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SAJAL KUMAR MUKHERJEE
ABSTRACT

This thesis is divided into two parts. The first part deals with some problems on power graphs of finite groups. The power graph $P(G)$ of a finite group $G$ is the graph with vertex set $G$ and two distinct vertices are adjacent if either of them is a power of the other. In first part, we address three problems on power graphs:

1. In Chapter 2, we give an elementary combinatorial proof of a conjecture on the automorphism group of the power graph of finite cyclic groups, proposed by Doostabadi, Erfanian and Jafarzadeh in 2015.

2. In Chapter 3, we investigate the relationship of the power graphs $P(G_1)$ and $P(G_2)$ with $P(G_1 \times G_2)$ for any two groups $G_1$ and $G_2$. A new product of graphs, namely generalized product of graphs, has been introduced and we prove that $P(G_1 \times G_2)$ is isomorphic to the generalized product of $P(G_1)$ and $P(G_2)$.

3. In Chapter 4, we characterize all finite groups whose power graphs are Cayley graphs of some groups.

Second part of the thesis is devoted to some problems in combinatorial algebraic topology. Here we consider the following problems:

1. In Chapter 6, we prove that the relative homology $H_{d-1}(F, R)$ is an elementary abelian $2$–group for any rooted forest $(F, R)$ in a triangulated closed $d$–manifold $M^d$. In addition, we have shown that if $M^d$ is orientable, $H_{d-1}(F, R)$ is trivial. We also give a combinatorial interpretation of the cardinality of the group $H_{d-1}(F, R)$, which answers an open question of Olivier Bernardi and Caroline Klivans for any closed manifold. We have shown that for any rooted forest $(F, R)$ in a closed manifold $M^d$, the forest $F$ simplicially collapses onto its root $R$ iff $H_{d-1}(F, R)$ is trivial.
2. In Chapter 7, we give a purely combinatorial proof of a Ryser-type formula for determinants inspired by multivariate finite operator calculus. This argument also includes a combinatorial proof of an interesting identity about Stirling number of second kind. Also we give a topological proof of Ryser’s formula for permanents.
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Introduction

Algebraic combinatorics is one of the most important subjects of modern mathematics. It deals with the problems, interconnecting combinatorics and algebra (mainly groups, rings, vector spaces, homological algebra etc). Some of the most dynamic and active branches of Algebraic Combinatorics are Algebraic graph theory, Combinatorial algebraic topology, Combinatorial Hopf algebras, Combinatorial representation theory etc. As the title suggests, this thesis addresses some problems in some of the aforesaid branches.

The first part of this thesis deals with some problems in algebraic graph theory, more specifically some problems on the power graph of finite groups. Given an algebraic structure $S$, there are different ways to associate a directed or undirected graph to $S$ in such a way that the vertices are associated with families of elements or subsets of $S$ and in which two vertices are joined by an arc or by an edge if and only if they satisfy a certain relation. Since a graph (directed or undirected) can be investigated in terms of the results from Graph Theory, one can obtain some information about the structure of $S$. One such graph is power graph of groups and semigroups. Kelarav and Quinn [57] defined the power graph of a semigroup $S$ as a directed graph in which the vertex set is $S$ and for $x, y \in S$ there is an arc from $x$ to $y$ if and only if $x \neq y$ and $y = x^m$ for some positive integer $m$. Following this Chakrabarty et al. [56] defined the (undirected) power graph $\mathcal{P}(S)$ of a semigroup $S$ with vertex set $S$ and two vertices $x, y \in S$ are adjacent if and only if $x \neq y$ and $x^m = y$ or $y^m = x$ for some positive integer $m$. 

1
In chapter 2, we have proved a conjecture of Doostabadi, Erfanian and Jafarzadeh, about the evaluation of the automorphism group of the power graph of finite cyclic groups. In 2015, Doostabadi, Erfanian and Jafarzadeh [59] conjectured that for \( n \neq p^m \),

\[
\text{Aut}(\mathcal{P}(\mathbb{Z}_n)) = \bigoplus_{d \mid n, d \neq 1, n} S_{\phi(d)} \bigoplus S_{\phi(n)+1},
\]

where \( \phi \) is the Euler's phi function. We have given an elementary combinatorial proof of this conjecture.

There are three fundamental products of graphs, namely, cartesian product, direct product and strong product of graphs. In Chapter 3, we have shown that, given any two groups \( G_1 \) and \( G_2 \), \( P(G_1 \times G_2) \) is not isomorphic to either of these products of \( P(G_1) \) and \( P(G_2) \) in general. So we introduce a new product of two graphs \( \Gamma_1 \) and \( \Gamma_2 \), which we call generalized product of \( \Gamma_1 \) and \( \Gamma_2 \). Reason behind such naming is that each of the cartesian, direct and strong products is a special case of the generalized product of two graphs. Our main theorem states that \( P(G_1 \times G_2) \) is isomorphic to a generalized product of the power graphs \( P(G_1) \) and \( P(G_2) \).

In the excellent survey [60] on power graph of groups and semigroups, Abawajy et.al. proposed the following problem:

\textit{Describe all directed and undirected power graphs of groups and semigroups that can be represented as Cayley graphs and, respectively, undirected Cayley graphs of groups or semigroups.}

We have described all groups whose power graphs can be represented as Cayley graphs of some groups, both directed and undirected in Chapter 4. In fact we prove that

1. Let \( G \) be a finite group. Then \( \mathcal{P}(G) \) is a Cayley graph if and only if \( G \) is a cyclic \( p \)-group.
2. There is no finite group $G$, whose directed power graph is a Cayley graph.

The second part of this thesis deals with some problems in combinatorial algebraic topology. Combinatorial algebraic topology is a very important branch of algebraic combinatorics. Very roughly speaking, the goal of this branch is to apply the tools of algebraic topology to the problems in combinatorics and vice-versa. The notion of forests and their roots in higher dimensions was introduced by Olivier Bernardi and Caroline Klivans in [61] to obtain a generalization of the ”Matrix Forest Theorem” in higher dimension. In fact, they proved that, for a $d$-dimensional simplicial complex $G$,

$$\sum_{(F,R)} |H_{d-1}(F,R)|x^{\lvert R \rvert} = det(L_G + xId),$$

where the summation runs over all rooted forests of $G$ and $Id$ is the identity matrix of dimension $\lvert G_{d-1} \rvert$, and $L_G$ is the Laplacian matrix. A combinatorial interpretation of the cardinality of the relative homology group $H_{d-1}(F,R)$ for a general simplicial complex $G$ is a challenging problem.(See [61]). In this part, we give a combinatorial interpretation of $\lvert H_{d-1}(F,R) \rvert$ for any closed $d$-manifold $M^d$ in terms of their "Poincare-dual graphs" (details later). We have also addressed the following problem. In graphs, a forest simplicially collapses onto any of its roots (in graph case, a root consists of a finite number of vertices, each belongs to its mother component). So it is tempting to conjecture that, in higher dimensional case also, a forest simplicially collapses onto any of its roots. But unfortunately, this is false in general. In fact a non-orientable, closed, triangulated manifold can not collapse onto any of its roots. In this paper, we have proved that, if $(F,R)$ is a rooted forest in a closed, connected, triangulated manifold $M^d$, then $F$ simplicially collapses onto its root $R$ if and only if $H_{d-1}(F,R)$ is trivial. As a corollary of this theorem we have proved that a “punctured” closed, connected, orientable triangulated manifold simplicially collapses onto any of its roots.
In [62], Erik Insko, Katie Johnson and Shaun Sullivan proved a Ryser-type formula for determinants. They called it “A terrible expansion of the determinant”. This expansion came from a conjecture about a transfer formula in Multivariate finite operator calculus [62]. Their expansion is the following:

For a square matrix \( A = (a_{ij})_{n \times n} \),

\[
|A| = \sum_{B \subseteq [n]} (-1)^{n-|B|} \prod_{\beta_k \in B} \prod_{i \in \beta_k} \sum_{j \in \beta_k} a_{ij},
\]

where the outer summation runs over all ordered partitions \( B = (\beta_1, \beta_2, \ldots, \beta_r) \) of the set \([n]\) and the inner summation runs over all integers \( j \) in the union \( \beta_k = \bigcup_{l=1}^{k} \beta_l \) of first \( k \) parts of the partition.

Now we spend a few words on the expansion and the proof of it given in [62].

For a function \( f : [n] \to [n] \), define \( a_f := \prod_{i=1}^{n} a_{f(i)} \).

In the light of this definition, the expansion says, \( |A| = \sum_{B \subseteq [n]} (-1)^{n-|B|}(\sum_f a_f) \); where the inner summation runs over all functions \( f : [n] \to [n] \) satisfying the following property: If \( a \in \beta_k \), then \( f(a) \in \bigcup_{j=1}^{k} \beta_j \), where \( B = (\beta_1, \beta_2, \ldots, \beta_r) \). Now if we expand, this will take the form \( \sum_f c_f a_f \), \( f \) running through all the functions \( f : [n] \to [n] \). In [62], the authors proved that if \( f \) is bijective \( c_f = sgn(f) \) and for non-bijective \( f \), they showed \( c_f \) to be zero by an elegant topological argument analyzing the Euler characteristics of subsets of the Permutahedron. Since this proof is highly topological and also their expansion resembles the Ryser’s formula for permanents, stating that for a square matrix \( A = (a_{ij})_{n \times n} \), \( \text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j \in S} a_{ij} \), the authors asked two questions at the end of their paper:

1. If there exists a combinatorial proof of their theorem.
2. If the Ryser’s formula can be proved topologically in the spirit of their argument.

We have answered these two questions in the last chapter of this thesis.
Part I

Some problems on the power graph of finite groups
Chapter 1

Introduction to part-I

In this chapter, we recall some preliminaries needed for the first part of this thesis. Here we recall basic definitions in power graphs of finite groups. We conclude this chapter with a brief summary of the results obtained in the first part of this thesis.

1.1 Some basic notions in graph theory

A simple graph $\Gamma = (V; E)$ consists of a nonempty set $V(\Gamma)$ or simply $V$ and a collection $E(\Gamma)$ or simply $E$ of 2-subsets of $V$. The elements of $V$ and $E$ are called the vertices and edges of $\Gamma$ respectively. The cardinality of $V$ and $E$ are called the order and size of $\Gamma$ respectively. A graph is said to be finite if both the order and size of the graph are finite. In this thesis, we consider finite simple graphs only. Two vertices $u$ and $v$ in a graph $\Gamma$ are said to be adjacent if the 2-subsets $uv$ is an edge in $\Gamma$, otherwise $u$ and $v$ are non-adjacent in $\Gamma$. We write $u \sim v$ for $u$ is adjacent to $v$. If $e = uv$ is an edge in $\Gamma$ then vertices $u$ and $v$ are called end vertices of $e$ and the edge $e$ is said to be incident on vertices $u$ and $v$. The degree of a vertex $v$ in a simple graph is the number of edges incident on $v$. A graph $\Gamma$ is called regular if all the vertices in $\Gamma$ have the same degree. If degree of all the vertices in $\Gamma$ is $r$ then $\Gamma$ is called an $r$-regular graph or a regular graph of degree $r$. A degree one vertex in a graph is known as a pendent vertex. The complement of a graph $\Gamma$, denoted by $\Gamma^\prime$, is the graph on the same vertex set as $\Gamma$ and two vertices are adjacent in $\Gamma^\prime$ if and only if they are non-adjacent in $\Gamma$. A subgraph of a graph $\Gamma = (V; E)$ is a graph $\Gamma_1 = (V_1; E_1)$ such that $V_1 \subset V$ and $E_1 \subset E$. A subgraph $\Gamma_1$ is called an
induced subgraph of $\Gamma$ if $\Gamma_1$ contains all the edges of $\Gamma$ whose end vertices are in $\Gamma_1$. A graph is said to be complete if each pair of vertices in the graph are adjacent. A complete graph on $n$ vertices is denoted by $K_n$. Two graphs $\Gamma_1$ and $\Gamma_2$ are said to be isomorphic if there exists a bijection $f : V(\Gamma_1) \to V(\Gamma_2)$ such that $u \sim v$ in $\Gamma_1$ if and only if $f(u) \sim f(v)$ in $\Gamma_2$. An automorphism of a graph is an isomorphism of the graph to itself. The set of all automorphisms of a graph $\Gamma$ form a group under the operation of composition of mappings. This group is called automorphism group of $\Gamma$, denoted by $\text{Aut}(\Gamma)$. Determination of the automorphism group of a graph is a challenging problem in general.

1.2 Literature survey on power graphs

During the last two decades, study of the interplay between the properties of an algebraic structure $S$ and the graph theoretic properties of $\Gamma(S)$, a graph associated to $S$, has been an exciting topic of research. Given an algebraic structure $S$, we can associate $S$ to a directed or undirected graph in different ways [6], [20], [36]. To study algebraic structures using graph theory, different graphs have been formulated namely, commuting graph associated to a group [51], [52], power graph of a semigroup [6], deleted power graph of a finite group [46], strong power graph of a group [49], [20], normal subgroup based power graph of a group [50], zero-divisor graph of a ring [59], [36] semiring [53], semigroup [37], poset [40], etc. The investigation of graphs related to various algebraic structures is important, because such graphs have valuable applications [47] and are related to automata theory [45].

The directed power graph of a semigroup was introduced in [41] for groups and in [43] for semigroups. Then the undirected power graph $P(S)$ of a semigroup $S$ was defined by Chakrabarty, Ghosh and Sen in [6]. For the semigroup $S$, the vertex set of the undirected power graph $\Gamma(S)$ is $S$ and two distinct vertices $u, v$ are adjacent if $u = v^m$ or $v = u^n$ for some positive integers $m, n$. They studied the completeness, Hamiltonicity etc of the power graphs in terms of the underlying
semigroups or groups. Also they calculated the edge number of the power graph of finite groups. In [54], Chelvam proved that the power graph of a finite group $G$ is Eulerian iff $|G|$ is odd. The vertex connectivity has been studied in [31], [7]. Most notably, in [4], Cameron and Ghosh showed that for two finite abelian groups $G_1$ and $G_2$, $\mathcal{P}(G_1) \cong \mathcal{P}(G_2)$ implies that $G_1 \cong G_2$; in other words the power graph of an abelian group uniquely determines the group up to isomorphism. There are many more directions in this topic yet to be explored.

1.3 Objective of part-I

In 2015, Doostabadi, Erfanian and Jafarzadeh [59] conjectured that for $n \neq p^m$, $\text{Aut}(\mathcal{P}(\mathbb{Z}_n)) = \left( \bigoplus_{d|n, d \neq 1, n} S_{\phi(d)} \right) \bigoplus S_{\phi(n)+1}$, where $\phi$ is the Euler’s phi function. Chapter 2 is devoted to a proof of this conjecture.

In Chapter 3 we show that the power graph $\mathcal{P}(G_1 \times G_2)$ of the direct product of two groups $G_1$ and $G_2$ is not isomorphic to either of the direct, cartesian and normal product of their power graphs $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$. A new product of graphs, namely generalized product, has been introduced and we prove that the power graph $\mathcal{P}(G_1 \times G_2)$ is isomorphic to a generalized product of $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$. There also we have shown that each of the cartesian, direct and strong products is a special case of the generalized product of two graphs.

In Chapter 4, we describe all groups whose power graphs can be represented as Cayley graphs of some groups, both directed and undirected, answering a question proposed by Jemal Abawajy, Andrei Kelarev and Morshed Chowdhury in [60].
Chapter 2

An elementary proof of a conjecture on graph-automorphism

In 2015, Doostabadi, Erfanian and Jafarzadeh [59] conjectured that for \( n \neq p^m \),
\[
\text{Aut}(\mathcal{P}(\mathbb{Z}_n)) = \left( \bigoplus_{d|n, d \neq 1, n} S_{\phi(d)} \right) \bigoplus S_{\phi(n)+1},
\]
where \( \phi \) is the Euler’s phi function. We know that, if \( n \) is a prime power, then \( \mathcal{P}(\mathbb{Z}_n) \) is complete [6], hence \( \text{Aut}(\mathcal{P}(\mathbb{Z}_n)) = S_n \).
So we need not worry about the prime power case any further. Our aim of this chapter is to provide an elementary combinatorial proof of the conjecture for \( n \neq p^m \).

2.1 Main theorem

In this section, first we prove several lemmas and as a consequence, we shall prove the following result, which is the main theorem of this chapter.

**Theorem 1.** For \( n \neq p^m \) (p prime), \( \text{Aut}(\mathcal{P}(\mathbb{Z}_n)) = \left( \bigoplus_{d|n, d \neq 1, n} S_{\phi(d)} \right) \bigoplus S_{\phi(n)+1} \)

First we prove a technical lemma, which is also the heart of our argument. Before stating it, we have to fix some notations.

Let \( S \) be a finite set of positive real numbers. For each subset \( B \subseteq S \), let \( \Pi(B) \) denote the product of all the elements of \( B \). Now let us state and prove the lemma.

**Lemma 2.** Let \( n \geq 2 \) and \( m_1, m_2, \cdots, m_n \) be \( n \) positive integers with \( m_1 > m_2 > \cdots > m_n \). Let \( m \) be any positive integer, and set \( A = \{ m_1, m_2, \cdots, m_n; m \} \) and
Let \( B = \{m_2, m_3, \ldots, m_n\} \). Then to every non empty subset \( S_B \) of \( B \), we can associate a proper subset \( S_A \) of \( A \), for which \( m_1, m \in S_A \) and \( \prod(S_B) < \prod(S_A \setminus \{m\}) \). The association can be made one to one.

Proof. We prove this by induction on \( n \). Let \( n = 2 \). Then \( A = \{m_1, m_2, m\} \), \( B = \{m_2\} \) and \( m_1 > m_2 \). The only nonempty subset of \( B \) is \( B \) itself. To \( B \), we associate \( \{m_1, m\} \) and the result holds.

Now let the statement be true for \( n = k \). We will prove for \( n = k + 1 \). Let \( m_1, m_2, \ldots, m_{k+1} \) be any \( k + 1 \) positive integers with \( m_1 > m_2 > \cdots > m_{k+1} \) and \( m \) be any positive integer. Then \( A = \{m_1, m_2, \ldots, m_{k+1}; m\} \) and \( B = \{m_2, m_3, \ldots, m_{k+1}\} \). Let \( B_1 \) be the collection of non empty subsets of \( B \), which do not contain \( m_{k+1} \) and \( B_2 \) be the collection of subsets of \( B \) containing \( m_{k+1} \). Now consider \( \hat{A} = \{m_1, m_2, \ldots, m_k; m_{k+1}\} \) and \( \hat{B} = \{m_2, m_3 \ldots m_k\} \). Then by the induction hypothesis, to each nonempty subsets \( S_B \) of \( \hat{B} \) we can associate a proper subset \( S_A \) of \( \hat{A} \), for which \( m_1, m_{k+1} \in S_A \) and \( \prod(S_B) < \prod(S_A \setminus \{m_{k+1}\}) \). Now let \( S_{B_1} \) be an arbitrary element of \( B_1 \). But \( S_{B_1} \) is also a non empty subset of \( \hat{B} \). So we have a proper subset \( S_{\hat{A}_1} \) of \( \hat{A} \) for which \( m_1, m_{k+1} \in S_{\hat{A}_1} \) and \( \prod(S_{\hat{B}_1}) < \prod(S_{\hat{A}_1} \setminus \{m_{k+1}\}) \) [by induction hypothesis]. Set \( S_{\hat{A}_1}(\not\in A) \) to be \( (S_{\hat{A}_1} \setminus \{m_{k+1}\}) \cup \{m\} \). Clearly \( \prod(S_{\hat{B}_1}) < \prod(S_{\hat{A}_1} \setminus \{m\}) \). Now let \( S_{B_2} \) be an arbitrary element of \( B_2 \setminus \{m_{k+1}\} \). Then \( S_{B_2} \setminus \{m_{k+1}\} \subset B_1 \). So for the sake of simplicity assume that \( S_{B_2} \setminus \{m_{k+1}\} = S_{B_1} \). But for \( S_{B_1} \), we have \( S_{\hat{A}_1} \), so that \( \prod(S_{\hat{B}_1}) < \prod(S_{\hat{A}_1} \setminus \{m\}) \). Let us take \( S_{A_2} \) to be \( S_{\hat{A}_1} \cup \{m_{k+1}\} \). It is easy to see that \( S_{A_2} \) is in fact a proper subset of \( A \) and \( \prod(S_{A_2}) < \prod(S_{A_2} \setminus \{m\}) \). Now the number of non empty subset of \( B \) is equal to the number of proper subset of \( A \) containing both \( m_1, m \) is equal to \( 2^k - 1 \). Hence for the set \( \{m_{k+1}\} \), there still remains exactly one proper subset \( S_A \) of \( A \), containing both \( m_1, m \). Clearly \( \prod(\{m_{k+1}\}) = m_{k+1} < m_1 \leq \prod(S_A \setminus \{m\}) \). This completes the induction as well as the proof.

Before plunging into a chain of lemmas, we once again fix some notations. Let \( X_d \) denote the set of all generators of the unique cyclic subgroup of \( \mathbb{Z}_n \) of order \( d \).
Then \(|X_d| = \phi(d)|. We shall denote an element of \(X_d\) by \(x_d\).

**Lemma 3.** If \(n = p_1p_2\cdots p_k\) where \(p_1 > p_2 \cdots > p_k\) are distinct primes then there does not exist any graph automorphism \(\sigma \in \text{Aut} (\mathcal{P}(\mathbb{Z}_n))\) such that \(\sigma(x_{p_i}) = x_{p_1p_2\cdots p_i}\) for any \(i, l, \alpha, \eta, \sigma\) satisfying \(i \geq i_1 > i_2 > \cdots > i_l\).

**Proof.** To prove the lemma, it suffices to prove that degree \((x_{p_i}) > \text{deg}(x_{p_1p_2\cdots p_i})\), satisfying the above stated condition. But, since \(\text{deg}(x_{p_i}) \geq \text{deg}(x_{p_1})\) [as \(i \geq i_1\)], it suffices to prove that degree \((x_{p_i}) > \text{deg}(x_{p_1p_2\cdots p_i})\). We will only prove that degree \((x_{p_1}) > \text{deg}(x_{p_1p_2\cdots p_{k-1}})\) for the sake of simplicity, because the other cases will follow exactly in the similar fashion. Now, in Lemma 2, take

\[
A = \{p_1 - 1, p_2 - 1, \ldots p_{k-1} - 1, p_k - 1\} \text{ and } B = \{p_2 - 1, p_3 - 1, \ldots p_{k-1} - 1\}.
\]

Then from Lemma 2, we see that the total number of vertices adjacent to \(x_{p_1}\) but not adjacent to \(x_{p_1p_2\cdots p_{k-1}}\) is strictly greater than the number of vertices adjacent to \(x_{p_1p_2\cdots p_{k-1}}\) but not adjacent to \(x_{p_1}\). And this proves the claim. \(\Box\)

**Lemma 4.** Let \(n = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}\) and \(p_1 > p_2^{m_1} > p_2^{m_2} > \cdots > p_k^{m_k}\). Then we have the following:

(i): There does not exist any automorphism \(\sigma\) such that \(\sigma(x_{p_i}) = x_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\) for any \(i, r\) and a proper subset \(\{i_1, i_2, \cdots i_r\}\) of \(\{1, 2, \cdots k\}\) satisfying \(i \geq i_1 > i_2 \cdots > i_r\).

(ii): There does not exist any automorphism \(\eta\) such that \(\eta(x_{p_i}) = x_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\) for any \(\alpha_1, \alpha_2, \cdots \alpha_k\) with \(1 \leq \alpha_i \leq m_i\) for all \(i = 1, 2, \cdots k\).

**Proof.** (i): Proof of this part is exactly similar to that of Lemma 3.

(ii): We divide this case into two cases.

**Case 1:** \(k \geq 3\), i.e. \(n = p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}\) where \(k \geq 3\) and \(p_1^{x_1} > p_2^{x_2} > \cdots > p_k^{x_k}\). We will eliminate the following two difficult cases. Rest will follow quite similarly.

**Subcase 1.1:** If possible let \(\eta(x_{p_i}) = x_{p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}}\) where \(1 \leq r < x_2\). Then the number of vertices adjacent to \(x_{p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}}\) but not adjacent to \(x_{p_1}\) is exactly equal to \((\phi(p_2) + \phi(p_2^2) + \cdots + \phi(p_2^r) + 1)(\phi(p_3) + \phi(p_3^2) + \cdots + \phi(p_3^s) + 1)\cdots (\phi(p_k) + \phi(p_k^r) + \cdots ) + 1\cdots (\phi(p_k) + \phi(p_k^{x_k}) + 1)\)
\[
\cdots + \phi(p_k^{x_k}) + 1) - 1 \text{ which is equal to } p_2^{x_2}p_3^{x_3} \cdots p_k^{x_k} - 1. \]

Now there are at least \((\phi(p_1) + \phi(p_2^2) + \cdots + \phi(p_1^{x_1}))(\phi(p_2^{x_2}) + \phi(p_2^{x_2+2}) + \cdots \phi(p_2^{x_2})\phi(p_3) + \phi(p_3^2) + \cdots + \\
\phi(p_3^{x_3}) + 1) \cdots (\phi(p_{k-1}) + \phi(p_{k-1}^{x_k}) \cdots + \phi(p_{k-1}^{x_k+1}) + 1) = (p_1^{x_1} - 1)(p_2^{x_2} - p_2^2)p_3^{x_3} \cdots p_{k-1}^{x_{k-1}}\]

number of vertices adjacent to \(x_{p_1}\) but not adjacent to \(x_{p_1}^{x_1}p_2^{x_2} \cdots p_k^{x_k}\). But we see that \((p_1^{x_1} - 1)(p_2^{x_2} - p_2^2) \cdots (p_{k-1}^{x_{k-1}}) > p_2^{x_2}p_3^{x_3} \cdots p_k^{x_k}\) which shows that degree of \(x_{p_1}\) is strictly greater than the degree of \(\eta(x_{p_1})\) which is a contradiction.

**Subcase(2.1):** If possible let \(\sigma(x_{p_1}) = x_{p_1}^{m_1}p_2^{x_2} \cdots p_k^{x_k}\) where \(1 \leq m_1 < x_1\) but degree \((x_{p_1}) > \text{degree } (x_{p_2}) > \text{degree } (x_{p_1}^{m_1}p_2^{x_2} \cdots p_k^{x_k})\). [Second inequality follows exactly same way as the previous case ] Hence a contradiction.

**Case(2):** \(k = 2\) i.e \(n\) is of the form \(n = p^a q^b\) where \(p^a > q^b\). Now we have the following situations.

**Subcase(2.1):** \(a = 1\), hence we may assume that \(b \geq 2\) (because the case \(b = 1\) has already been dealt with). Now if possible let there exists an automorphism \(\sigma\) such that \(\sigma(x_{p}) = x_{pq^t}, 1 < t < b\). Now since \(x_q\) is not adjacent to \(x_p\), \(\sigma(x_q)\) should be some \(x_{qs}\) where \(s > t\). But this is impossible, because degree \((x_q) > \text{degree } (x_{qs})\).

**Subcase(2.2):** \(a > 1\). Now if possible let \(\sigma(x_{p}) = x_{p^m q^b}\) where \(m \leq a\) and \(n \leq b\). Let us assume that \(n = b\) and also assume that \(b > 1\) (the case \(n < b\) can be done by a similar argument used in the subcases 1.1 and 1.2). Then the number of vertices adjacent to \(x_{p^m q^b}\) but not adjacent to \(x_p\) is \(\phi(q) + \phi(q^2) + \cdots + \phi(q^b) = q^b - 1\). The number of vertices adjacent to \(x_p\) but not adjacent to \(x_{p^m q^b}\) is \((\phi(p^{m+1}) + \phi(p^{m+2}) + \cdots + \phi(p^a)) + \phi(q) + \phi(q^2) + \cdots + \phi(q^{b-1}) + 1) = (p^a - p^m)(q^{b-1})\) which can not be equal to \(q^b - 1\) if \(b > 1\).

So assume that \(b = 1\). Now if possible let \(\sigma(x_p) = x_{p^m q}\). Now by the same logic as subcase 2.1, \(\sigma(x_q)\) is of the form \(x_{p^s}\) where \(s > m\). Now the number of vertices that are adjacent to \(x_{p^s}\) but not adjacent to \(x_q\) is \(p^a - 1\) but the number of vertices adjacent to \(x_q\) but not adjacent to \(x_{p^s}\) is equal to \((q - 1)p^{s-1}\) which is not equal to \(p^a - 1\). Hence a contradiction and the proof is complete.

\[\square\]
Now we state our main lemma, which immediately implies the theorem.

**Lemma 5.** If \( \sigma \in \text{Aut}(P(\mathbb{Z}_n)) \), then \( \sigma(X_d) = X_d \) for all \( d(\neq 1, n) \) dividing \( n \).

**Proof.** We will illustrate the proof for \( n = p_1p_2p_3 \cdots p_k \). The general case is similar. Suppose if possible, \( \sigma(x_{d_1}) = x_{d_2} \) for some automorphism \( \sigma \) and \( d_1, d_2 \) two distinct non trivial divisors of \( n \). Then there exists some prime \( p \in \{p_1, p_2, \cdots p_k\} \) such that \( p \) divides \( d_2 \) but does not divide \( d_1 \). So by the definition of automorphism there exists an automorphism which sends \( p \) to a vertex say \( v \) adjacent to \( d_1 \). Note that \( v \neq p \). If \( p \) is greater than or equal to each prime divisors of \( v \) then we get a contradiction from Lemma 3. So there exists at least one prime divisor of \( v \) say \( q \) which is strictly greater than \( p \). Now there exists an automorphism, which sends \( q \) to a vertex \( v_1 \) adjacent to \( p \). Again note that \( v_1 \) is not equal to \( q \). And we proceed similarly as before to eventually reach a contradiction using Lemma 3.

\( \square \)
Chapter 3

On the power graph of the direct product of two groups

Here we investigate relationship of $\mathcal{P}(G_1 \times G_2)$ with $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$ for any two groups $G_1$ and $G_2$. There are three fundamental products of graphs, namely, cartesian product, direct product and strong product of graphs [58]. Here we show that, in general, $\mathcal{P}(G_1 \times G_2)$ is not isomorphic to either of these products of $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$. So we introduce a new product of two graphs $\Gamma_1$ and $\Gamma_2$, which we call generalized product of $\Gamma_1$ and $\Gamma_2$. Reason behind such naming is that each of the cartesian, direct and strong products is a special case of the generalized product of two graphs.

Our main theorem states that $\mathcal{P}(G_1 \times G_2)$ is isomorphic to a generalized product of the power graphs $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$.

3.1 Main results

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{Z}^\sharp = \mathbb{N} \cup \{0\}$. For any two integers $a$ and $b$, $AP(a, b)$ is the arithmetic progression with initial term $a$ and common difference $b$.

Let $\Gamma$ be a graph. Then we define a generalization on $\Gamma$ to be a function $W : A(\Gamma) \cup \Delta \to \mathbb{Z}^\sharp \times \mathbb{Z}^\sharp$, where $A(\Gamma)$ is the arc set of $\Gamma$ and $\Delta = \{(v, v) : v \text{ is a vertex of } \Gamma\}$. 

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Definition: Let \((G_1, W_1)\) and \((G_2, W_2)\) be two graphs equipped with two generalizations \(W_1, W_2\) respectively. Then the generalized product \(G_1 \times_W G_2\) is a graph with vertex set \(V(G_1) \times V(G_2)\) and \((g_1, g_2) \sim (g_1', g_2')\) if and only if the following two conditions hold simultaneously:

(i) \((g_1, g_2) \neq (g_1', g_2')\)

(ii) \(AP(W_1(g_1, g_1')) \cap AP(W_2(g_2, g_2')) \cap N \neq \emptyset\) or \(AP(W_1(g_1, g_1')) \cap AP(W_2(g_2, g_2')) \cap N \neq \emptyset\)

We shall show below that the generalized product of two graphs \(G_1\) and \(G_2\) generalizes the aforesaid products on graphs, namely the direct product, cartesian product and normal product. First, we recall the definitions of these products.

Definition: Let \(G_1, G_2\) be two graphs. Then:

(i) The direct product \(G_1 \times G_2\) is defined as follows:
\[V(G_1 \times G_2) = V(G_1) \times V(G_2)\] and \((g_1, g_2) \sim (g_1', g_2')\) if and only if \(g_1 \sim g_1'\) and \(g_2 \sim g_2'\)

(ii) The cartesian product \(G_1 \boxtimes G_2\) is defined follows:
\[V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)\] and \((g_1, g_2) \sim (g_1', g_2')\) if and only if \(g_1 = g_1'\) and \(g_2 \sim g_2'\) or \(g_1 \sim g_1'\) and \(g_2 = g_2'\)

(iii) The normal product \(G_1 * G_2\) is defined follows:
\[V(G_1 * G_2) = V(G_1) \times V(G_2)\] and \((g_1, g_2) \sim (g_1', g_2')\) if and only if \(g_1 \sim g_1'\) and \(g_2 \sim g_2'\) or \(g_1 = g_1'\) and \(g_2 \sim g_2'\) or \(g_1 \sim g_1'\) and \(g_2 = g_2'\)

Theorem 6. Each of the products, defined above is a particular generalized product.

Proof. Let \(G_1\) and \(G_2\) be two graphs.

(i) Take \(W_1 : A(G_1) \cup \Delta \to \mathbb{Z}^2 \times \mathbb{Z}^2\) to be \(W_1(x, y) = (1, 1)\) for \(x \neq y\) and \(W_1(x, x) = (0, 0)\) for all \(x \in V(G_1)\) and take \(W_2\) similarly. Then it is not difficult to verify that \(G_1 \times_W G_2 = G_1 \times G_2\).

(ii) Take \(W_1 : A(G_1) \cup \Delta \to \mathbb{Z}^2 \times \mathbb{Z}^2\) to be \(W_1(x, y) = (1, 0)\) for \(x \neq y\) and \(W_1(x, x) = (1, 1)\) for all \(x \in V(G_1)\) and take \(W_2(x, y) = (2, 0)\) for \(x \neq y\) and \(W_2(x, x) = (1, 1)\) for all \(x \in V(G_2)\). Then \(G_1 \times_W G_2 = G_1 \boxtimes G_2\).
(iii) Take $W_1 : A(G_1) \cup \triangle \to \mathbb{Z}^2 \times \mathbb{Z}^2$ to be $W_1(x, y) = (1, 0)$ for $x \neq y$ and $W_1(x, x) = (1, 1)$ for all $x \in V(G_1)$ and take $W_2(x, y) = (1, 0)$ for $x \neq y$ and $W_2(x, x) = (1, 1)$ for all $x \in V(G_2)$. Then $G_1 \times W_2 G_2 = G_1 * G_2$. \hfill \Box

Now we try to uncover the relationship of the power graphs $P(G_1)$ and $P(G_2)$ with $P(G_1 \times G_2)$ for any two groups $G_1$ and $G_2$.

**Theorem 7.** Let $G_1$ and $G_2$ be two proper finite groups. Then $P(G_1 \times G_2)$ is never isomorphic to $P(G_1) \boxtimes P(G_2)$

**Proof.** Suppose, on the contrary that the two graphs, stated in the theorem are isomorphic. Let $e_{G_i}$ be the identity of the group $G_i$. Let $g_1, g_2$ be two elements of $G_1$ and $G_2$ respectively, such that $g_i \neq e_{G_i}$ for $i = 1, 2$. Then by the definition of the cartesian product of graphs, we see that $(e_{G_1}, e_{G_2})$ is not adjacent to $(g_1, g_2)$ in $P(G_1) \boxtimes P(G_2)$, whereas $(e_{G_1}, e_{G_2}) \sim (g_1, g_2)$ in $P(G_1 \times G_2)$. A contradiction! \hfill \Box

**Example :** In this example we show that $P(G_1 \times G_2)$ is not generally isomorphic to either of the direct or normal products of $P(G_1)$ and $P(G_2)$. Take, for example $G_1 = G_2 = \mathbb{Z}_2$. Then $P(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has precisely three edges, each edge emanating from the identity of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and connects the remaining three vertices, whereas $P(\mathbb{Z}_2) * P(\mathbb{Z}_2)$ is the complete graph $K_4$ and $P(\mathbb{Z}_2) \times P(\mathbb{Z}_2)$ is a graph with precisely two edges.

Now, we prove the main theorem of this chapter. But before that, we state the following simple lemma.

**Lemma 8.** Let $G$ be a group and $a, b \in G$. Let $n$ be the smallest positive integer such that $a^n = b$. Then $\{m \in \mathbb{N} : a^m = b\} = AP(n, o(a))$.

**Theorem 9.** For two groups $G_1$ and $G_2$, $P(G_1 \times G_2) = P(G_1) \times W_1 P(G_2)$ for some choice of generalizations $W_1$ and $W_2$ on $P(G_1)$ and $P(G_2)$ respectively.

**Proof.** For a group $G$, first we specify the choice of the generalization $W$ as $W(a, b) = (m, o(a))$, where $m$ is the smallest positive integer for which $a^m = b$; provided such
an $m$ exists, and otherwise set $W(a,b) = (0,0)$. Now we show that with respect to this particular generalization $W$, $\mathcal{P}(G_1 \times G_2) = \mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$. Let $(g_1, g_2) \sim (g'_1, g'_2)$ in $\mathcal{P}(G_1 \times G_2)$. This implies that $(g_1, g_2)^m = (g'_1, g'_2)$ or $(g_1, g_2) = (g'_1, g'_2)^n$ for some $m, n > 0$. Consider the first case i.e. $(g_1, g_2)^m = (g'_1, g'_2)$. This implies that $g_1^m = g'_1$ and $g_2^n = g'_2$. Combining this and the previous lemma we get that $AP(W_1(g_1, g'_1)) \cap AP(W_2(g_2, g'_2)) \cap N \neq \emptyset$ and as a consequence $(g_1, g_2) \sim (g'_1, g'_2)$ in $\mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$. Similarly we can show the converse i.e. if $(g_1, g_2) \sim (g'_1, g'_2)$ in $\mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$, $(g_1, g_2) \sim (g'_1, g'_2)$ in $\mathcal{P}(G_1 \times G_2)$. \hfill $\square$
Chapter 4

On the power graphs which are Cayley graphs of some groups

Cayley graph is a widely studied graph, associated with finite groups. Let $G$ be a group and $C$ be a subset of $G \setminus \{e\}$. Then the directed Cayley graph $\overrightarrow{X(G,C)}$ is defined to be a directed graph with vertex set $G$ and arc set $\{(g,h) : g^{-1}h \in C\}$. If in addition, $C$ is an inverse closed subset, then the undirected Cayley graph $\overline{X(G,C)}$ is defined to be the underlying undirected graph of $\overrightarrow{X(G,C)}$. In the excellent survey [60] on power graph of groups and semigroups, Abawajy et.al. proposed the following problem: Describe all directed and undirected power graphs of groups and semigroups that can be represented as Cayley graphs and, respectively, undirected Cayley graphs of groups or semigroups.

Here we describe all groups whose power graphs can be represented as Cayley graphs of some groups, both directed and undirected.

4.1 Main result

An undirected graph $\Gamma$ is called vertex transitive if its automorphism group acts transitively on the vertex set of $\Gamma$. Every Cayley graph is vertex transitive and every vertex transitive undirected graph is regular. If $\Gamma$ is a directed vertex transitive graph then the in-degree of all the vertices are equal and the same holds for out-degree. Thus we have:
Theorem 10. (i) Let $G$ be a finite group. Then $\mathcal{P}(G)$ is a Cayley graph if and only if $G$ is a cyclic $p$–group.

(ii) There is no finite group $G$, whose directed power graph is a Cayley graph.

Proof. Proof of (i): Suppose $G$ is a cyclic $p$–group, then the undirected power graph $\mathcal{P}(G)$ is complete and hence is a Cayley graph. Conversely, assume that $\mathcal{P}(G)$ is a Cayley graph of some group. then $\mathcal{P}(G)$ is vertex transitive and so regular. If the order of $G$ is $n$ and $e$ is the identity element of $G$, then degree of $e$ is $n – 1$ in $\mathcal{P}(G)$ and hence degree of $v$ is $n – 1$ for every $v$ in $G$. It follows that $\mathcal{P}(G)$ is complete. Therefore $G$ is a cyclic $p$-group.

Proof of (ii): If possible, on the contrary, assume that $\overrightarrow{\mathcal{P}(G)}$ is a directed Cayley graph for some group $G$ of order $n$. Then $\overrightarrow{\mathcal{P}(G)}$ is vertex transitive and hence in-degree of all the vertices are equal and the same holds for out-degree. Now in $\overrightarrow{\mathcal{P}(G)}$, out-degree of $e$ is 0 and in-degree of $e$ is $n – 1$. Hence it follows that the out-degree of each vertex is 0 and the in-degree of each vertex is $n – 1$, which is impossible. Therefore, there is no finite group $G$ such that the directed power graph $\overrightarrow{\mathcal{P}(G)}$ is Cayley graph of some group.

$\square$
Part II

Some problems in combinatorial algebraic topology
Chapter 5

Introduction to part-II

In this chapter, we recall some topological preliminaries needed for the rest of this thesis. We conclude this chapter with a brief summary of the results obtained in this part of the thesis.

5.1 Topological preliminaries

The most fundamental object of study in combinatorial algebraic topology is *abstract simplicial complex*. An abstract simplicial complex (or simply a simplicial complex) $\Delta$ is defined to be an ordered pair $(V, \mathcal{F})$, where $V$ is any finite set (we only consider finite case here) and $\mathcal{F} \subset \mathcal{P}(V)$, the power set of $V$ satisfying the condition: if $\sigma \in \mathcal{F}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{F}$.

The non-empty elements of $\mathcal{F}$ are called the *faces* of the simplicial complex $\Delta$. *Dimension* of the face $\sigma$, denoted by $\text{dim}(\sigma)$ is defined to be $|\sigma| - 1$, where $|\sigma|$ is the cardinality of $\sigma$. The zero-dimensional faces of $\Delta$ are called *vertices* of $\Delta$. A $k-$dimensional face of $\Delta$ is also called a $k-$simplex of $\Delta$. The *dimension of a simplicial complex* is defined to be the maximum of the dimensions of its simplices. Note that, graphs are one dimensional simplicial complexes. The *Euler Characteristic* of a simplicial complex $\Delta$, denoted by $\chi(\Delta)$ is defined to be the summation $\sum_{\sigma \in \Delta} (-1)^{\text{dim}(\sigma)}$.

Suppose that $\Delta$ is an abstract simplicial complex. Let $\tau, \sigma$ are two faces of $\Delta$, such that $\tau \subset \sigma$ and $\sigma$ is a maximal face of $\Delta$ and no other maximal face of $\Delta$ contains $\tau$. then $\tau$ is called a *free face* of $\Delta$. A *simplicial collapse* of $\Delta$ is the removal of all simplices $\gamma$ such that $\tau \subseteq \gamma \subseteq \sigma$, where $\tau$ is a free face. If additionally we have
\[ \dim(\tau) = \dim(\sigma) - 1, \] then this is called an \textit{elementary collapse}. A simplicial complex that has a sequence of collapses leading to a point is called \textit{collapsible}.

Let \( \sigma \) be a \( k \)-simplex of \( \Delta \). Two orderings of its vertex set are equivalent if they differ by an even permutation. If \( \dim(\sigma) > 0 \) then the orderings of the vertices of \( \sigma \) fall into two equivalence classes. Each class is called an \textit{orientation} of \( \sigma \). An \textit{oriented simplex} is a simplex \( \sigma \) together with an orientation of \( \sigma \). Let \( a_0, a_1, \ldots, a_k \) be an ordering of the vertices of \( \sigma \). then we shall use the symbol \([a_0, a_1, \ldots, a_k]\) to denote the oriented simplex. Suppose that \( b_0, b_1, \ldots, b_k \) be another vertex-ordering, which differs from the previous ordering by an odd permutation, then we write \([a_0, a_1, \ldots, a_k] = -[b_0, b_1, \ldots, b_k]\). Note that vertices have only one orientation. Let \( C_k(\Delta) \) be the free \( \mathbb{Z} \)-Module generated by all oriented \( k \)-simplices of \( \Delta \). Now for \( k > 0 \), we define a homomorphism \( \partial_k : C_k(\Delta) \to C_{k-1}(\Delta) \), called the \textit{boundary operator} as follows:

Let \( \sigma = [v_0, v_1, \ldots, v_k] \) be an oriented simplex. Then \( \partial_k(\sigma) = \sum_{i=0}^{k} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k] \), where \( \hat{v}_i \) means that the vertex \( v_i \) has been omitted.

Now it is well known and can be checked easily that \( \partial_{k-1} \circ \partial_k \equiv 0 \) for all \( k \geq 1 \). This in fact implies that \( \text{Im}(\partial_{k+1}) \subset \text{ker}(\partial_k) \). Then the \( k \)-th \textit{homology group} of \( \Delta \), denoted by \( H_k(\Delta) \) is defined to be \( \text{ker}(\partial_k)/\text{Im}(\partial_{k+1}) \).

Suppose that \( X \) be a simplicial complex and \( S \) be a subcomplex of \( X \). Then \( \partial_k \) induces a map \( \hat{\partial}_k : C_k(X)/C_k(S) \to C_{k-1}(X)/C_{k-1}(S) \). We also have \( \partial_{k-1} \circ \hat{\partial}_k \equiv 0 \). And similarly as before, the \( k \)-th \textit{relative homology group of the pair} \( (X, S) \), denoted by \( H_k(X, S) \) is defined to be \( \text{ker}(\hat{\partial}_k)/\text{Im}(\hat{\partial}_{k+1}) \).

A \textit{combinatorial} \( d \)-\textit{manifold} (or sometimes called a \textit{pseudo manifold}) is a \( d \)-dimensional simplicial complex \( M \), such that the following conditions hold:

1. \( M \) is \textit{homogeneous}, i.e. each simplex of \( M \) is a face of some \( d \)-simplex;

2. \( M \) is \textit{unramified}, i.e. each \((d-1)\)-simplex of \( M \) is a face of atmost two \( d \)-simplices;

3. \( M \) is \textit{strongly connected}, i.e. for any two \( d \)-simplices \( \Delta_1 \) and \( \Delta_2 \), there exists a
sequence of $d$–simplices starting and ending with $\Delta_1$ and $\Delta_2$ respectively with two neighboring simplices in the sequence share a common $(d - 1)$–face.

A combinatorial manifold $M$ is said to be orientable, if its top dimensional homology group is nontrivial, otherwise $M$ is said to be non-orientable.

5.2 Objective of part-II

The notion of forests and their roots in higher dimensions was introduced by Olivier Bernardi and Caroline Klivans in [61] to obtain a generalization of the “Matrix Forest Theorem” in higher dimension. In fact, they proved that, for a $d$–dimensional simplicial complex $G$,

$$\sum_{(F,R)} |H_{d-1}(F,R)|^2 x^{|R|} = det(L_G + xId),$$

where the summation runs over all rooted forests of $G$ and $Id$ is the identity matrix of dimension $|G_{d-1}|$, and $L_G$ is the Laplacian matrix. A combinatorial interpretation of the factor $|H_{d-1}(F,R)|$ for a general simplicial complex $G$ is a challenging problem. (See [61]). In Chapter 6 we prove that the relative homology $H_{d-1}(F,R)$ is a 2–group for any rooted forest $(F,R)$ in a triangulated closed $d$–manifold $M^d$. We also give a combinatorial interpretation of the cardinality of the group $H_{d-1}(F,R)$, which answers an open question of Olivier Bernardi and Caroline Klivans for any closed manifold. We have shown that for any rooted forest $(F,R)$ in a closed manifold $M^d$, the forest $F$ simplicially collapses onto its root $R$ iff $H_{d-1}(F,R)$ is trivial.

In Chapter 7, we give a purely combinatorial proof of a Ryser-type formula for determinants inspired by multivariate finite operator calculus. This argument also includes a combinatorial proof of an interesting identity about Stirling number of second kind. Also we give a topological proof of Ryser’s formula for permanents.
Chapter 6

On the rooted forests in triangulated closed manifolds

The notion of forests and their roots in higher dimensions was introduced by Olivier Bernardi and Caroline Klivans in [61] to obtain a generalization of the ”Matrix Forest Theorem” in higher dimension. In fact, they proved that, for a $d$-dimensional simplicial complex $G$,

$$
\sum_{(F,R)} |H_{d-1}(F,R)|^2 x^{|R|} = det(L_G + xId),
$$

where the summation runs over all rooted forests of $G$ and $Id$ is the identity matrix of dimension $|G_{d-1}|$, and $L_G$ is the Laplacian matrix. We call this theorem the “Bernardi-Klivans Forest Theorem”. A combinatorial interpretation of the factor $|H_{d-1}(F,R)|$ for a general simplicial complex $G$ is a challenging problem.(See [61]). In this chapter, we give a combinatorial interpretation of $|H_{d-1}(F,R)|$ for any closed $d$-manifold $M^d$ in terms of their “Poincare-dual graphs” (details later). We have also addressed the following problem. In graphs, a forest simplicially collapses onto any of its roots (in graph case, a root consists of a finite number of vertices, each belongs to its mother component). So it is tempting to conjecture that, in higher dimensional case also, a forest simplicially collapses onto any of its roots. But unfortunately, this is false in general. In fact a non-orientable, closed, triangulated manifold can not collapse onto any of its roots. In Section 6.4, we have proved that, if $(F, R)$ is a rooted
forest in a closed, connected, triangulated manifold $M^d$, then $F$ simplicially collapses onto its root $R$ if and only if $H_{d-1}(F, R)$ is trivial. As a corollary of this theorem we have proved that a "punctured" closed, connected, orientable triangulated manifold simplicially collapses onto any of its roots. In the course of the proof of our main theorem, we prove a lemma (Lemma-12), which partially answer a question of Olivier Bernardi and Caroline Klivans. In [61], Olivier Bernardi and Caroline Klivans posed a problem of finding a sign-reversing involution to prove that the sum of the signs of all fitting orientations (definition later) of a non-rooted forest is zero. In Lemma 12, we have proved that, the sum of the signs of all fitting orientations of a simplicial complex containing a ‘geometric cycle’ is zero by a sign-reversing involution argument. The motivation for considering ‘geometric cycle’ is the following. In one dimensional simplicial complexes (i.e. graphs), a forest does not contain any cycle. In higher dimensional simplicial complex, the analogue of the graphical cycle is closed orientable manifold or more generally ‘geometric cycle’. See the next section for the definitions. We have used the technique of Discrete Morse Theory to prove some of the important theorems in this chapter. We refer to [69], [70], [71] for necessary background in Discrete Morse Theory.

6.1 Basic definitions

Let $k \geq 1$. The $k$–th incidence matrix of a simplicial complex $G$ is the matrix $\partial_k(G)$ (for short $\partial_k$) defined as follows:

The rows of $\partial_k$ are indexed by the $(k-1)$–faces and the columns are indexed by $k$–faces.

The entry $\partial_k[r, f]$ corresponding to a $(k-1)$–face $r$ and a $k$–face $f$ is $(-1)^j$ if $f = \{v_0, v_1, \cdots, v_k\}$ with $v_0 < v_1 < \cdots < v_k$ and $r = f \setminus \{v_j\}$ for some $j$ and 0 otherwise. Moreover, we will order the rows and columns of $\partial_k(G)$ according to the lexicographic order of the faces.

Let $G$ be a $d$–dimensional complex and let $F$ be a subset of the $d$–faces and
$R$ be a subset of the $(d-1)$-faces. We say that $F$ is a forest of $G$ if the corresponding columns of $\partial_d(G)$ are linearly independent. We say that $R$ is a root of $G$ if the rows of $\partial_d(G)$ corresponding to the faces in $\bar{R} = G_{d-1} \setminus R$ form a basis of the rows. A rooted forest of $G$ is a pair $(F, R)$, where $F$ is a forest of $G$ and $R$ is a root of the simplicial complex generated by $F$, which is also denoted by $F$. For the root $R$ of $F$ the root complex $R$ is defined to be $R \cup F_{d-2}$.

Let $R$ be a ring, and $C_k(G,R)$ denote the free $R$-module with basis consisting of $k$-faces of $G$. Note that $\partial_k(G)$ can be interpreted as the matrix of a linear map from $C_k(G,R)$ to $C_{k-1}(G,R)$. This map is called the boundary map in simplicial topology. A $k$-cycle of $G$ is a non-zero element of $C_k(G,R)$ in the kernel of $\partial_k$. From now on, we will take $R = \mathbb{Z}$, the ring of integers. Because we are working with $\mathbb{Z}$-coefficient, a $k$-cycle $C$ of $G$ can be re-interpreted as a collection of $k$-faces of $G$, namely $\{f_1, f_2, \ldots, f_m\}$ such that there exist non-zero integers $\{c_1, c_2, \ldots, c_m\}$ for which $\partial_k(\sum_{i=1}^m c_i f_i) = 0$. In other words, a $k$-cycle $C$ can be thought of as a multiset $\tilde{C} = \{f_1, f_1, \ldots, f_1, f_2, f_2, \ldots, f_2, \ldots, f_m, f_m, \ldots, f_m\}$, where, for each $i$, $f_i$ has multiplicity $|c_i|$. The sign of $f_i$ in $C$ is defined to be $\frac{c_i}{|c_i|}$ and this is denoted by $\langle [C], f_i \rangle$. Note that $\langle [C], f_i \rangle \in \{1, -1\}$. In this terminology, the cycle condition can be restated as $\partial_k(\sum_{f \in \tilde{C}} \langle [C], f \rangle f) = 0$. A geometric $d$-cycle is a $d$-dimensional simplicial complex, whose $d$-faces form a cycle and each $(d-1)$-face is contained in precisely two $d$-faces.

Now we reproduce the definition of the fitting orientations and their signs from [61]. Let $F$ be a $d$-dimensional simplicial complex. The set of all $d$-faces of $F$ is also denoted by $F$. Let $R$ be a subset of $(d-1)$-faces such that $|F| = |\bar{R}|$, where $\bar{R} = F_{d-1} \setminus R$ and where $F_{d-1}$ is the $(d-1)$-skeleton of $F$. A fitting orientation (or pairing) of $(F, R)$ is a bijection $\phi$ between $\bar{R}$ and $F$, such that each face $r \in \bar{R}$ is mapped to a face $f \in F$ containing $r$. For any fitting orientation of $(F, R)$, i.e. any bijection $\phi : \bar{R} \to F$, we associate the unique permutation $\pi_\phi \in S_{|F|}$ such that $\phi$ maps the $i$-th face in $\bar{R}$ in the lexicographic order to the $\pi_\phi(i)$-th
face in $F$ in the lexicographic order. Now define the sign of $\phi$, denoted by $\Lambda(\phi)$ as $sgn(\pi_\phi)\prod_{r \in R} \partial_d[r, \phi(r)]$. The notion of fitting orientation was introduced to give a combinatorial interpretation (in terms of vector fields) of the homological quantity $|H_{d-1}(F, R)|$, occurred in the “Matrix forest theorem” for higher dimensions (See the introduction). In fact in [61], the authors have shown that $|\sum_\phi \Lambda(\phi)| = |H_{d-1}(F, R)|$, where $\phi$ runs over all fitting orientations of $(F, R)$.

6.2 Rooted forests in orientable manifolds

In this section, we prove that a ‘punctured’ closed, connected, orientable triangulated manifold is simple homotopy equivalent to any of its roots. To be more precise, let $M^d$ be a closed, connected, orientable, triangulated $d$–manifold. Note that, in the triangulation of $M^d$, each $(d-1)$–simplex is contained in exactly two $d$–simplices. Let $F$ be the $d$–dimensional simplicial complex formed by removing a finite, nonempty collection of $d$–simplices from $M^d$. Clearly $F$ is a forest. We show that $F$ simplicially collapses onto any of its roots.

**Theorem 11.** Let $F$ be a punctured, closed, connected, orientable triangulated manifold and $R$ be a root of $F$. Then $F$ simplicially collapses onto $R$.

Before proving this, we need several lemmas which are of independent interest.

**Lemma 12.** If $F$, a $d$–dimensional simplicial complex contains a geometric $d$–cycle as a subcomplex and let $R$ be a subset of $(d-1)$–faces such that $|F| = |\bar{R}|$, then $\sum_\phi \Lambda(\phi) = 0$, where $\phi$ runs over all the fitting orientations of $(F, R)$.

**Proof.** Let $C$ be the geometric $d$–cycle. Fix a fitting orientation $\phi$. We claim that $C$ contains some $d$–faces $\{f_1, f_2, \cdots, f_k\}$ and $\bar{R}$ contains some $(d-1)$–faces $\{r_1, r_2, \cdots, r_k\}$ such that the following holds:

1. $f'_i$s and $r'_i$s can be rearranged in a sequence of simplices
\[ r_{i_1} \rightarrow f_{i_1} \rightarrow r_{i_2} \rightarrow f_{i_2} \rightarrow \cdots \rightarrow r_{i_k} \rightarrow f_{i_k} \rightarrow r_{i_1} \] such that \( r_{i_m}, r_{i_{m+1}} \in \partial f_{i_m} \) and \( \phi(r_{i_m}) = f_{i_m} \) for \( m \neq k \) and \( \phi(r_{i_k}) = f_{i_k} \) and \( \{ r_{i_k}, r_{i_1} \} \in \partial f_{i_k} \) for \( m = k \).

Here the direction of an arrow from an \( r \) to an \( f \) indicates that \( \phi(r) = f \) and the direction of an arrow from an \( f \) to an \( r \) indicates that \( r \subset f \).

2. For \( m \neq 1, \partial[r_{i_m}, f_{i_m}] \cdot \langle [C], f_{i_m} \rangle + \partial[r_{i_m}, f_{i_{m-1}}] \cdot \langle [C], f_{i_{m-1}} \rangle = 0 \)

and for \( m = 1, \partial[r_{i_1}, f_{i_1}] \cdot \langle [C], f_{i_1} \rangle + \partial[r_{i_1}, f_{i_k}] \cdot \langle [C], f_{i_k} \rangle = 0 \).

The sequence, defined in condition 1 is called a ‘cycle’ and if in addition, it satisfies 2, the cycle is called an ‘alternating sign cycle’ (or ASC for short). So our claim is that, \( \phi \) has an ASC in \( C \). To prove this, take a \( d \)--simplex, say \( f_1 \in C \). Let \( r = \phi^{-1}(f_1) \). Now the boundary-vanishing condition of the geometric cycle \( C \) ensures that, there exists a (unique) face \( f_2 \in C \) such that \( f_2 \neq f_1 \) and \( r_1 \) is face of \( f_2 \) and \( \langle [C], f_1 \rangle \partial[r_1, f_1] = -\langle [C], f_2 \rangle \partial[r_1, f_2] \). Let \( r_2 = \phi^{-1}(f_2) \). Again by the boundary-vanishing condition of the cycle \( C \) ensures that, there exists a face \( f_3 \in C \) such that \( f_3 \neq f_2 \) and \( r_2 \) is a face of \( f_3 \) and \( \langle [C], f_2 \rangle \partial[r_2, f_2] = -\langle [C], f_3 \rangle \partial[r_2, f_3] \). Note that, \( f_3 \neq f_1 \). Again let \( r_3 = \phi^{-1}(f_3) \) and proceed further. This \( r_3 \) gives rise to another face \( f_4 \) satisfying the ‘alternating sign’ condition. If \( f_4 \) equals any of the faces \( \{ f_1, f_2, f_3 \} \) in this case the only possibility is \( f_1 \), we stop; and continue otherwise. Note that we can not continue forever. We must stop at a face say \( f_m \) which coincides with a face, previously encountered, say \( f_1 \) without loss of generality. And thus we get an ASC.

We can order all ASC’s of \( \phi \) as follows:

To each ASC, we associate the set of all \( d \)--dimensional simplices involved in that particular ASC and write down the elements of the set one by one in their increasing lexicographic order to make a word. Thus, to each ASC we associate a word, whose letters come from the set of all \( d \)--simplices of \( C \). Now order the words in the dictionary order and order the ASC’s in the order of their associated words.

Now we are ready to define a sign-reversing involution on the set of all fitting orientations \( \phi \) of \( (F, R) \).
Take a $\phi$. Let $A$ be the smallest $ASC$ (in the dictionary order, we just defined) in $C$. Now reverse the direction of each arrow of $A$ and keep the other paired simplices unturned (see the figure below).

The arrows of $A$, getting reversed give rise to another pairing (or fitting orientation) say $\tilde{\phi}$ and an $ASC$, say $\tilde{A}$. This is clearly an involution. See the following diagram.

In the above diagram, the white vertices indicate the $(d - 1)$–faces and the black vertices indicate the $d$–faces. An arrow from a white vertex to a black vertex indicates that the two vertices are paired and an arrow from a black vertex to a white vertex indicates a containment. Note that, the above configurations are forbidden in a fitting orientation of a geometric cycle. The left figure is not admissible, because at the rightmost black vertex, the bijectivity of the fitting orientation is violated and
the right figure is not admissible, because the upper rightmost white vertex has total degree 3, which is not possible in a geometric cycle. Hence, reversing the arrows does not give rise to any new ASC (in fact any cycle) in C except the inverted one. We prove that, this is in fact sign-reversing.

Let $S$ and $T$ be the set of all $(d - 1)$ and $d-$simplices involved in $A$ (or $\tilde{A}$) respectively.

Now we have

$$\text{sgn}(\pi_\phi)(\prod_{f \in T}([C], f))\prod_{r \in \overline{R}} \partial_d[r, \phi(r)]$$

$$= \text{sgn}(\pi_\phi)(\prod_{f \in T}([C], f)) \prod_{r \in \overline{R} \setminus S} \partial_d[r, \phi(r)] \prod_{r \in S} \partial_d[r, \phi(r)]$$

$$= (\prod_{r \in \overline{R} \setminus S} \partial_d[r, \phi(r)]) \text{sgn}(\pi_\phi)(\prod_{r \in S} \partial_d[r, \phi(r)]([C], \phi(r)))$$

$$= (\prod_{r \in \overline{R} \setminus S} \partial_d[r, \phi(r)]) \text{sgn}(c_\partial^\circ \pi_\phi)(\prod_{r \in S} (-\partial_d[r, \tilde{\phi}(r)]([C], \tilde{\phi}(r))))$$

$[c_\partial]$ is a permutation-cycle of length $|T|$

$$= (-1)^{|T| - 1}(\prod_{r \in \overline{R} \setminus S} \partial_d[r, \tilde{\phi}(r)]) \text{sgn}(\pi_\phi)(\prod_{r \in S} \partial_d[r, \tilde{\phi}(r)]([C], \tilde{\phi}(r)))$$

$[\text{for } r \in \overline{R} \setminus S, \phi(r) = \tilde{\phi}(r)]$

$$= -\prod_{r \in \overline{R} \setminus S} \partial_d[r, \tilde{\phi}(r)] \text{sgn}(\pi_\phi)(\prod_{r \in S} \partial_d[r, \tilde{\phi}(r)]([C], \tilde{\phi}(r)))$$

$$= -\text{sgn}(\pi_\phi)(\prod_{f \in T}([C], f))(\prod_{r \in \overline{R}} \partial_d[r, \tilde{\phi}(r)]).$$

So, $\text{sgn}(\pi_\phi)(\prod_{r \in \overline{R}} \partial_d[r, \phi(r)]) = -\text{sgn}(\pi_\phi)(\prod_{r \in \overline{R}} \partial_d[r, \tilde{\phi}(r)]).$

\[ \square \]

Here we briefly summarize some basics of discrete Morse theory, that will be used subsequently. Let us begin with the following essential definitions.

**Definition:** A discrete Morse function $f$ on a simplicial complex $X$ is a real valued function on the set of simplices of $X$ such that for each simplex $\beta$, atmost one face $\alpha$ of $\beta$ has a value greater than or equal to that of $\beta$ and atmost one coface $\gamma$ of $\beta$ has a value less than or equal to that of $\beta$.

A critical simplex $\beta$ of a discrete Morse function $f$ on $X$ is a simplex for which no face $\alpha$ of $\beta$ has a value greater than or equal to that of $\beta$ and no coface $\gamma$ of $\beta$ has a value less than or equal to that of $\beta$. 

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The following theorem, often said to be the fundamental theorem of discrete Morse theory is due to Robin Forman [69].

**Theorem 13.** Let $X$ be a simplicial complex with a discrete Morse function $f$. Then $X$ is homotopy equivalent to a cell complex containing the same number of cells of a given dimension as there are critical simplices of $f$ of that dimension.

Now we define the notion of a discrete vector field (or simply a pairing) on a simplicial complex $X$.

**Definition:** A discrete vector field (or a pairing) $\phi$ on a simplicial complex $X$ is a set containing pairs of simplices $\{\alpha, \beta\}$ of $X$, such that $\text{dim}(\alpha)$ and $\text{dim}(\beta)$ differ by one and for each simplex $\gamma$ of $X$, there is atmost one pair in $\phi$ containing $\gamma$.

Note that a fitting orientation is a special kind of discrete vector field.

**Definition:** A pairing $\phi$ is cyclic if there exists a cycle i.e. a sequence of simplices $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n, \alpha_{n+1}$ where $n > 1$ such that the following conditions hold:

1. $\alpha_1 = \alpha_{n+1}$
2. Both $\alpha_i, \alpha_{i+1} \in \partial \beta_i$, but $\alpha_i \neq \alpha_{i+1}$ and $\{\alpha_i, \beta_i\} \in \phi$ for all $i = 1, 2, \ldots, n$

A pairing $\psi$ is called acyclic if it is not cyclic.

A standard result of discrete Morse theory [69] states that a discrete Morse function $f$ on a simplicial complex $X$ gives rise to an acyclic pairing $\phi_f$, where $\{\alpha, \beta\} \in \phi_f$ if and only if $\alpha \in \partial \beta$ and $f(\alpha) \geq f(\beta)$. The simplices of $X$ not occurring in any pair in $\phi_f$ are precisely the critical simplices of $f$. $\phi_f$ is said to be the discrete gradient vector field of $f$.

Conversely, if a pairing $\phi$ is acyclic then $\phi$ is a discrete gradient vector field of some discrete Morse function on $X$.

Two discrete Morse functions are said to be equivalent if their corresponding discrete gradient vector fields are the same.

At this stage, we reproduce some terminology from [71]. The Hasse diagram of a simplicial complex $X$ is a directed acyclic graph on the poset of simplices of $X$,
ordered by the relation $\alpha < \beta$ if and only if $\alpha \in \partial \beta$. Given the Hasse diagram $H$ of $X$ and a pairing $\phi$ of simplices, we can construct a modified directed graph $\tilde{H}$ by flipping the edge between any two simplices that appear as a pair in $\phi$. We call this modified graph ‘modified Hasse diagram’. For details, see [71].

Now we state a fundamental theorem, proved in [71].

**Theorem 14.** ([71], 3.14): A pairing $\phi$ is acyclic if and only if the corresponding modified Hasse diagram $\tilde{H}$ is acyclic.

With this background in hand, let us prove the following crucial lemma.

**Lemma 15.** Let $F$ be a forest as described in the theorem 11 and $R$ be a root of $F$. Then $|\sum_{\phi} \Lambda(\phi)| = |H_{d-1}(F, R)|$, where $\phi$ runs over all distinct discrete gradient vector fields, whose set of all critical simplices is precisely the root complex $R$.

**Proof.** In [61], it was proved that for any rooted forest, $|\sum_{\phi} \Lambda(\phi)| = |H_{d-1}(F, R)|$, where $\phi$ runs over all fitting orientations of $(F, R)$. Now the crucial observation is that, in the left hand side of the above equation, the cyclic pairings (i.e. fitting orientations containing a cycle) have no contribution as they cancel each other. Orientability implies that any cycle is an ASC, so the previous argument (Lemma 12) applies and the surviving terms correspond to precisely the acyclic pairings, i.e. discrete gradient vector fields with the prescribed properties. \hfill \Box

**Corollary 16.** Let $(F, R)$ be a rooted forest as in the previous lemma. Then the number of non-equivalent discrete morse functions on $F$ with $R$ as the set of critical simplices is at least $|H_{d-1}(F, R)|$.

**Corollary 17.** Let $(F, R)$ be a rooted forest as in the Theorem 11. Then $F$ has at least one discrete morse function with the property in Corollary 16.

**Proof.** Since $(F, R)$ is a rooted forest, $\det(\partial_{R, F}) \neq 0$, where $\partial_{R, F}$ is the submatrix of $\partial_d(G)$ obtained by keeping the rows corresponding to $R$ and the columns corresponding to $F$ ([61], Lemma 14). Again $|\det(\partial_{R, F})| = |H_{d-1}(F, R)|$ ([61], Lemma 17). So
we have $|H_{d-1}(F,R)| \neq 0$. Hence, by Corollary 16 we get that $F$ has at least one discrete morse function with the prescribed properties.

Now we are in a position to prove Theorem 11.

Proof. By Corollary 17, there exists at least one acyclic pairing $\phi$ of $(F,R)$. Let $\widetilde{H}$ be the modified Hasse diagram of the simplices of $F$ associated to the pairing $\phi$. Hence by Theorem 14, $\widetilde{H}$ is acyclic. Let $\widetilde{H}_1$ be the subgraph of $\widetilde{H}$, induced by the vertices of the face poset of $F$ corresponding to all $d$–faces of $F$ and all $(d-1)$–faces of $\tilde{R}$. Since $\widetilde{H}$ is acyclic, $\widetilde{H}_1$ is also acyclic. Hence $\widetilde{H}_1$ has a vertex with no inward-oriented arrows. Since each $d$–face is paired, the vertex should be a $(d-1)$–face $r_1$ of $\tilde{R}$, which implies that the coface $f_1$ with which $r_1$ is paired is the only matched face of $F$ containing $r_1$, i. e. $r_1$ is a ‘free face’ of $f_1$. Hence $F$ collapses onto $F \setminus (\text{int}(r_1) \cup \text{int}(f_1)) = F_1$. We now consider $\widetilde{H}_2 = \widetilde{H}_1 \setminus \{r_1, f_1\}$ (also removing the corresponding edges). Again $\widetilde{H}_2$ is acyclic and so we can apply the above process to the simplicial complex $F_1$. Continuing this process, we eventually end up when all the $d$–faces and all $(d-1)$–faces of $\tilde{R}$ are removed. So what is left is precisely the simplicial complex $R$. This completes the proof.

Theorem 11 is interesting in the sense that this does not hold in general for arbitrary rooted forests. In fact, if $F$ is a closed, connected, non-orientable, triangulated $d$–manifold, $F$ is automatically a forest. Let $R$ be a root of $F$. Then $F$ and $R$ are not homotopy equivalent, as the codimension one homology of $F$, $H_{d-1}(F,\mathbb{Z})$ contains $\mathbb{Z}_2$ as a summand, whereas $H_{d-1}(R,\mathbb{Z})$, being the top homology group of $R$, can not contain any torsion. In section 6.4, we develop a general theory of rooted-forests in arbitrary triangulated closed manifolds. As a recipe to do so, we prove a “rooted-labeled matrix pseudo forest Theorem” in the next section.
6.3 A rooted-labeled matrix pseudo forest Theorem

Let $G$ be a graph (or a one dimensional simplicial complex). Let $E$ be a subset of $E(G)$ (edge set of $G$) and $V$ be a subset of $V(G)$ (vertex set of $G$) such that $|E| = |V|$. A discrete vector-field of $G$, with critical edges $E(G) \setminus E$ and critical vertices $V(G) \setminus V$ is defined to be a bijection $\phi$ between $V$ and $E$ such that each vertex $v \in V$ is mapped to an edge $e \in E$ containing the vertex $v$. A flow line of $\phi$ is a sequence of vertices and edges:

$$v_1, e_1, v_2, e_2, v_3, e_3, \cdots, v_n, e_n, v_{n+1},$$

where $e_i = \phi(v_i) \forall i = 1, 2, \cdots, n$ and each $v_{i+1} \in e_i \forall i = 1, 2, \cdots, n$ and $v_{n+1}$ not necessarily belongs to the set $V$.

For a graph $G$, select a subset $L$ of edges $E(G)$ and call them “labeled edges”. Now, define the vertex-edge Pseudo Incidence Matrix $\tilde{\partial}(G)$ as the same as the Incidence Matrix $\partial(G)$ except, for the columns corresponding to the edges in $L$, where for each edge $e$ in $L$, each non-zero entry of the corresponding column to $e$ is 1. Note that, as already stated in Section 2, the columns of $\partial(G)$ are ordered in the lexicographic order. Similarly define the Pseudo Laplacian Matrix of $G$, denoted by $\tilde{L}(G)$ is defined to be $(\tilde{\partial})(\tilde{\partial})^T$. Let us define $\tilde{\partial}_{V,E}$ to be the submatrix of $\tilde{\partial}$, with rows and columns of $\tilde{\partial}_{V,E}$ are $V$ and $E$ respectively.

Before stating the next very crucial lemma, let us recall that, a pseudo forest is a graph, each of whose component is either a unicyclic graph (i.e a connected graph with exactly one cycle) or a tree.

**Lemma 18.** Suppose that $G$ is a graph. Let $E \subseteq E(G)$ and $V \subseteq V(G)$ such that $|E| = |V|$. Then $\tilde{\partial}_{V,E}$ is nonsingular if and only if the subgraph $G_E$ of $G$, formed by the edges in $E$ is a pseudo forest, having at least $|V(G)| - |V|$ components, some consisting of trees and each of the remaining components, consisting of the unicyclic graph has an odd number labeled edges in its cycle. Call this pseudo forest, a rooted-labeled pseudo forest, where the roots correspond to the vertices in $V(G) \setminus V$. Moreover, $|\det(\tilde{\partial}_{V,E})| = 2^m$, where $m$ is the number of the components, consisting of the unicyclic graphs.
Proof. First assume that, $\partial_{V,E}$ is nonsingular. Also assume that $V(G)\setminus V \neq \emptyset$. So, let $v \in V(G)\setminus V$. Now, since $\partial_{V,E}$ is nonsingular, $\exists$ at least a discrete vector field $\phi$ of $G$, with critical edges $E(G)\setminus E$ and critical vertices $V(G)\setminus V$. Otherwise $|\det(\partial_{V,E})| = 0$. Clearly, $v$ is a critical vertex of $\phi$. Consider the set of all vertices $\hat{v}$, which, following a flow line of the vector field $\phi$, eventually reach the vertex $v$. See the following figure.

Denote this set of vertices by $\hat{V}$. Clearly the vertex set $\hat{V} \cup \{v\}$ and the edges involved in the aforesaid flow lines form a tree and the tree is the connected component of $G_{E}$, containing the vertex $v$. This tree can be thought to be a rooted-tree with $v$ as the root vertex. So, for each $v \in V(G)\setminus V$, we have a connected component of $G_{E}$, containing a rooted-tree.

Now consider a component(if at all exists) of $G_{E}$, containing no vertices of $V(G)\setminus V$. Since no vertices and edges in this component are critical, they are equal.
in number. and hence, this component is a unicyclic graph. Clearly this component has precisely two distinct discrete vector fields, namely the restriction of \( \phi \) on this component and the other, denoted by \( \dot{\phi} \) is same as \( \phi \), but the arrows reversed in the cycle. See the following figure.

Consider the cycle \( C \) in the unicyclic graph. Let the vertex and edge set of the cycle be \( \{v_1, v_2, \cdots, v_k\} \) and \( \{e_1, e_2, \cdots, e_k\} \) respectively. Without loss of generality assume that, among the the edges \( \{e_1, e_2, \cdots, e_k\} \), \( e_1, e_2, \cdots, e_t \) are the labeled edges. Now, to the discrete vector field \( \phi \) restricted to \( C \), we can associate a unique permutation \( \pi_{\phi|C} \in S_k \) (in short \( \pi_\phi \)),such that \( \phi \) maps the \( i \)th vertex in \( \{v_1, v_2, \cdots, v_k\} \) to the \( \pi_\phi(i) \)th edge in \( \{e_1, e_2, \cdots, e_k\} \) in the lexicographic order and denote the product \( sgn(\pi_\phi)\prod_{i=1}^{k} \tilde{\partial}(v_i, \phi(v_i)) \) by \( \Lambda(\phi, C) \). Now we have

\[
\begin{align*}
\Lambda(\phi, C) &= sgn(\pi_\phi)\prod_{i=1}^{k} \tilde{\partial}(v_i, \phi(v_i)) \\
&= sgn(\pi_\phi)\prod_{i=1}^{k} \tilde{\partial}(\phi^{-1}(e_i), e_i) \\
&= sgn(\pi_\phi)\prod_{i=1}^{t} \tilde{\partial}(\phi^{-1}(e_i), e_i)\prod_{i=t+1}^{k} \tilde{\partial}(\phi^{-1}(e_i), e_i) \\
&= sgn(\pi_\phi)\prod_{i=1}^{t} \tilde{\partial}(\phi^{\prime -1}(e_i), e_i)\prod_{i=t+1}^{k} (-\tilde{\partial}(\phi^{\prime -1}(e_i), e_i))
\end{align*}
\]
\[= \text{sgn}(c_k \circ \pi_\phi) \prod_{i=1}^{t} \partial(\dot{\phi}^{-1}(e_i), e_i) \prod_{i=t+1}^{k}(-\partial(\dot{\phi}^{-1}(e_i), e_i)) \] [\(c_k\) is a permutation-cycle of length \(k\)]

\[= (-1)^{(k-1)+(k-t)} \text{sgn}(\pi_\phi) \prod_{i=1}^{k} \partial(\dot{\phi}^{-1}(e_i), e_i)\]

\[= (-1)^{t-1} \text{sgn}(\pi_\phi) \prod_{i=1}^{k} \partial(v_i, \dot{\phi}(v_i))\]

\[= (-1)^{t-1} \Lambda(\dot{\phi}, C).\]

See the following figure.

In the above figure, the bold edges are labeled edges.

Now remembering that, for each \(v \in V\), \(\partial(v_i, \psi(v_i)) = \pm 1\), for any general vector field \(\psi\) on \(G\), we easily have

\[|\text{det}(\partial_{V,E})| = \prod_{i=1}^{m} |\Lambda(\phi, C_i) + \Lambda(\dot{\phi}, C_i)|,\]

where \(\{C_1, C_2, \cdots, C_m\}\) is the set of cycles in \(G_E\).

Now we have

\[|\Lambda(\phi, C_i) + \Lambda(\dot{\phi}, C_i)|\]
\[ \Lambda(\phi, C_i) = |1 + (-1)^{t_{C_i}}| \] where \( t_{C_i} \) is the number of labeled edges in \( C_i \).

But as \( \text{det}(\tilde{\partial}_{V,E}) \neq 0 \), it follows that \( t_{C_i} \) is odd for all \( i = 1, 2, 3, \cdots, m \) and also we have \( |\text{det}(\tilde{\partial}_{V,E})| = 2^m \). This proves one part of the lemma. For the converse part, the computation of the determinant is exactly similar.

Now let us recall the “Generalized Cauchy-Binet theorem” due to Oliver Knill [67].

**Theorem 19.** Let \( A \) and \( B \) be two \( n \times m \) matrices. Then
\[
\text{det}(x \text{Id} + AB^T) = \sum_{k=0}^{n} \sum_{|P|=k} \text{det}(A_P)\text{det}(B_P), \text{ where, } \text{det}(H_P) \text{ is the minor in } H, \text{ masked by a square pattern } P = I \times J.
\]

Combining Lemma 18 and Theorem 19 and remembering that \( \tilde{\mathcal{L}}(G) = (\tilde{\partial})(\tilde{\partial})^T \), we have the following Rooted-Labeled Matrix Pseudo-Forest Theorem:

**Theorem 20.** \( \text{det}(\tilde{\mathcal{L}}(G) + x \text{Id}) = \sum_{\Gamma} 4^{m_{\Gamma}} x^{|R_{\Gamma}|} \), where the summation runs over all rooted labeled pseudo forests \( \Gamma \), \( R_{\Gamma} \) (a subset of vertices) is the root set of \( \Gamma \) and \( m_{\Gamma} \) is the number of cycles in \( \Gamma \).

### 6.4 Rooted forests in a triangulated closed manifold

Let \( M \) be a triangulated, closed, \( d \)-dimensional manifold. Now the Poincare Dual Graph, denoted by \( PD(M) \) is a graph, whose vertices are the \( d \)-simplices of \( M \) and two vertices are adjacent if and only if their corresponding \( d \)-simplices \((f_1, f_2)(\text{say})\) share a common \((d-1)\)-face \((r, \text{say})\). More over, if \( \partial_d[r, f_1] = \partial_d[r, f_2](\partial_d \text{ is the } d \text{-th incidence matrix of } M) \), then declare the edge in \( PD(M) \), corresponding to \( r \) to be a labeled edge in the sense of the previous section. Note that, if \( M \) is non-orientable, its Poincare Dual graph \( PD(M) \) always contains labeled edges (vaguely speaking, traversing along a labeled edge reverses the local orientation of \( M \)).
So we see that, there are two bijections, one from the set of \(d\)–simplices of \(M\) to \(V(PD(M))\) and the other from the set of \((d-1)\)–simplices of \(M\) to \(E(PD(M))\).

As there would be no additional confusion, let us denote these two bijections by the same notation, \(D\), say.

Now, a pseudo-incidence matrix \(\tilde{\partial}(PD(M))\) of \(PD(M)\) is defined as follows:

\[
\tilde{\partial}[v, e] := \partial_d[D^{-1}(e), D^{-1}(v)], \text{ if } e \text{ is not a labeled edge of } PD(M), \text{ otherwise } \\
\tilde{\partial}[v, e] := |\partial_d[D^{-1}(e), D^{-1}(v)]|.
\]

We observe that, if \(F\) is a subset of the set of all \(d\)–faces of \(M\) and \(R\) is a subset of \((d-1)\)–faces of \(M\), such that \(|F| = |\tilde{R}|\), then there is a natural one to one correspondence between the set of all fitting orientations of \((F, R)\) and the set of discrete vector fields of \(PD(M)\), with the critical edges \(E(PD(M)) \setminus D(\tilde{R})\) and critical vertices \(V(PD(M)) \setminus D(F)\).

Now suppose that \((F, R)\) is a rooted forest of \(M\). It was shown in [61] that, \(|H_{d-1}(F, R)| = |\det(\tilde{\partial}_{\tilde{R}, F})|\). Again we can easily see that \(|\det(\partial_{\tilde{R}, F})| = |\det(\tilde{\partial}_{D(F), D(\tilde{R})})|\).

Hence from Lemma 18, we have the following theorem:

**Theorem 21.** Let \((F, R)\) be a rooted forest of \(M\). Then \(PD(M)_{D(\tilde{R})}\) is a rooted-labeled pseudo forest, where the roots correspond to the vertices in \(V(PD(M)) \setminus D(F)\).

Moreover, \(|H_{d-1}(F, R)| = 2^m\), where \(m\) is the number of components of \(PD(M)_{D(\tilde{R})}\), consisting of the unicyclic graphs.

**Remark:** In fact, modifying the argument of Lemma 18 and using the fact that the Smith Normal forms of a matrix and its transpose are the same, one can easily show that the group \(H_{d-1}(F, R)\) is an elementary abelian group.

As a corollary of this theorem, we get the following interesting result:

**Theorem 22.** Let \(M\) be an orientable manifold and \((F, R)\) be any rooted forest of \(M\). Then \(H_{d-1}(F, R)\) is trivial.

**Proof.** Since \(M\) is orientable, we can always orient its faces, so that there would be no labeled edges in \(PD(M)\). Hence no component of \(PD(M)_{D(\tilde{R})}\) consists of unicyclic
Now we prove the following theorem, which is one of the main results of this chapter.

**Theorem 23.** Let \((F, R)\) be a rooted forest of a manifold \(M\). Then \(F\) simplicially collapses onto \(R\) if and only if \(H_{d-1}(F, R)\) is trivial.

**Proof.** One direction is easy, i.e. if \(F\) simplicially collapses onto \(R\), then \(H_{d-1}(F, R)\) is trivial. One can use standard topological arguments (say, for example homology long exact sequence).

The interesting part is the converse, i.e. suppose that \(H_{d-1}(F, R)\) is trivial. By Theorem 21, we can say that, each component of \(PD(M)_{D(R)}\) is a tree and hence \(PD(M)\) has a unique discrete vector field with the critical edges \(E(PD(M)) \setminus D(R)\) and critical vertices \(V(PD(M)) \setminus D(F)\). More over this vector field is **acyclic**. So, \((F, R)\) has a unique fitting orientation \(\phi\), which is also a **discrete gradient vector field**, whose set of all critical simplices is precisely the root complex \(R\). Hence mimicking the proof of theorem 11, we get the desired result.

Combining Theorems 22 and 23 we can reestablish theorem 11.

**Remark:** Theorem 23 does not hold for general simplicial complexes. To see exactly what happens for a general simplicial complex seems to be a challenging problem. This problem seems to have a direct connection to the enumeration of perfect matchings in bipartite graphs and the famous **Zeeman’s conjecture**.

### 6.5 An application to the enumeration of rooted-forests in surfaces

In this section we apply the theorems of the previous sections to obtain an interesting new result on closed triangulated surfaces. Before stating the next theorem we fix some notations. From now on, till the end of this chapter we will use
the notation $\Sigma$ to denote a general closed surface. If $\Sigma$ is non-orientable, then the maximum number of pairwise disjoint Möbius bands in $\Sigma$ is denoted by $g(\Sigma)$. This number is independent of the triangulation of $\Sigma$. We also use the notation $|(F, R)|_{\Sigma}$ to denote the number of rooted forests $(F, R)$ in $\Sigma$.

**Theorem 24.** Let $\Sigma$ be a closed surface and $L_\Sigma$ be its Laplacian matrix. Then:

1. If $\Sigma$ is orientable, then $\det(I + L_\Sigma) = |(F, R)|_{\Sigma}$.

2. If $\Sigma$ is non-orientable, then $1 \leq \frac{\det(I + L_\Sigma)}{|(F, R)|_{\Sigma}} \leq 4g(\Sigma)$.

**Proof. Proof of 1 :** This immediately follows from Theorem 22 and the Bernardi-Klivans forest theorem.

**Proof of 2 :** Suppose that $(F, R)$ is a rooted forest in $\Sigma$. Then Theorem 21 implies that $|H_1(F, R)| = 2^m$, where $m$ is the number of components of $PD(\Sigma)|_{D(\overline{R})}$, consisting of the unicyclic graphs. Now the crucial observation is that the cycle in the unicyclic graph corresponds to a Möbius band in $\Sigma$. Hence $|H_1(F, R)| = 2^m \leq 2g(\Sigma)$. Now putting $x = 1$ in the Bernardi-Klivans forest theorem we get the desired inequality. $\square$
Chapter 7

A combinatorial proof of the Ryser-type formula for determinants given by Insko, Johnson and Sullivan

In [62], Erik Insko, Katie Johnson and Shaun Sullivan proved a Ryser-type formula for determinants. They called it “A terrible expansion of the determinant”. This expansion evolved from a conjecture about a transfer formula in multivariate finite operator calculus. Before stating their theorem, let us define some notations and terminologies. We will denote the set \( \{1, 2, 3, \cdots, n\} \) by \([n]\). Let \( S \) be a finite set. Then an ordered set partition (in short, an ordered partition) of \( S \) is an ordered tuple \((\beta_1, \beta_2, \cdots, \beta_r)\) of pairwise disjoint subsets, whose union is \( S \). For example \((\{1, 3\}, \{2\}), (\{2\}, \{1, 3\})\) are two (there are many others) ordered partitions of [3]. Now let us state the theorem.

**Theorem 25.** For a square matrix \( A = (a_{ij})_{n \times n} \), \( |A| = \sum_{B \in [n]} (-1)^{|B|} \prod_{\beta_k \in B} \prod_{i \in \beta_k} \sum_{j \in \beta_k} a_{ij} \), where the outer summation runs over all ordered partitions \( B = (\beta_1, \beta_2, \cdots, \beta_r) \) of \([n]\) and the inner summation runs over all integers \( j \) in the union \( \beta_k = \bigcup_{i=1}^k \beta_i \) of first \( k \) parts of the partition \( B \).

For a function \( f : [n] \to [n] \), define \( a_f := \prod_{i=1}^n a_{f(i)} \). Then the above expansion becomes, \( |A| = \sum_{B \in [n]} (-1)^{|B|} (\sum_f a_f) \); where the inner summation runs over all functions \( f : [n] \to [n] \) satisfying the following property: if \( i \in \beta_k \), then \( f(i) \in \bigcup_{j=1}^k \beta_j \), where \( B = (\beta_1, \beta_2, \cdots, \beta_r) \). Now after expansion, this will take the form
\[
\sum_{f} c_f a_f, \ f \text{ running through all the functions } f : [n] \to [n]. \]

In [62], Erik Insko, Katie Johnson and Shaun Sullivan proved that if \( f \) is bijective \( c_f = \text{sign}(f) \) and for nonbijective \( f \), they showed \( c_f \) to be zero by an elegant topological argument analyzing the Euler characteristics of subsets of a suitably chosen permutahedron. Since this proof is highly topological and also their expansion resembles the Ryser’s formula for permanents, stating that for a square matrix \( A = (a_{ij})_{n \times n} \),

\[
\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^n \sum_{j \in S} a_{ij},
\]

the authors asked two questions at the end of their paper [62]:

1. Is there any combinatorial proof of Theorem 1 in [62]?

2. Is it possible to prove the Ryser’s formula topologically in the spirit of their argument in Theorem 1 in [62]?

In this chapter, we answer these two questions, i.e. we give a simple combinatorial argument to prove the “terrible expansion” and also we give a topological proof of Ryser’s formula at the end. From now on we will use IJS to abbreviate the names of the authors of [62].

### 7.1 Combinatorial proof of the expansion

Before proceeding, we recall a definition from [62]. For a function \( f : [n] \to [n] \), define \( S_f := \{ B \in [n] : i \in \beta_k \Rightarrow f(i) \in \bigcup_{i=1}^{k} \beta_j \} \), where \( B = (\beta_1, \beta_2, \cdots, \beta_r) \).

Let us give a short description of the structure of the poset of ordered partitions. Let \( P_n \) denote the poset of ordered partitions of the set \( [n] \). At the top of the poset \( P_n \), we have \( n! \) ordered partitions consisting of singletons. IJS call them ‘singleton partitions’. Directly below a given ordered partition \( B = (\beta_1, \beta_2, \cdots, \beta_r) \) in \( P_n \) are the ordered partitions \( B_i = (\beta_1, \beta_2, \cdots, \beta_i \cup \beta_{i+1}, \cdots \beta_r) \), where \( 1 \leq i \leq r - 1 \), i.e. directly below a given ordered partition \( B \) in \( P_n \) are the ordered partitions formed by taking the union of two consecutive parts in \( B \). We label in \( P_n \) as follows: “1st label” consists of all the singleton partitions. “2nd label” consists of all the partitions lying
just below the “1st label” partitions and so on.

Now to prove the theorem combinatorially, we just need to prove that if \( f \) is bijective then \( c_f = \text{sign}(f) \), and if \( f \) is not bijective, then \( c_f = 0 \). Now, for the case when \( f \) is not bijective, it suffices to prove combinatorially that \( c_f = 0 \) whenever \( f \) is acyclic (i.e. \( f^k(i) = i \) implies \( k = 1 \) for all \( i \in [n] \)). IJS [62] showed how to reduce the problem of calculating coefficients \( c_f \), when \( f \) is not bijective, to that of calculating the coefficients \( c_f \) corresponding to acyclic functions \( \overline{f} : [n] \rightarrow [n] \) (see Lemma 8, in [62]; in fact it can be easily shown that, if \( f \) has a cycle, then collapsing the cycle to a point does not change the set \( S_f \)). Hence to address Question 1, we need only to prove combinatorially the following result.

**Proposition 26.** Let \( f : [n] \rightarrow [n] \) be a function. Then

1. \( c_f = \sum_{B \in S_f} (-1)^{n-|B|} = 0 \) if \( f \) is an acyclic non-identity function.

2. \( c_f = \text{sign}(f) \), if \( f \) is bijective.

**Proof.** 1. Since \( f \) is acyclic, \( f \) can be very naturally viewed as a rooted forest as described in [62], where the fixed points are the roots and each component tree contains a single root. We also call the rooted forest associated to \( f \) as \( f \).

Now since \( f \) is nonidentity, \( f \) has a component having more than one element. Without loss of generality let 1 be the root of this component.

Let \( \hat{S}_f = \{ B \in S_f : \beta_1, \text{the first part of } B \text{ contains } 1 \} \). If a singleton partition belongs to \( \hat{S}_f \), then every partition below it must belongs to \( \hat{S}_f \). Conversely, let \( B \in \hat{S}_f \) and \( B = (\beta_1, \beta_2, \cdots, \beta_r) \), where \( 1 \in \beta_1 \). We prove that there exists a singleton partition \( \hat{B} \in \hat{S}_f \) such that \( \hat{B} \geq B \) in \( \mathcal{P}_n \). To do this first place the 1 at the beginning. Now look at the remaining elements of \( \beta_1 \). Let \( \beta_1 \setminus \{1\} = A_1 \cup A_2 \cup \cdots \cup A_k \), where for any particular \( i \) each element of \( A_i \) belongs to the same component of \( f \) and \( A_i \cap A_j = \emptyset \) for \( i \neq j \). Order each element of \( A_1 \) by their distance from the root of their mother component (elements having the
same distance from the root can be put in any order). Do this for $A_2, A_3$ and so on. Continue the same method for $\beta_2, \beta_3$ and so on and we get our desired \( \hat{B} \). So \( \hat{S}_f \) consists of all the singleton partitions in \( S_f \) with 1 at the beginning and all the partitions below them.

Now we pair up all the partitions of \( \hat{S}_f \) in an interesting way. Let \( x_1, x_2, x_3, \cdots, x_n \) be a typical 1st label partition in \( \hat{S}_f \) (this is an abuse of notation. By \( x_1, x_2, x_3, \cdots, x_n \) we mean the ordered tuple \( (x_1, x_2, x_3, \cdots, x_n) \) ). We pair this element with \( (1, x_2), x_3, x_4, \cdots, x_n, \) a 2nd label partition. Do this for all 1st label partitions in \( \hat{S}_f \). Now, which 2nd label partitions in \( \hat{S}_f \) are still unpaired? A typical partition of this type is of the form \( y_1, y_2, y_3, \cdots, (y_i, y_{i+1}) \cdots, y_n \) (this is also an abuse of notation. By \( (y_i, y_{i+1}) \) we really mean \( \{y_i\} \cup \{y_{i+1}\}\)). We pair these partitions with the 3rd label partitions as follows: \( 1, y_2, y_3, \cdots, (y_i, y_{i+1}) \cdots, y_n \) is paired with \( (1, y_2), y_3, y_4, \cdots, (y_i, y_{i+1}), \cdots, y_n \). Now we ask the same question i.e. which 3rd label partitions are still unpaired? The answer is the partitions with singleton one as their first part i.e. of the form \( 1, \alpha_1, \alpha_2, \cdots, \alpha_k \) and we pair these partitions with the 4th label partitions as follows: \( 1, \alpha_1, \alpha_2, \cdots, \alpha_k \) is paired with \( \{1\} \cup \alpha_1, \alpha_2, \cdots, \alpha_k \). We proceed this way until there will be no unpaired partitions. The way we paired up the partitions in \( \hat{S}_f \) immediately tells us that the contribution of them in \( c_f \) is zero i.e. \( \sum_{B \in \hat{S}_f} (-1)^{n-|B|} = 0 \).

Now let \( k \geq 1 \) and \( \beta = \beta_1, \beta_2, \cdots, \beta_k, \beta_k+1, \beta_k+2, \cdots, \beta_r \) be a partition in \( S_f \), where \( 1 \in \beta_{k+1} \). We fix this partition. Let \( S^3_f = \{ \gamma \in S_f : \gamma = \beta_1, \beta_2, \cdots, \beta_k, \gamma_1, \gamma_2, \cdots, \gamma_s, 1 \in \gamma_1 \} \). Note that \( \beta \in S^3_f \). Let \( [n] \setminus (\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k) = M \). Since 1 is a root of \( f \), \( |M| > 1 \). Let us denote \( P^1(M) \) to be the set of all partitions of the set \( M \), with 1 belonging to the first part. So \( S^3_f = \{ \gamma \in S_f : \gamma = \beta_1, \beta_2, \cdots, \beta_k, \gamma_1, \gamma_2, \cdots, \gamma_s \gamma_s \in P^1(M) \} \). So each partition of \( S^3_f \) is of the form \( (\beta_1, \beta_2, \cdots, \beta_k) \) followed by an element of \( P^1(M) \). Now since \( |M| > 1 \), we can apply our previous argument to pair up each element of \( S^3_f \) just forgetting the part \( (\beta_1, \beta_2, \cdots, \beta_k) \) of each element, so that
their contribution in $c_f$ is zero i.e. $\sum_{B \in S_f} (-1)^{|B|-|B|} = 0$. Since $\beta$ is arbitrary, we conclude that $c_f = 0$.

2. Assume that $f$ is bijective. In this case Corollary 10 of [62] proved that $c_f = \text{sgn}(f)$ by a direct consequence of the identity $\sum_{k=0}^{n} (-1)^{n-k} k! S(n, k) = 1$, where $S(n, k)$ is the stirling number of 2nd kind. Here we give a combinatorial proof of this identity almost similarly to that of the acyclic case. Actually we have $\sum_{k=0}^{n} (-1)^{n-k} k! S(n, k) = \sum_{B \in \mathcal{P}_n} (-1)^{|B|-|B|}$. Note that $S_1 = \mathcal{P}_n$, where $I$ is the identity mapping from $[n] \rightarrow [n]$. Let us consider the set of all partitions in $\mathcal{P}_n$ with singleton $\{n\}$ as the last part and call this set $\mathcal{P}_n'$. Since the elements of $\mathcal{P}_n \setminus \mathcal{P}_n'$ can be paired up exactly the same way as the previous argument (replacing 1 by $n$), we have $\sum_{B \in \mathcal{P}_n} (-1)^{|B|-|B|} = \sum_{B \in \mathcal{P}_n'} (-1)^{|B|-|B|} = 1$ (by induction, since initially $(-1)^{1-1}! S(1, 1) = 1$), where we assume $B = \hat{B} \cup \{n\}$.

7.2 Topological proof of Ryser’s formula

Recall that the Ryser’s formula for the permanent of a square matrix $A = (a_{ij})_{n \times n}$ is: $\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i=1}^{n} \sum_{j \in S} a_{ij}$, which can be restated as $\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{|S|} (\sum_{f : [n] \rightarrow S} a_f)$. In the following result we give a topological proof for the Ryser’s formula for permanents of a square matrix.

**Proposition 27.** If $A = (a_{ij})_{n \times n}$ is square matrix, then

$$\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{|S|} (\sum_{f : [n] \rightarrow S} a_f).$$

**Proof.** Consider the expression $\sum_{S \subseteq [n]} (-1)^{|S|} (\sum_{f : [n] \rightarrow S} a_f)$. Upon expansion this would be of the form $\sum_{f : [n] \rightarrow [n]} d_f a_f$. If $f$ is surjective then it is immediate that $d_f = 1$. So to prove Ryser’s formula, it suffices to prove that $d_f = 0$, if $f$ is not surjective. Suppose that $\text{range}(f) = S \subset [n]$. Then $d_f = \sum_{S \supseteq S} (-1)^{|S|}$. Now
\[ \sum_{\mathcal{S} \subseteq [n]} (-1)^{n-|\mathcal{S}|} = \sum_{\mathcal{S} \supseteq [n]} (-1)^{n-|\mathcal{S}|} + \sum_{\mathcal{S}} (-1)^{n-|\mathcal{S}|}, \] where the 2nd summation on the R.H.S of the above equation runs through all subsets of \([n]\) not containing \(S\). Now
\[ \sum_{\mathcal{S} \subseteq [n]} (-1)^{n-|\mathcal{S}|} = (-1)^n \sum_{\mathcal{S} \subseteq [n]} (-1)^{|\mathcal{S}|} = (-1)^n \chi(\Delta^{n-1}) = (-1)^n, \] where \(\Delta^{n-1}\) is the standard simplex and \(\chi(\Delta^{n-1})\) is its Euler characteristic, which is equal to 1. Again the subsets of \([n]\), not containing \(S\) form the faces of an abstract simplicial complex \(\Delta\) with vertex set \([n]\). But how does \(\Delta\) look like? \(\Delta\) is actually \(x_1 * x_2 * \cdots * x_{n-|S|} * (\partial \Delta^{|[S]|-1})\), where \(\Delta^{|[S]|-1}\) is the simplex with vertex set \(S\) and \(\partial \Delta^{|[S]|-1}\) is its boundary sphere, '*' is the usual simplicial join operation(see [63], page 12) and \([n] \setminus S = \{x_1, x_2, \cdots x_{n-|S|}\}\). Since \(\Delta\) is an iterated cone, its geometric realization \(|\Delta|\) (see [63], page 16) is contractible and so \(\chi(|\Delta|) = 1\). Hence
\[ d_f = \sum_{\mathcal{S} \supseteq [n]} (-1)^{n-|\mathcal{S}|} = (-1)^n - (-1)^n = 0. \]
Bibliography


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