(qB)_{h,g,n}] = h^{(1,0)} \frac{\partial}{\partial y}[(qB)_{h,g,n+1}]$.
3. $qq \frac{\partial}{\partial y}[(qB)_{h,g,n}] + h \frac{\partial}{\partial y}[(qB)_{h,g,n+1}] = (q-1)(n - (q-1))(qB)_{h,g,n} g^{(0,1)}$.
4. $qq \frac{\partial}{\partial x}[(qB)_{h,g,n}] h^{(1,0)} + h \frac{\partial}{\partial x}[(qB)_{h,g,n}] g^{(0,1)} = (q-1)(n-(q-1))(qB)_{h,g,n} h^{(1,0)} g^{(0,1)}$.

Proof. (1) Note that for $0 \leq n \leq q - 2$, L.H.S. = 0 = R.H.S.

Now let $n \geq q - 1$ and take $n = qm$. Using (5.44) and L.H.S. of (1), we have

$$qq \frac{\partial}{\partial x}[(qB)_{h,g,qm}] + h \frac{\partial}{\partial x}[(qB)_{h,g,qm+1}] = [q-1](qm - (q-2)) h^{(q-1)(qm-(q-2))} + \sum_{i=1}^{q-1} (q-1)m-(q-2) \left[ qr \frac{((q-1)(qm-(q-2)-r))^{i}}{i!} \right]$$

$$+ \sum_{r=1}^{q-1} (q-1)(qm-(q-2)-r) h^{(q-1)(qm-(q-2)-r)-r} g^{r} h^{(1,0)}$$

$$= [q-1](qm - (q-2)) h^{(q-1)(qm-(q-2))} + \sum_{r=1}^{q-1} (q-1)m-(q-2) \frac{((q-1)(qm-(q-2)-r))^{i}}{(q-1)!}$$

$$\cdot [qr + (q-1)(qm - (q-2) - r) - r] h^{(q-1)(qm-(q-2)-r)-r} g^{r} h^{(1,0)}$$

$$= (q-1)(qm - (q-2))$$

$$\sum_{r=0}^{q-1} (q-1)m-(q-2) \left[ (q-1)(qm-(q-2)-r))^{i} \right] h^{(q-1)(qm-(q-2)-r)-r} g^{r} h^{(1,0)}$$

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\[= (q - 1)(qm - (q - 2))(\text{q}_B)_{h,g,qm+1}h^{(1,0)}.
\]

Therefore, the result is true for \(n = qm\).

Similarly, the result can be proved for \(n = qm + 1, \cdots, qm + q - 1\). Hence (1) is proved.

Identity (2) can be verified by differentiating \((\text{q}_B)_{h,g,n}\) and \((\text{q}_B)_{h,g,n+1}\) respectively with respect to \(x\) and with respect to \(y\). Identity (3) can be proved using Identities (1) and (2). Identity (4) can be deduced from (2) and (3).

\[\square\]

### 5.7 Generalized bivariate \(B\)-\(q\) Lucas polynomials

In this section, we define generalized bivariate \(B\)-\(q\) Lucas polynomials and obtain some identities related to these polynomials.

**Definition 5.7.1.** Let \(n \in \mathbb{N} \cup \{0\}\). The generalized bivariate \(B\)-\(q\) Lucas polynomials \((\text{q}_L)_{h,g,n}(x, y)\) are defined by

\[
(\text{q}_L)_{h,g,n+q-1}(x, y) = \sum_{r=0}^{q-1} \frac{(q - 1)^r}{r!} h^{q-1-r}(x) g^r(y) (\text{q}_L)_{h,g,n+q-2-r}(x, y), \text{ for all } n \geq 1,
\]

\[(5.49)\]

with \((\text{q}_L)_{h,g,i}(x, y) = 0, i = 0, 1, 2, 3, \cdots q - 3, (\forall q \geq 3), (\text{q}_L)_{h,g,q-2}(x, y) = 2\) and \((\text{q}_L)_{h,g,q-1}(x, y) = h^{q-1}(x)\), where \((\text{q}_L)_{h,g,n}(x, y)\) is \(n^{th}\) \(B\)-\(q\) Lucas polynomial.

For \(q \geq 2\) and \(q - 1 \leq n \leq q+1\), the terms of (5.49) are \((\text{q}_L)_{h,g,q-1}(x, y) = h^{q-1}(x)\),

\((\text{q}_L)_{h,g,q}(x, y) = h^{2(q-1)}(x) + 2(q - 1)h^{q-2}(x) g(y)\) and

\((\text{q}_L)_{h,g,q+1}(x, y) = h^{3(q-1)}(x) + 3(q - 1)h^{2q-3}(x) g(y) + (q - 1)(q - 2)h^{q-3}(x)g^2(y)\).

For simplicity, we use \((\text{q}_L)_{h,g,n}(x, y) = (\text{q}_L)_{h,g,n}\) and \(h(x) = h\) and \(g(y) = g\).

We state below identities related to \((\text{q}_L)_{h,g,n}\).
Theorem 5.7.2. The $n^{th}$ term of (5.49) is given by

$$(qL)_{h,g,n} = \sum_{k=1}^{q}(-1)^{k+1} \prod_{1 \leq i,j \leq q,i \neq j}(\phi_i - \phi_j)\phi_k^{n}(2\phi_k - h^{q-1}),$$  \hspace{1cm} (5.50)

where $\phi_p, p = 1, 2, \ldots, q$ are $q$ distinct roots of characteristic equation corresponding to (5.49).

Equation (5.50) is called a Binet type formula for (5.49).

Theorem 5.7.3. The $n^{th}$ term $(qL)_{h,g,n}$ of (5.49) is given by

$$(qL)_{h,g,n} = \sum_{r=0}^{(q-1)(n-(q-2))} \left[ \frac{(q-1)(n-(q-2))}{(q-1)(n-(q-2)-r)} \frac{(q-1)(n-(q-2)-r)}{r!} \right] \phi^{(q-1)(n-(q-2))-qr}g^{r}$$

$- \sum_{r=2}^{(q-1)(n-(q-2))} \left[ \sum_{s=1}^{q-1} \frac{(q-1)(n-(q-1)-r)+s-2}{(r-2)!} \right] \phi^{(q-1)(n-(q-2))-qr}g^{r},$$

for all $n \geq q - 1$.

Theorem 5.7.4. The sum of the first $n+1$ terms of (5.49)

$$\sum_{r=0}^{n} (qL)_{h,g,r} = \frac{(qL)_{h,g,n+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)r}{r!} \phi^{r-1-r}g^{r} (qL)_{h,g,n-i} + (qL)_{h,g,q-1} - (qL)_{h,g,q-2}}{(h+g)q-1 - 1},$$  \hspace{1cm} (5.52)

provided \( \begin{cases} h + g \neq 1, & \text{if } q \text{ is even;} \\ h + g \neq \pm 1, & \text{if } q \text{ is odd.} \end{cases} \)

Theorem 5.7.5. The generating function for (5.49) is given by

$$(qG(L))_{h,g}(z) = \frac{2 - h^{q-1}z}{1 - z(h + gz)^{q-1}}.$$  \hspace{1cm} (5.53)

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Following theorem gives the relation between bivariate $B$-{$q$}bonacci and $B$-{$q$}Lucas polynomials.

**Theorem 5.7.6.** For all $n \geq q - 1$,

$$(q^g)_{h,g,n} = (q^B)_{h,g,n+1} + \sum_{r=1}^{q-1} \frac{(q-1)_r}{r!} h^{q-1-r} g^r \ (q^B)_{h,g,n-r}.$$  \hspace{1cm} (5.54)

**Proof.** We prove the theorem by mathematical induction on $n$.

Note that (5.54) is true for $n = q - 1$. Assume now that the result is true for $n \leq m$.

Consider, $$(q^g)_{h,g,m+1} = \sum_{r=0}^{q-1} \frac{(q-1)_r}{r!} h^{q-1-r} g^r \ (q^g)_{h,g,m-r}$$

$$= \sum_{r=0}^{q-1} \frac{(q-1)_r}{r!} h^{q-1-r} g^r \left[ (q^B)_{h,g,m+1-r} + \sum_{s=1}^{q-1} \frac{(q-1)_s}{s!} h^{q-1-s} g^s \ (q^g)_{h,g,m-r-s} \right]$$

$$= (q^B)_{h,g,m+2} + \sum_{s=1}^{q-1} \frac{(q-1)_s}{s!} h^{q-1-s} g^s \ (q^B)_{h,g,m+1-s}.$$ 

Hence the result follows. \hfill \Box

Following result follows immediately.

**Corollary 5.7.7.** For all $n \geq q - 2$,

$$(q^g)_{h,g,n} = 2 (q^B)_{h,g,n+1} - h^{q-1} (q^B)_{h,g,n}.$$  \hspace{1cm} (5.55)

**Proof.** Note that $2 (q^B)_{h,g,q-1} - h^{q-1} (q^B)_{h,g,q-2} = 2 = (q^g)_{h,g,q-2}$.

Hence equation (5.55) is true for $n = q - 2$.

For $n \geq q - 1$, the result can be proved using equations (5.42) and (5.54). \hfill \Box

Next two theorems are related to the recurrence properties of $(q^g)_{h,g,n}$.

**Theorem 5.7.8.** For all $s \geq 1$,

$$(q^g)_{h,g,n+qs} = \sum_{i=0}^{(q-1)s} \frac{(q-1)_s}{s!} \left( \frac{(q-1)s}{i} \right) (q^g)_{h,g,n+i} h^{i} g^{(q-1)s-i}.$$  \hspace{1cm} (5.56)

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Theorem 5.7.9. For all \(s \geq 1\) and \(q \geq 2\),

\[
(qL)_{h,g,n+(q-1)+s} - h^{(q-1)s}(qL)_{h,g,n+q-1} = \sum_{i=0}^{s-1} \sum_{j=1}^{q-1} \frac{(q-1)^i}{j!} g^jh^{(q-1)(s-i)-j}(qL)_{h,g,n+i+j}.
\]

(5.57)

Next, we prove the identities related to first order partial derivatives of \((qL)_{h,g,n}\) with respect to \(x\) and \(y\).

Theorem 5.7.10. For all \(n \geq 0\),

1. \(qg \frac{\partial}{\partial x}[(qL)_{h,g,n}] + h \frac{\partial}{\partial x}[(qL)_{h,g,n+1}] = h^{(1,0)}((q-1)(n-(q-3))(qL)_{h,g,n+1} - q(q-1)h^{q-2}g(qB)_{h,g,n}).\)

2. \(g^{(0,1)} \frac{\partial}{\partial x}[(qL)_{h,g,n}] = h^{(1,0)}(\frac{\partial}{\partial y}[(qL)_{h,g,n+1}] - (q-1)h^{q-2}(qB)_{h,g,n}).\)

3. \(qg \frac{\partial}{\partial y}[(qL)_{h,g,n}] + h \frac{\partial}{\partial y}[(qL)_{h,g,n+1}] = g^{(0,1)}((q-1)(n-(q-1))(qL)_{h,g,n} + 2(q-1)(qB)_{h,g,n+1}).\)

4. \(qg \frac{\partial}{\partial y}[(qL)_{h,g,n}] h^{(1,0)} + h \frac{\partial}{\partial x}[(qL)_{h,g,n}]g^{(0,1)} = (q-1)(n-(q-2))(qL)_{h,g,n}h^{(1,0)}g^{(0,1)}.\)

Proof. Equation (5.55) implies,

\((qL)_{h,g,n} = 2(qB)_{h,g,n+1} - h^{(q-1)}(qB)_{h,g,n} \).

Differentiating both sides with respect to \(x\), we get

\[
\frac{\partial}{\partial x}[(qL)_{h,g,n}] = 2 \frac{\partial}{\partial x}[(qB)_{h,g,n+1}] - h^{(q-1)} \frac{\partial}{\partial x}[(qB)_{h,g,n}] - (q-1)h^{q-2}(qB)_{h,g,n}
\]

Also, \(\frac{\partial}{\partial x}[(qL)_{h,g,n+1}] = 2 \frac{\partial}{\partial x}[(qB)_{h,g,n+2}] - h^{(q-1)} \frac{\partial}{\partial x}[(qB)_{h,g,n+1}] - (q-1)h^{q-2}(qB)_{h,g,n+1}.\)

Thus, using (1) of Theorem 5.6.8, we get
\[
qy \frac{\partial}{\partial x}[(qL)_{h, g, n}] + x \frac{\partial}{\partial x}[(qL)_{h, g, n+1}]
\]

\[
= h^{(1, 0)}(2(q - 1)(n - (q - 3))(qL)_{h, g, n+2} - h^{q-1}(q - 1)(n - (q - 3))(qB)_{h, g, n+1}
\]

\[
= h^{q-1}(q - 1)(qB)_{h, g, n+1} - (q - 1)h^{q-2}(qB)_{h, g, n} + h^{(1, 0)}(q - 1)(n - (q - 3)(qL)_{h, g, n+1} - q(q - 1)h^{q-2}(qB)_{h, g, n}).
\]

Similarly, other identities can be proved. \(\square\)

We now prove some identities involving \(k^{th}\) order partial derivative with respect to \(x\) and \(j^{th}\) order partial derivative with respect to \(y\) of bivariate polynomials \((qB)_{h, g, n}\) and \((qL)_{h, g, n}\) respectively, where \(k, j \geq 0\). Let \((qB)_{h, g, n}^{(k,j)}\) and \((qL)_{h, g, n}^{(k,j)}\) denote the \(k^{th}\) order partial derivative with respect to \(x\) and \(j^{th}\) order partial derivative with respect to \(y\) of \((qB)_{h, g, n}\) and \((qL)_{h, g, n}\) respectively. Let \((\cdot)^{(s, 0)}\) denote the \(s^{th}\) order derivative of \((\cdot)\) with respect to \(x\) and \((\cdot)^{(0, p)}\) denote the \(p^{th}\) order derivative of \((\cdot)\) with respect to \(y\). We have following identities.

**Theorem 5.7.11.** For all \(n \geq q - 1\),

1. \((qL)_{h, g, n}^{(k,j)} = (qB)_{h, g, n+1}^{(k,j)}
\]

\[
+ \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1} \sum_{p=0}^r \frac{k^p s!}{p!} (h^{q-1-r})_{s,0} (g^r)^{(0,p)} (qB)_{h, g, n-r}^{(k-s, j-p)}.
\]

2. \((qB)_{h, g, n}^{(k,j)} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1} \sum_{p=0}^r \frac{k^p s!}{p!} (h^{q-1-r})_{s,0} (g^r)^{(0,p)} (qB)_{h, g, n-1-r}^{(k-s, j-p)}.
\]

3. \((qL)_{h, g, n}^{(k,j)} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1} \sum_{p=0}^r \frac{k^p s!}{p!} (h^{q-1-r})_{s,0} (g^r)^{(0,p)} (qL)_{h, g, n-1-r}^{(k-s, j-p)}.
\]

**Proof.**

1. Note that \((qL)_{h, g, n} = (qB)_{h, g, n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} g^r (qB)_{h, g, n-r}.
\]

Differentiating both sides \(k\) times with respect to \(x\) and \(j\) times with respect to \(y\) and using Leibnitz theorem for derivatives, we get

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\((qL)^{(k,j)}_{h,g,n} = (qB)^{(k,j)}_{h,g,n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \frac{\partial^k}{\partial x^k} \left( h^{q-1-r} \sum_{p=0}^{r} \frac{r!}{p!} (g^r)^{(0,p)} (qB)^{(0,j-p)}_{h,g,n-r} \right) \)

\[= (qB)^{(k,j)}_{h,g,n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \frac{k^s}{s!} \left( h^{q-1-r} \right)^{(s,0)} \sum_{p=0}^{r} \frac{r!}{p!} (g^r)^{(0,p)} (qB)^{(k-s,j-p)}_{h,g,n-r} \]

\[= (qB)^{(k,j)}_{h,g,n+1} \]

\[+ \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^s}{s!} \frac{r!}{p!} \left( h^{q-1-r} \right)^{(s,0)} (g^r)^{(0,p)} (qB)^{(k-s,j-p)}_{h,g,n-r}. \]

Hence (1) is proved.

(2) We have from (5.42), \((qB)_{h,g,n} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} g^r (qB)_{h,g,n-1-r}. \)

Differentiating both sides \(k\) times with respect to \(x\) and \(j\) times with respect to \(y\) and using Leibnitz theorem for derivatives, we get

\[(qB)^{(k,j)}_{h,g,n} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \frac{\partial^k}{\partial x^k} \left( h^{q-1-r} \sum_{p=0}^{r} \frac{r!}{p!} (g^r)^{(0,p)} (qB)^{(0,j-p)}_{h,g,n-1-r} \right) \]

\[= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \frac{k^s}{s!} \left( h^{q-1-r} \right)^{(s,0)} \sum_{p=0}^{r} \frac{r!}{p!} (g^r)^{(0,p)} (qB)^{(k-s,j-p)}_{h,g,n-1-r} \]

\[= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^s}{s!} \frac{r!}{p!} \left( h^{q-1-r} \right)^{(s,0)} (g^r)^{(0,p)} (qB)^{(k-s,j-p)}_{h,g,n-1-r}. \]

Hence (2) is proved. Similarly, we can prove the identity (3). □

**Remark 5.7.12.** Using Leibnitz theorem for derivatives we can established similar type of identities using identities in Theorem 5.6.8 and Theorem 5.7.10.

**Theorem 5.7.13.** *(Convolution property for \((qB)_{h,g,n}\))*

\[(qB)^{(1,0)}_{h,g,n} = (q - 1) h^{(1,0)} \left( \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{q-2-r} g^r \sum_{i=0}^{n+q-2-r} (qB)_{h,g,i} (qB)_{h,g,n+q-2-r-i} \right). \]

\[(5.58)\]
Proof. Equation (5.46) implies,

$$(^qG_{(B)})(z) = \frac{1}{1 - z (h + gz)^{-1}}.$$  

Therefore,

$$\sum_{n=0}^\infty (^qB)_{h,g,n} z^{n-(q-1)} = \frac{1}{1 - z(h + gz)^{(q-1)}}.$$  

Differentiating both sides with respect to $x$ we get,

$$\sum_{n=0}^\infty (^qB)_{h,g,n} z^{n-(q-1)}$$

$$= (z(q-1)(h + gz)^{q-2} \frac{1}{[1-z(h+gz)^{-1}]^2}) h^{(1,0)}$$

$$= \left((q - 1) \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2),r} g^rz^{r+1} \left[\sum_{n=0}^\infty (^qB)_{h,g,n} z^{n-(q-1)}\right]^2\right) h^{(1,0)}$$

$$= \left((q - 1) \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2),r} g^rz^{-2(q-1)+r+1} \left[\sum_{n=0}^\infty (^qB)_{h,g,n} z^n\right]^2\right) h^{(1,0)}$$

$$= (q-1)h^{(1,0)} \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2),r} g^r \sum_{n=0}^\infty \left(\sum_{i=0}^n (^qB)_{h,g,i} (^qB)_{h,g,n-i} z^{n-2(q-1)+r+1}\right).$$

Comparing the coefficients of $z^{n-(q-1)}$ we get,

$$(^qB)_{h,g,n}^{(1,0)} = (q-1)h^{(1,0)} \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2),r} g^r \sum_{i=0}^n \left(\sum_{i=0}^n (^qB)_{h,g,i} (^qB)_{h,g,n-i} z^{n-2(r+1)}\right)\right).$$  

\[\square\]

Theorem 5.7.14. (Convolution property for $(^qL)_{h,g,n}$)

$$(^qL)_{h,g,n}^{(1,0)}$$

$$= (q-1)h^{(1,0)} \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2),r} g^r \sum_{i=0}^n \left(\sum_{i=0}^n (^qB)_{h,g,i} (^qL)_{h,g,n-i} z^{n-2(r+1)}\right)\right).$$  

\[5.59\]
\textit{Proof.} Equation (5.55) implies,

\[(qL)_{h,g,n} = 2 (qB)_{h,g,n+1} - h^{q-1}(qB)_{h,g,n}.\]

Differentiating both sides with respect to \(x\) and then using (5.58), we get

\[(qL)^{(1,0)}_{h,g,n} = 2 (qB)^{(1,0)}_{h,g,n+1} - h^{q-1}(qB)^{(1,0)}_{h,g,n} - (q-1)h^{q-2}(qB)^{(1,0)}_{h,g,n}\]

\[= (q-1)h^{(1,0)} \left[ \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} x^r \sum_{i=0}^{n+q-2-r} (qB)_{h,g,i} \right] - (q-1)h^{q-2}h^{(1,0)}_{h,g,n}\]

\[= (q-1)h^{(1,0)} \left[ \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} x^r \sum_{i=0}^{n+q-2-r} (qL)_{n+q-2-r-i} (qB)_{h,g,i} \right]\]

\[-h^{q-2}(qB)_{h,g,n}.\]

\[\square\]

### 5.8 Incomplete generalized bivariate \(B-q\) bonacci polynomials

In this section, we introduce the extension of incomplete generalized bivariate \(B\)-Tribonacci polynomials (5.27) to \(q\)th order incomplete generalized bivariate polynomials and call it, incomplete generalized bivariate \(B-q\) bonacci polynomials. We also study their various identities.

\textbf{Definition 5.8.1.} The \textit{incomplete generalized bivariate \(B-q\) bonacci polynomials} are defined by

\[\begin{align*}
(qB)^{(l)}_{h,g,n}(x, y) &= \sum_{r=0}^{l} \frac{(q-1)(n-(q-1)-r)}{r!} \frac{z^r}{r} h^{(q-1)(n-(q-1)-r)-r}(x)g^r(y), \\
\forall 0 \leq l \leq \left[\frac{(q-1)(n-(q-1))}{q}\right] \text{ and } n \geq q-1.
\end{align*}\]
Consider,

Proof. \( \forall \)

Theorem 5.8.2. \( n \geq \left( \frac{q}{B_{h,g,n}} \right) = \sum_{j=0}^{\infty} h_j g^r (q B_{h,g,n})^{l+q-1-r} g^r \), \( 0 \leq l \leq \left( \frac{q}{B_{h,g,n}} \right) \), \( (q B_{h,g,n})^{l+q-1-r} g^r \), \( \forall n \geq q. \)

For Simplicity, we use \( (q B_{h,g,n})^{l+q-1-r} g^r \), \( (q B_{h,g,n})^{l+q-1-r} g^r \), \( h(x) = h \) and \( g(y) = g \).

We prove identities related to recurrence relations of \( (q B_{h,g,n})^{l+q-1-r} g^r \).

**Theorem 5.8.2.** The recurrence relation of \( (q B_{h,g,n})^{l+q-1-r} g^r \) is given by

\[
(q B_{h,g,n})^{l+q-1-r} g^r = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} g^r \sum_{i=0}^{l+q-1-r} \left( \frac{(q-1)(n+q-1-r-(q-1)-i)}{d!} \right)^i h^{(q-1)(n+q-1-r-(q-1)-i)} g^i
\]

\[
= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} \sum_{i=0}^{l+q-1-r} \left( \frac{(q-1)(n-r-i)}{d!} \right)^i h^{(q-1)(n-r-i)} g^{r+i}
\]

\[
= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{i=0}^{l+q-1-r} \left( \frac{(q-1)(n-r+(r+i))}{d!} \right)^i h^{(q-1)(n+1)-q(i+r)} g^{r+i}
\]

Taking \( j = i + r \), we get

\[
\sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{j=0}^{\infty} h^{q-1-r} g^r
\]

\[
= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{j=r+1}^{\infty} \frac{(q-1)(n-j)}{(j-r)!} h^{(q-1)(n+1)-qj} g^j
\]

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\[
\sum_{j=0}^{l+q-1} \frac{(q-1)(n+1-j)^2}{j!} h_{(q-1)(n+1)-qj}^j \\
= (qB)_{h,g,n+q}^{l+q-1}.
\]

**Theorem 5.8.3.** \( s \geq 1, \)

\[
(qB)_{h,g,n+qs}^{l+(q-1)s} = \sum_{i=0}^{(q-1)s} \sum_{j=0}^{(q-1)(n+1)-j} \frac{(q-1)^2}{i!} (qB)_{h,g,n+i}^{l+i} h^i g^{(q-1)(n+1)-i},
\]

(5.62)

\[0 \leq l \leq \left\lfloor \frac{(q-1)(n-s-(q-1))}{q} \right\rfloor.\]

**Proof.** By mathematical induction on \( s. \) Clearly (5.62) holds for \( s = 1. \) Assume that the result holds for all \( s \leq m. \)

Consider, \( \sum_{i=0}^{(q-1)(m+1)} \frac{(q-1)(m+1)^2}{i!} (qB)_{h,g,n+i}^{l+i} h^i g^{(q-1)(m+1)-i}. \)

\[
= \sum_{i=0}^{(q-1)(m+1)} \sum_{r=0}^{q-1} (q-1)^{r+1} \frac{(q-1)m^2}{r!} (qB)_{h,g,n+i}^{l+i} h^i g^{(q-1)(m+1)-i} \\
= \sum_{r=0}^{q-1} \sum_{i=r}^{(q-1)m} (q-1)^{r+1} \frac{(q-1)m^2}{r!} (qB)_{h,g,n+i}^{l+i} h^i g^{(q-1)(m+1)-i} \\
= \sum_{r=0}^{q-1} (q-1)^{r+1} \sum_{j=0}^{(q-1)m-r} \frac{(q-1)m^2}{j!} (qB)_{h,g,n+r+j}^{l+r+j} h^j g^{(q-1)(m+1)-(j+r)} \\
= \sum_{r=0}^{q-1} \sum_{j=0}^{(q-1)m-r} \frac{(q-1)^{r+1}}{r!} (qB)_{h,g,n+r+j}^{l+r+j} h^j g^{(q-1)m-j} \\
= \sum_{r=0}^{q-1} \frac{(q-1)^{r+1}}{r!} (qB)_{h,g,n+r+qm}^{l+r+(q-1)m} h^j g^{(q-1)-r}. \]

Hence the result is true for \( s = m + 1. \)

Thus, by mathematical induction, the theorem is proved. \( \square \)
Theorem 5.8.4. For $n \geq \left\lfloor \frac{q^{l+2(q-1)}}{q-1} \right\rfloor$,

\[
(qB)_{h,g,n+(q-1)+s}^{l+(q-1)} - h(q-1)s(qB)_{h,g,n+q-1}^{l+q-1} = \sum_{i=0}^{s-1} \sum_{r=1}^{q-1} \frac{(q-1)^{2r}}{r!} \cdot h(q-1)^{s-(q-1)i-r} g^r (qB)_{h,g,n+(q-1)+i-r}^{l+(q-1)-r}.
\]

(5.63)

Proof. By mathematical induction on $s$. Note that (5.63) clearly holds for $s = 1$.

Now let the result be true for $s \leq m$. We prove it for $s = m + 1$.

Consider,

\[
\sum_{i=0}^{m} \sum_{r=1}^{q-1} \frac{(q-1)^{2r}}{r!} \cdot h(q-1)^{(m+1)-(q-1)i-r} g^r (qB)_{h,g,n+(q-1)+i-r}^{l+(q-1)-r} = \sum_{i=0}^{m-1} \sum_{r=1}^{q-1} \frac{(q-1)^{2r}}{r!} \cdot h(q-1)^{(m+1)-(q-1)i-r} g^r (qB)_{h,g,n+(q-1)+i-r}^{l+(q-1)-r} + \sum_{r=1}^{q-1} \frac{(q-1)^{2r}}{r!} \cdot h(q-1)^{(m+1)-(q-1)m-r} g^r (qB)_{h,g,n+(q-1)+m+1-r}^{l+(q-1)-r}
\]

\[
= h(q-1)^{m-1} \sum_{r=1}^{q-1} \frac{(q-1)^{2r}}{r!} \cdot h(q-1)^{m-(q-1)i-r} g^r (qB)_{h,g,n+(q-1)+i-r}^{l+(q-1)-r} + \sum_{r=1}^{q-1} \frac{(q-1)^{2r}}{r!} \cdot h(q-1)^{m} g^r (qB)_{h,g,n+q+m-r}^{l+(q-1)-r}
\]

\[
= \frac{(qB)_{h,g,n+(q-1)+m+1}^{l+(q-1)}}{q} - \frac{(qB)_{h,g,n+q-1}^{l+q-1}}{q}.
\]

Lemma 5.8.5. For $n \geq q - 1$,

\[
\sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} r \cdot \frac{(q-1)(n-(q-1)-r)^{2r}}{r!} \cdot h(q-1)^{n-(q-1)-qr} g^r = \frac{(q-1)(n-(q-1))}{q} (qB)_{h,g,n}
\]

\[
- \frac{h}{q} \sum_{i=0}^{n} \frac{h(q-1)^{n-i}}{i!} \cdot g^r \sum_{i=0}^{n+q-1} \sum_{r=0}^{n+q-1-r} \frac{(q-2)^{2r}}{r!} \cdot h(q-2)^{n+q-1-r} g^r (qB)_{h,g,n+q-2-r+i} \cdot \sum_{i=0}^{(qB)_{h,g,n+q-2-r+i} = (qB)_{h,g,n-i}}.
\]

(5.64)
Proof. Equation (5.44) implies,

\[ (qB)_{h,g,n} = \sum_{r=0}^{(q-1)(n-(q-1))} \left(\frac{(q-1)(n-(q-1))}{q}\right)^r h^{(q-1)(n-(q-1))-r} g^r. \]

Differentiating both sides with respect to \(x\), we get

\[ (qB)^{(1,0)}_{h,g,n} h \]

\[ = \sum_{r=0}^{(q-1)(n-(q-1))} \left(\frac{(q-1)(n-(q-1))}{q}\right)^r \left(\frac{(q-1)(n-(q-1)-r)}{r!}\right) h^{(q-1)(n-(q-1))-q} g^r. \]

Thus, \( h^{(1,0)} \sum_{r=0}^{(q-1)(n-(q-1))} \left(\frac{(q-1)(n-(q-1))}{q}\right)^r \left(\frac{(q-1)(n-(q-1)-r)}{r!}\right) h^{(q-1)(n-(q-1))-q} g^r \]

\[ = \frac{(q-1)(n-(q-1))}{q} (qB)_{h,g,n} h^{(1,0)} - h \frac{(qB)^{(1,0)}}{h,g,n}. \]

Hence, \( \sum_{r=0}^{(q-1)(n-(q-1))} \left(\frac{(q-1)(n-(q-1))}{q}\right)^r \left(\frac{(q-1)(n-(q-1)-r)}{r!}\right) h^{(q-1)(n-(q-1))-q} g^r \]

\[ = \frac{(q-1)(n-(q-1))}{q} (qB)_{h,g,n} \]

\[ - \frac{h}{q} (q-1) \left( \sum_{r=0}^{\frac{q-2}{q-1}} \frac{(q-2)^r}{r!} h^{(q-2)-q} g^r \sum_{i=0}^{n+q-2-r} (qB)_{h,g,i} (qB)_{h,g,n+q-2-r-i} \right). \]

\[ \square \]
Theorem 5.8.6. For all $n \geq q - 1$,

$$
\sum_{i=0}^{\lfloor (q-1)(n-(q-1)) \rfloor} (qB)^i_{h,g,n} = \left( \lfloor (q-1)(n-(q-1)) \rfloor + \frac{q-(q-1)(n-(q-1))}{q} \right) (qB)_{h,g,n}
$$

$$
+ \frac{h}{q} (q-1) \left( \sum_{r=0}^{q-2} \frac{(q-2)\xi}{r!} h^{(q-2)-r} g^r \sum_{i=0}^{n+q-2-r} (qB)_{h,g,i} (qB)_{h,g,n+q-2-r-i} \right). \tag{5.65}
$$

Proof. \[ \sum_{i=0}^{\lfloor (q-1)(n-(q-1)) \rfloor} (qB)^i_{h,g,n} \]

\[ = (qB)^0_{h,g,n} + \frac{(qB)^1_{h,g,n}}{0!} + \ldots + \frac{(qB)^{\lfloor (q-1)(n-(q-1)) \rfloor}_{h,g,n}}{\lfloor (q-1)(n-(q-1)) \rfloor!} \]

\[ = \frac{(q-1)(n-(q-1))!}{0!} h^{(q-1)(n-(q-1))} \]

\[ + \left[ \frac{(q-1)(n-(q-1))!}{1!} h^{(q-1)(n-(q-1))} + \frac{(q-1)(n-(q-1)-1)!}{1!} h^{(q-1)(n-(q-1))} g \right] + \ldots \]

\[ + \left[ \frac{(q-1)(n-(q-1)-1)!}{1!} h^{(q-1)(n-(q-1))} + \ldots + \frac{(q-1)(n-(q-1)-r)!}{r!} h^{(q-1)(n-(q-1))} g^r \right] \]

\[ + \ldots \]

\[ + \left[ \frac{(q-1)(n-(q-1)-r)!}{r!} h^{(q-1)(n-(q-1))} + \ldots + \frac{(q-1)(n-(q-1)-r)!}{r!} h^{(q-1)(n-(q-1))} g^r \right] \]

\[ + \ldots \]

\[ + \left[ \frac{(q-1)(n-(q-1)-r)!}{r!} h^{(q-1)(n-(q-1))} + \ldots + \frac{(q-1)(n-(q-1)-r)!}{r!} h^{(q-1)(n-(q-1))} g^r \right] \]

\[ = \left( \frac{(q-1)(n-(q-1))!}{0!} \right) h^{(q-1)(n-(q-1))} + \left( \frac{(q-1)(n-(q-1))!}{1!} \right) h^{(q-1)(n-(q-1))} g \]

\[ = \left( \frac{(q-1)(n-(q-1))!}{q} \right) h^{(q-1)(n-(q-1))} + 1) \frac{(q-1)(n-(q-1))!}{0!} \]

\[ h^{(q-1)(n-(q-1))} \]

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+ \left( \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor \right) \left( \frac{(q-1)(n-(q-1)-1)}{r!} \right) h_{(q-1)(n-(q-1))} g + \ldots 

+ \left( \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + 1 - r \right) \left( \frac{(q-1)(n-(q-1)-r)}{r!} \right) h_{(q-1)(n-(q-1))} g^{q-r} 

+ \ldots 

+ \left( \frac{(q-1)(n-(q-1)-(q-1(n-(q-1))))}{q} \right) \left( \frac{(q-1)(n-(q-1))}{q} \right) \left( \frac{(q-1)(n-(q-1)-1)}{r!} \right) h_{(q-1)(n-(q-1))} g^{q-r} 

= \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \left( \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + 1 - r \right) \left( \frac{(q-1)(n-(q-1)-r)}{r!} \right) h_{(q-1)(n-(q-1))} g^{q-r} 

= \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \left( \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + 1 \right) \left( \frac{(q-1)(n-(q-1)-r)}{r!} \right) h_{(q-1)(n-(q-1))} g^{q-r} 

= \left( \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + 1 \right) \left( \frac{(q-1)(n-(q-1))}{q} \right) \left( \frac{(q-1)(n-(q-1)-1)}{r!} \right) h_{(q-1)(n-(q-1))} g^{q-r} 

= \sum_{i=0}^{n} \left( q-1 \right) \sum_{r=0}^{\left\lfloor \frac{(q-2)(n-q-1)}{q} \right\rfloor} \left( q^{q-2}-r \right) g^{q-r} \sum_{i=0}^{n} \left( q^{q-2}-r \right) g^{q-r} \left( qB \right)_{h,g,n} 

+ \sum_{i=0}^{n} \left( q-1 \right) \sum_{r=0}^{\left\lfloor \frac{(q-2)(n-q-1)}{q} \right\rfloor} \left( q^{q-2}-r \right) g^{q-r} \sum_{i=0}^{n} \left( q^{q-2}-r \right) g^{q-r} \left( qB \right)_{h,g,n-1} 

= \left( \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + 1 - \frac{(q-1)(n-(q-1))}{q} \right) \left( \frac{(q-1)(n-(q-1))}{q} \right) \left( \frac{(q-1)(n-(q-1)-1)}{r!} \right) h_{(q-1)(n-(q-1))} g^{q-r} 

= \left( \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + q-\frac{(q-1)(n-(q-1))}{q} \right) \left( \frac{(q-1)(n-(q-1))}{q} \right) \left( \frac{(q-1)(n-(q-1)-1)}{r!} \right) h_{(q-1)(n-(q-1))} g^{q-r} 

= \sum_{i=0}^{n} \left( q-1 \right) \sum_{r=0}^{\left\lfloor \frac{(q-2)(n-q-1)}{q} \right\rfloor} \left( q^{q-2}-r \right) g^{q-r} \sum_{i=0}^{n} \left( q^{q-2}-r \right) g^{q-r} \left( qB \right)_{h,g,n-1} 

+ \sum_{i=0}^{n} \left( q-1 \right) \sum_{r=0}^{\left\lfloor \frac{(q-2)(n-q-1)}{q} \right\rfloor} \left( q^{q-2}-r \right) g^{q-r} \sum_{i=0}^{n} \left( q^{q-2}-r \right) g^{q-r} \left( qB \right)_{h,g,n-1} 

\square
5.9 Incomplete generalized bivariate $B$-$q$ Lucas polynomials

In this section we introduce the extension of incomplete generalized bivariate $B$-Tri Lucas polynomials (5.34) to $q^{th}$ order incomplete generalized bivariate polynomials and call it incomplete generalized bivariate $B$-$q$ Lucas polynomials. We also study their various identities.

Definition 5.9.1. The incomplete generalized bivariate $B$-$q$ Lucas polynomials are defined by

\[(qL)_{h,g,n}^l(x, y) = \sum_{r=0}^{l} \left[ \frac{(q-1)(n-(q-2))}{(q-1)(n-(q-2)-r)} \left( \frac{(q-1)(n-(q-2)-r)}{r!} \right)^{q} \right] h^{(q-1)(n-(q-2))-qr(x)g^r(y)} \]

\[- \sum_{r=2}^{l} \left[ \sum_{s=1}^{q-1} \frac{(q-1)(n-(q-1)-r)+(r-2)}{(r-2)!} \right] h^{(q-1)(n-(q-2))-qr(x)g^r(y)}, \quad (5.66)\]

\forall n \geq q \text{ and } 0 \leq l \leq \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor.

Next three theorems give results on recurrence properties of incomplete generalized bivariate $B$-$q$ Lucas polynomials (5.72). Proof of these results can be obtained using a procedure similar to that used in the relative identities of incomplete generalized bivariate $B$-$q$ bonacci sequence (5.60).

Theorem 5.9.2. The recurrence relation for incomplete generalized bivariate $B$-$q$ Lucas polynomials $(qL)_{h,g,n}^l$ is given by

\[(qL)_{h,g,n+q}^{l+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{q}}{r!} \frac{(qL)_{h,g,n+q-1-r}}{(qL)_{h,g,n+q-1-r}^{q-1-r} g^r}, \quad (5.67)\]
∀ 0 ≤ l ≤ \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor \text{ and } n ≥ q - 2.

**Theorem 5.9.3.** For all 0 ≤ l ≤ \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor,

\[(qL)_{h,g,n+qs}^{l+(q-1)s} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^i}{i!} (qL)_{h,g,n+(q-1)s-i}^{l+(q-1)s-i} h^{(q-1)s-i} g^i. \quad (5.68)\]

**Theorem 5.9.4.** For n ≥ \left\lfloor \frac{ql}{q-1} + q - 2 \right\rfloor,

\[(qL)_{h,g,n+(q-1)+s}^{l+(q-1)} = \sum_{i=0}^{s-1} \sum_{r=1}^{q-1} \frac{(q-1)^s}{r!} \left( h^{(q-1)s-r-(q-1)i} g^r (qL)_{h,g,n+(q-1)+i-r}^{l+(q-1)-r} \right). \quad (5.69)\]

Next two results gives the relation between n\textsuperscript{th} term \((qB)_n^l\) and \((qL)_n^l\).

**Theorem 5.9.5.** The relation between the n\textsuperscript{th} term \((qL)_n^l\) and n\textsuperscript{th} term \((qB)_n^l\) is given by

\[(qL)_{h,g,n}^l = (qB)_{h,g,n+1}^l + \sum_{r=1}^{q-1} \frac{(q-1)^s}{r!} h^{q-1-r} g^r (qB)_{h,g,n-r}^{l-r}, \quad 0 ≤ l ≤ \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor. \quad (5.70)\]

Proof of the Theorem 5.9.5 is similar to that of Theorem 5.5.2.

**Corollary 5.9.6.**

\[(qL)_{h,g,n}^l = 2 (qB)_{h,g,n+1}^l - h^{q-1} (qB)_{h,g,n}^l, \quad 0 ≤ l ≤ \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor. \quad (5.71)\]

Proof. Using equations (5.61) and (5.70), the Corollary can be proved.
Lemma 5.9.7. For all $n \geq q - 2$,

$$
= \sum_{r=0}^{l} r \left[ \frac{(q-1)(n-(q-2))}{(q-1)(n-(q-2)-r)} \frac{(q-1)(n-(q-2)-r)}{r!} \right] h^{(q-1)}(n-(q-2))^{q-2} g^r
$$

$$
- \sum_{r=2}^{l} \left[ \sum_{s=1}^{q-1} (s-1) \left( (q-1)(n-(q-1)-r) + s-2 \right)^{r-2} \frac{r-2}{(r-2)!} \right] h^{(q-1)}(n-(q-2))^{q-2} g^r,
$$

(5.72)

$$
= \frac{(q-1)(n-(q-2))}{q} (qL)_{h,g,n} - \frac{h}{q} (q-1)
$$

(5.73)

$$
\left( \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} g^r \sum_{i=0}^{n+q-2-r} (qB)_{h,g,i} (qL)_{h,g,n+q-2-r-i} \right) - h^{q-2} (qB)_{h,g,n},
$$

where $l = \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor$.

Theorem 5.9.8. For all $n \geq q - 2$,

$$
\sum_{l=0}^{\left\lceil \frac{(q-1)(n-(q-2))}{q} \right\rceil} \frac{(qL)^l}{q} (qL)_{h,g,n}
$$

$$
= \left( \left\lceil \frac{(q-1)(n-(q-2))}{q} \right\rceil + \frac{q-\left\lceil \frac{(q-1)(n-(q-2))}{q} \right\rceil}{q} \right) (qL)_{h,g,n}
$$

$$
\frac{h}{q} (q-1) \left( \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} g^r \sum_{i=0}^{n+q-2-r} (qB)_{h,g,i} (qL)_{h,g,n+q-2-r-i} \right) - h^{q-2} (qB)_{h,g,n},
$$

(5.74)

Proof. Use Lemma 5.9.7 and procedure similar to that of Theorem 5.5.8.  □