Introduction

Algebraic combinatorics is one of the most important subjects of modern mathematics. It deals with the problems, interconnecting combinatorics and algebra(mainly groups, rings, vector spaces, homological algebra etc). Some of the most dynamic and active branches of Algebraic Combinatorics are Algebraic graph theory, Combinatorial algebraic topology, Combinatorial Hopf algebras, Combinatorial representation theory etc. As the title suggests, this thesis addresses some problems in some of the aforesaid branches.

The first part of this thesis deals with some problems in algebraic graph theory, more specifically some problems on the power graph of finite groups. Given an algebraic structure $S$, there are different ways to associate a directed or undirected graph to $S$ in such a way that the vertices are associated with families of elements or subsets of $S$ and in which two vertices are joined by an arc or by an edge if and only if they satisfy a certain relation. Since a graph (directed or undirected) can be investigated in terms of the results from Graph Theory, one can obtain some information about the structure of $S$. One such graph is power graph of groups and semigroups. Kelarav and Quinn [57] defined the power graph of a semigroup $S$ as a directed graph in which the vertex set is $S$ and for $x, y \in S$ there is an arc from $x$ to $y$ if and only if $x \neq y$ and $y = x^m$ for some positive integer $m$. Following this Chakrabarty et al. [56] defined the (undirected) power graph $\mathcal{P}(S)$ of a semigroup $S$ with vertex set $S$ and two vertices $x, y \in S$ are adjacent if and only if $x \neq y$ and $x^m = y$ or $y^m = x$ for some positive integer $m$. 

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In chapter 2, we have proved a conjecture of Doostabadi, Erfanian and Jafarzadeh, about the evaluation of the automorphism group of the power graph of finite cyclic groups. In 2015, Doostabadi, Erfanian and Jafarzadeh [59] conjectured that for \( n \neq p^m \),

\[
\text{Aut}(P(\mathbb{Z}_n)) = \left( \bigoplus_{d|n, d \neq 1, n} S_{\phi(d)} \right) \bigoplus S_{\phi(n) + 1},
\]

where \( \phi \) is the Euler’s phi function. We have given an elementary combinatorial proof of this conjecture.

There are three fundamental products of graphs, namely, cartesian product, direct product and strong product of graphs. In Chapter 3, we have shown that, given any two groups \( G_1 \) and \( G_2 \), \( P(G_1 \times G_2) \) is not isomorphic to either of these products of \( P(G_1) \) and \( P(G_2) \) in general. So we introduce a new product of two graphs \( \Gamma_1 \) and \( \Gamma_2 \), which we call generalized product of \( \Gamma_1 \) and \( \Gamma_2 \). Reason behind such naming is that each of the cartesian, direct and strong products is a special case of the generalized product of two graphs. Our main theorem states that \( P(G_1 \times G_2) \) is isomorphic to a generalized product of the power graphs \( P(G_1) \) and \( P(G_2) \).

In the excellent survey [60] on power graph of groups and semigroups, Abawajy et.al. proposed the following problem:

Describe all directed and undirected power graphs of groups and semigroups that can be represented as Cayley graphs and, respectively, undirected Cayley graphs of groups or semigroups.

We have described all groups whose power graphs can be represented as Cayley graphs of some groups, both directed and undirected in Chapter 4. In fact we prove that

1. Let \( G \) be a finite group. Then \( P(G) \) is a Cayley graph if and only if \( G \) is a cyclic \( p \)-group.
2. There is no finite group $G$, whose directed power graph is a Cayley graph.

The second part of this thesis deals with some problems in combinatorial algebraic topology. Combinatorial algebraic topology is a very important branch of algebraic combinatorics. Very roughly speaking, the goal of this branch is to apply the tools of algebraic topology to the problems in combinatorics and vice-versa. The notion of forests and their roots in higher dimensions was introduced by Olivier Bernardi and Caroline Klivans in [61] to obtain a generalization of the ”Matrix Forest Theorem” in higher dimension. In fact, they proved that, for a $d$–dimensional simplicial complex $G$,

$$
\sum_{(F,R)} |H_{d-1}(F, R)|^2 x^{|R|} = \det(L_G + xId),
$$

where the summation runs over all rooted forests of $G$ and $Id$ is the identity matrix of dimension $|G_{d-1}|$, and $L_G$ is the Laplacian matrix. A combinatorial interpretation of the cardinality of the relative homology group $H_{d-1}(F, R)$ for a general simplicial complex $G$ is a challenging problem.(See [61]). In this part, we give a combinatorial interpretation of $|H_{d-1}(F, R)|$ for any closed $d$–manifold $M^d$ in terms of their “Poincare-dual graphs” (details later). We have also addressed the following problem. In graphs, a forest simplicially collapses onto any of its roots (in graph case, a root consists of a finite number of vertices, each belongs to its mother component). So it is tempting to conjecture that, in higher dimensional case also, a forest simplicially collapses onto any of its roots. But unfortunately, this is false in general. In fact a non-orientable, closed, triangulated manifold can not collapse onto any of its roots.

In this paper, we have proved that, if $(F, R)$ is a rooted forest in a closed, connected, triangulated manifold $M^d$, then $F$ simplicially collapses onto its root $R$ if and only if $H_{d-1}(F, R)$ is trivial. As a corollary of this theorem we have proved that a “punctured” closed, connected, orientable triangulated manifold simplicially collapses onto any of its roots.
In [62], Erik Insko, Katie Johnson and Shaun Sullivan proved a Ryser-type formula for determinants. They called it “A terrible expansion of the determinant”. This expansion came from a conjecture about a transfer formula in Multivariate finite operator calculus [62]. Their expansion is the following:

For a square matrix $A = (a_{ij})_{n \times n}$,

$$|A| = \sum_{B \subseteq [n]} (-1)^{n-|B|} \prod_{\beta_k \in B} \prod_{i \in \beta_k} \sum_{j \in \beta_k} a_{ij},$$

where the outer summation runs over all ordered partitions $B = (\beta_1, \beta_2, \ldots, \beta_r)$ of the set $[n]$ and the inner summation runs over all integers $j$ in the union $\beta_k = \bigcup_{l=1}^k \beta_l$ of first $k$ parts of the partition.

Now we spend a few words on the expansion and the proof of it given in [62]. For a function $f : [n] \to [n]$, define $a_f := \prod_{i=1}^n a_{i f(i)}$.

In the light of this definition, the expansion says, $|A| = \sum_{B \subseteq [n]} (-1)^{n-|B|}(\sum_f a_f)$;

where the inner summation runs over all functions $f : [n] \to [n]$ satisfying the following property: If $a \in \beta_k$, then $f(a) \in \bigcup_{j=1}^k \beta_j$, where $B = (\beta_1, \beta_2, \ldots, \beta_r)$. Now if we expand, this will take the form $\sum_f c_f a_f$, $f$ running through all the functions $f : [n] \to [n]$. In [62], the authors proved that if $f$ is bijective $c_f = \text{sgn}(f)$ and for non-bijective $f$, they showed $c_f$ to be zero by an elegant topological argument analyzing the Euler characteristics of subsets of the Permutahedron. Since this proof is highly topological and also their expansion resembles the Ryser’s formula for permanents, stating that for a square matrix $A = (a_{ij})_{n \times n}$, $\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^n \sum_{j \in S} a_{ij}$, the authors asked two questions at the end of their paper:

1. If there exists a combinatorial proof of their theorem.
2. If the Ryser’s formula can be proved topologically in the spirit of their argument.

We have answered these two questions in the last chapter of this thesis.