Chapter 6

On the rooted forests in triangulated closed manifolds

The notion of forests and their roots in higher dimensions was introduced by Olivier Bernardi and Caroline Klivans in [61] to obtain a generalization of the ”Matrix Forest Theorem” in higher dimension. In fact, they proved that, for a $d$–dimensional simplicial complex $G$,

$$
\sum_{(F,R)} |H_{d-1}(F,R)|^2 x^{|R|} = \det(L_G + xId),
$$

where the summation runs over all rooted forests of $G$ and $Id$ is the identity matrix of dimension $|G_{d-1}|$, and $L_G$ is the Laplacian matrix. We call this theorem the “Bernardi-Klivans Forest Theorem”. A combinatorial interpretation of the factor $|H_{d-1}(F,R)|$ for a general simplicial complex $G$ is a challenging problem. (See [61]).

In this chapter, we give a combinatorial interpretation of $|H_{d-1}(F,R)|$ for any closed $d$–manifold $M^d$ in terms of their “Poincare-dual graphs” (details later). We have also addressed the following problem. In graphs, a forest simplicially collapses onto any of its roots (in graph case, a root consists of a finite number of vertices, each belongs to its mother component). So it is tempting to conjecture that, in higher dimensional case also, a forest simplicially collapses onto any of its roots. But unfortunately, this is false in general. In fact a non-orientable, closed, triangulated manifold can not collapse onto any of its roots. In Section 6.4, we have proved that, if $(F, R)$ is a rooted
forest in a closed, connected, triangulated manifold $M^4$, then $F$ simplicially collapses onto its root $R$ if and only if $H_{d-1}(F, R)$ is trivial. As a corollary of this theorem we have proved that a “punctured” closed, connected, orientable triangulated manifold simplicially collapses onto any of its roots. In the course of the proof of our main theorem, we prove a lemma (Lemma-12), which partially answer a question of Olivier Bernardi and Caroline Klivans. In [61], Olivier Bernardi and Caroline Klivans posed a problem of finding a sign-reversing involution to prove that the sum of the signs of all fitting orientations (definition later) of a non-rooted forest is zero. In Lemma 12, we have proved that, the sum of the signs of all fitting orientations of a simplicial complex containing a ‘geometric cycle’ is zero by a sign-reversing involution argument. The motivation for considering ‘geometric cycle’ is the following. In one dimensional simplicial complexes (i.e. graphs), a forest does not contain any cycle. In higher dimensional simplicial complex, the analogue of the graphical cycle is closed orientable manifold or more generally ‘geometric cycle’. See the next section for the definitions. We have used the technique of Discrete Morse Theory to prove some of the important theorems in this chapter. We refer to [69], [70], [71] for necessary background in Discrete Morse Theory.

### 6.1 Basic definitions

Let $k \geq 1$. The $k$–th incidence matrix of a simplicial complex $G$ is the matrix $\partial_k(G)$ (for short $\partial_k$) defined as follows:

The rows of $\partial_k$ are indexed by the $(k-1)$–faces and the columns are indexed by $k$–faces.

The entry $\partial_k[r, f]$ corresponding to a $(k-1)$–face $r$ and a $k$–face $f$ is $(-1)^j$ if $f = \{v_0, v_1, \ldots, v_k\}$ with $v_0 < v_1 < \cdots < v_k$ and $r = f \setminus \{v_j\}$ for some $j$ and 0 otherwise. Moreover, we will order the rows and columns of $\partial_k(G)$ according to the lexicographic order of the faces.

Let $G$ be a $d$–dimensional complex and let $F$ be a subset of the $d$–faces and
$R$ be a subset of the $(d - 1)$–faces. We say that $F$ is a forest of $G$ if the corresponding columns of $\partial_d(G)$ are linearly independent. We say that $R$ is a root of $G$ if the rows of $\partial_d(G)$ corresponding to the faces in $R = G_{d-1} \setminus R$ form a basis of the rows. A rooted forest of $G$ is a pair $(F, R)$, where $F$ is a forest of $G$ and $R$ is a root of the simplicial complex generated by $F$, which is also denoted by $F$. For the root $R$ of $F$ the root complex $\mathbf{R}$ is defined to be $R \cup F_{d-2}$.

Let $R$ be a ring, and $C_k(G, R)$ denote the free $R$-module with basis consisting of $k$–faces of $G$. Note that $\partial_k(G)$ can be interpreted as the matrix of a linear map from $C_k(G, R)$ to $C_{k-1}(G, R)$. This map is called the boundary map in simplicial topology. A $k$–cycle of $G$ is a non-zero element of $C_k(G, R)$ in the kernel of $\partial_k$. From now on, we will take $R = \mathbb{Z}$, the ring of integers. Because we are working with $\mathbb{Z}$–coefficient, a $k$–cycle $C$ of $G$ can be re-interpreted as a collection of $k$–faces of $G$, namely $\{f_1, f_2, \cdots, f_m\}$ such that there exist non-zero integers $\{c_1, c_2, \cdots, c_m\}$ for which $\partial_k(\sum_{i=1}^m c_if_i) = 0$. In other words, a $k$–cycle $C$ can be thought of as a multiset $\tilde{C} = \{f_1, f_1, \cdots, f_1, f_2, f_2, \cdots, f_2, f_3, f_3, \cdots, f_m, f_m, \cdots, f_m\}$, where, for each $i$, $f_i$ has multiplicity $|c_i|$. The sign of $f_i$ in $C$ is defined to be $\frac{c_i}{|c_i|}$ and this is denoted by $\langle [C], f_i \rangle$. Note that $\langle [C], f_i \rangle \in \{1, -1\}$. In this terminology, the cycle condition can be restated as $\partial_k(\sum_{f \in \tilde{C}} \langle [C], f \rangle f) = 0$. A geometric $d$–cycle is a $d$ – dimensional simplicial complex, whose $d$–faces form a cycle and each $(d - 1)$–face is contained in precisely two $d$–faces.

Now we reproduce the definition of the fitting orientations and their signs from [61]. Let $F$ be a $d$–dimensional simplicial complex. The set of all $d$–faces of $F$ is also denoted by $F$. Let $R$ be a subset of $(d - 1)$–faces such that $|F| = |\bar{R}|$, where $\bar{R} = F_{d-1} \setminus R$ and where $F_{d-1}$ is the $(d - 1)$–skeleton of $F$. A fitting orientation (or pairing) of $(F, R)$ is a bijection $\phi$ between $\bar{R}$ and $F$, such that each face $r \in \bar{R}$ is mapped to a face $f \in F$ containing $r$. For any fitting orientation of $(F, R)$, i.e for any bijection $\phi : \bar{R} \rightarrow F$, we associate the unique permutation $\pi_\phi \in S_{|F|}$ such that $\phi$ maps the $i$–th face in $\bar{R}$ in the lexicographic order to the $\pi_\phi(i)$–th
face in $F$ in the lexicographic order. Now define the sign of $\phi$, denoted by $\Lambda(\phi)$ as $\text{sgn}(\pi_{\phi}) \prod_{r \in \bar{R}} \partial_{d}[r, \phi(r)]$. The notion of fitting orientation was introduced to give a combinatorial interpretation (in terms of vector fields) of the homological quantity $|H_{d-1}(F, R)|$, occurred in the “Matrix forest theorem” for higher dimensions (See the introduction). In fact in [61], the authors have shown that $|\sum_{\phi} \Lambda(\phi)| = |H_{d-1}(F, R)|$, where $\phi$ runs over all fitting orientations of $(F, R)$.

6.2 Rooted forests in orientable manifolds

In this section, we prove that a ‘punctured’ closed, connected, orientable triangulated manifold is simple homotopy equivalent to any of its roots. To be more precise, let $M^d$ be a closed, connected, orientable, triangulated $d$–manifold. Note that, in the triangulation of $M^d$, each $(d - 1)$–simplex is contained in exactly two $d$–simplices. Let $F$ be the $d$–dimensional simplicial complex formed by removing a finite, nonempty collection of $d$–simplices from $M^d$. Clearly $F$ is a forest. We show that $F$ simplicially collapses onto any of its roots.

**Theorem 11.** Let $F$ be a punctured, closed, connected, orientable triangulated manifold and $R$ be a root of $F$. Then $F$ simplicially collapses onto $R$.

Before proving this, we need several lemmas which are of independent interest.

**Lemma 12.** If $F$, a $d$–dimensional simplicial complex contains a geometric $d$–cycle as a subcomplex and let $R$ be a subset of $(d - 1)$–faces such that $|F| = |\bar{R}|$, then $\sum_{\phi} \Lambda(\phi) = 0$, where $\phi$ runs over all the fitting orientations of $(F, R)$.

**Proof.** Let $C$ be the geometric $d$–cycle. Fix a fitting orientation $\phi$. We claim that $C$ contains some $d$–faces $\{f_1, f_2, \cdots, f_k\}$ and $\bar{R}$ contains some $(d - 1)$–faces $\{r_1, r_2, \cdots, r_k\}$ such that the following holds:

1. $f'_i$s and $r'_i$s can be rearranged in a sequence of simplices
\[ r_{i_1} \rightarrow f_{i_1} \rightarrow r_{i_2} \rightarrow f_{i_2} \rightarrow \cdots \rightarrow r_{i_k} \rightarrow f_{i_k} \rightarrow r_{i_1} \] such that \( r_{i_m}, r_{i_{m+1}} \in \partial f_{i_m} \) and 
\[ \phi(r_{i_m}) = f_{i_m} \quad \text{for } m \neq k \quad \text{and} \quad \phi(r_{i_k}) = f_{i_k} \quad \text{and} \quad \{ r_{i_k}, r_{i_1} \} \in \partial f_{i_k} \quad \text{for } m = k. \]

Here the direction of an arrow from an \( r \) to an \( f \) indicates that \( \phi(r) = f \) and the direction of an arrow from an \( f \) to an \( r \) indicates that \( r \subset f \).

2. For \( m \neq 1, \quad \partial[r_{i_m}, f_{i_m}] \cdot \langle [C], f_{i_m} \rangle + \partial[r_{i_m}, f_{i_{m-1}}] \cdot \langle [C], f_{i_{m-1}} \rangle = 0 \)

and for \( m = 1, \partial[r_{i_1}, f_{i_1}] \cdot \langle [C], f_{i_1} \rangle + \partial[r_{i_1}, f_{i_k}] \cdot \langle [C], f_{i_k} \rangle = 0. \)

The sequence, defined in condition 1 is called a ‘cycle’ and if in addition, it satisfies 2, the cycle is called an ‘alternating sign cycle’ (or ASC for short). So our claim is that, \( \phi \) has an ASC in \( C \). To prove this, take a \( d \)--simplex, say \( f_1 \in C \). Let \( r = \phi^{-1}(f_1) \). Now the boundary-vanishing condition of the geometric cycle \( C \) ensures that, there exists a(unique)face \( f_2 \in C \) such that \( f_2 \neq f_1 \) and \( r_1 \) is face of \( f_2 \) and 
\[ \langle [C], f_1 \rangle \partial[r_1, f_1] = -\langle [C], f_2 \rangle \partial[r_1, f_2]. \]

Let \( r_2 = \phi^{-1}(f_2) \). Again by the boundary-vanishing condition of the cycle \( C \) ensures that, there exists a face \( f_3 \in C \) such that
\[ f_3 \neq f_2 \] and \( r_2 \) is a face of \( f_3 \) and 
\[ \langle [C], f_2 \rangle \partial[r_2, f_2] = -\langle [C], f_3 \rangle \partial[r_2, f_3]. \]
Note that, \( f_3 \neq f_1 \). Again let \( r_3 = \phi^{-1}(f_3) \) and proceed further. This \( r_3 \) gives rise to another face \( f_4 \) satisfying the ‘alternating sign’ condition. If \( f_4 \) equals any of the faces \( \{ f_1, f_2, f_3 \} \) in this case the only possibility is \( f_1 \), we stop; and continue otherwise. Note that we can not continue forever. We must stop at a face say \( f_m \) which coincides with a face, previously encountered, say \( f_1 \) without loss of generality. And thus we get an ASC.

We can order all ASC’s of \( \phi \) as follows:

To each ASC, we associate the set of all \( d \)--dimensional simplices involved in that particular ASC and write down the elements of the set one by one in their increasing lexicographic order to make a word. Thus, to each ASC we associate a word, whose letters come from the set of all \( d \)--simplices of \( C \). Now order the words in the dictionary order and order the ASC’s in the order of their associated words.

Now we are ready to define a sign-reversing involution on the set of all fitting orientations \( \phi \) of \((F, R)\).
Take a $\phi$. Let $A$ be the smallest ASC (in the dictionary order, we just defined) in $C$. Now reverse the direction of each arrow of $A$ and keep the other paired simplices unturned (see the figure below).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{The arrows of $A$, getting reversed give rise to another pairing (or fitting orientation) say $\tilde{\phi}$ and an ASC, say $\tilde{A}$. This is clearly an involution. See the following diagram.}
\end{figure}

In the above diagram, the white vertices indicate the $(d - 1)$–faces and the black vertices indicate the $d$–faces. An arrow from a white vertex to a black vertex indicates that the two vertices are paired and an arrow from a black vertex to a white vertex indicates a containment. Note that, the above configurations are forbidden in a fitting orientation of a \textbf{geometric cycle}. The left figure is not admissible, because at the rightmost black vertex, the bijectivity of the fitting orientation is violated and
the right figure is not admissible, because the upper rightmost white vertex has total degree 3, which is not possible in a geometric cycle. Hence, reversing the arrows does not give rise to any new ASC (in fact any cycle) in $C$ except the inverted one. We prove that, this is in fact sign-reversing.

Let $S$ and $T$ be the set of all $(d - 1)$ and $d$-simplices involved in $A$ (or $\tilde{A}$) respectively.

Now we have

$$sgn(\pi_\phi)(\prod_{f \in T}<[C], f>)(\prod_{r \in R} \partial_d[r, \phi(r)])$$

$$= sgn(\pi_\phi)(\prod_{f \in T}<[C], f>)(\prod_{r \in R \setminus S} \partial_d[r, \phi(r)])(\prod_{r \in S} \partial_d[r, \phi(r)])$$

$$= (\prod_{r \in R \setminus S} \partial_d[r, \phi(r)])(\prod_{r \in S} \partial_d[r, \phi(r)])([C], \phi(r)))$$

$$= (\prod_{r \in R \setminus S} \partial_d[r, \phi(r)])(\prod_{r \in S} \partial_d[r, \phi(r)])([C], \phi(r)))$$

$$= (\prod_{r \in R \setminus S} \partial_d[r, \phi(r)])(\prod_{r \in S} \partial_d[r, \phi(r)])([C], \phi(r)))$$

$$= (-1)^{|T|-1}(\prod_{r \in R \setminus S} \partial_d[r, \phi(r)])(\prod_{r \in S} \partial_d[r, \phi(r)])([C], \phi(r)))$$

[ for $r \in R \setminus S, \phi(r) = \tilde{\phi}(r)$]

$$= -\prod_{r \in R \setminus S} \partial_d[r, \phi(r)](\prod_{r \in S} \partial_d[r, \phi(r)])([C], \phi(r)))$$

$$= -sgn(\pi_\phi)(\prod_{f \in T}<[C], f>)(\prod_{r \in R} \partial_d[r, \phi(r)])$$

So, $sgn(\pi_\phi)(\prod_{r \in R} \partial_d[r, \phi(r)]) = -sgn(\pi_\phi)(\prod_{r \in R} \partial_d[r, \phi(r)])$.

Here we briefly summarize some basics of discrete Morse theory, that will be used subsequently. Let us begin with the following essential definitions.

**Definition:** A discrete Morse function $f$ on a simplicial complex $X$ is a real valued function on the set of simplices of $X$ such that for each simplex $\beta$, at most one face $\alpha$ of $\beta$ has a value greater than or equal to that of $\beta$ and at most one coface $\gamma$ of $\beta$ has a value less than or equal to that of $\beta$.

A critical simplex $\beta$ of a discrete Morse function $f$ on $X$ is a simplex for which no face $\alpha$ of $\beta$ has a value greater than or equal to that of $\beta$ and no coface $\gamma$ of $\beta$ has a value less than or equal to that of $\beta$.
The following theorem, often said to be the fundamental theorem of discrete Morse theory is due to Robin Forman [69].

**Theorem 13.** Let $X$ be a simplicial complex with a discrete Morse function $f$. Then $X$ is homotopy equivalent to a cell complex containing the same number of cells of a given dimension as there are critical simplices of $f$ of that dimension.

Now we define the notion of a discrete vector field (or simply a pairing) on a simplicial complex $X$.

**Definition:** A discrete vector field (or a pairing) $\phi$ on a simplicial complex $X$ is a set containing pairs of simplices $\{\alpha, \beta\}$ of $X$, such that $\text{dim}(\alpha)$ and $\text{dim}(\beta)$ differ by one and for each simplex $\gamma$ of $X$, there is at most one pair in $\phi$ containing $\gamma$.

Note that a fitting orientation is a special kind of discrete vector field.

**Definition:** A pairing $\phi$ is cyclic if there exists a cycle i.e. a sequence of simplices $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n, \alpha_{n+1}$ where $n > 1$ such that the following conditions hold:

1. $\alpha_1 = \alpha_{n+1}$

2. Both $\alpha_i, \alpha_{i+1} \in \partial \beta_i$, but $\alpha_i \neq \alpha_{i+1}$ and $\{\alpha_i, \beta_i\} \in \phi$ for all $i = 1, 2, \cdots, n$

A pairing $\psi$ is called acyclic if it is not cyclic.

A standard result of discrete Morse theory [69] states that a discrete Morse function $f$ on a simplicial complex $X$ gives rise to an acyclic pairing $\phi_f$, where $\{\alpha, \beta\} \in \phi_f$ if and only if $\alpha \in \partial \beta$ and $f(\alpha) \geq f(\beta)$. The simplices of $X$ not occurring in any pair in $\phi_f$ are precisely the critical simplices of $f$. $\phi_f$ is said to be the discrete gradient vector field of $f$.

Conversely, if a pairing $\phi$ is acyclic then $\phi$ is a discrete gradient vector field of some discrete Morse function on $X$.

Two discrete Morse functions are said to be equivalent if their corresponding discrete gradient vector fields are the same.

At this stage, we reproduce some terminology from [71]. The Hasse diagram of a simplicial complex $X$ is a directed acyclic graph on the poset of simplices of $X$, 31
ordered by the relation $\alpha < \beta$ if and only if $\alpha \in \partial \beta$. Given the Hasse diagram $H$ of $X$ and a pairing $\phi$ of simplices, we can construct a modified directed graph $\tilde{H}$ by flipping the edge between any two simplices that appear as a pair in $\phi$. We call this modified graph ‘modified Hasse diagram’. For details, see [71].

Now we state a fundamental theorem, proved in [71].

**Theorem 14.** ([71], 3.14): A pairing $\phi$ is acyclic if and only if the corresponding modified Hasse diagram $\tilde{H}$ is acyclic.

With this background in hand, let us prove the following crucial lemma.

**Lemma 15.** Let $F$ be a forest as described in the theorem 11 and $R$ be a root of $F$. Then $|\sum_\phi \Lambda(\phi)| = |H_{d-1}(F, R)|$, where $\phi$ runs over all distinct discrete gradient vector fields, whose set of all critical simplices is precisely the root complex $R$.

**Proof.** In [61], it was proved that for any rooted forest, $|\sum_\phi \Lambda(\phi)| = |H_{d-1}(F, R)|$, where $\phi$ runs over all fitting orientations of $(F, R)$. Now the crucial observation is that, in the left hand side of the above equation, the cyclic pairings (i.e. fitting orientations containing a cycle) have no contribution as they cancel each other. Orientability implies that any cycle is an ASC, so the previous argument (Lemma 12) applies and the surviving terms correspond to precisely the acyclic pairings, i.e. discrete gradient vector fields with the prescribed properties. \qed

**Corollary 16.** Let $(F, R)$ be a rooted forest as in the previous lemma. Then the number of non-equivalent discrete morse functions on $F$ with $R$ as the set of critical simplices is at least $|H_{d-1}(F, R)|$.

**Corollary 17.** Let $(F, R)$ be a rooted forest as in the Theorem 11. Then $F$ has at least one discrete morse function with the property in Corollary 16.

**Proof.** Since $(F, R)$ is a rooted forest, $\det(\partial_{R,F}) \neq 0$, where $\partial_{R,F}$ is the submatrix of $\partial_d(G)$ obtained by keeping the rows corresponding to $R$ and the columns corresponding to $F$ ([61], Lemma 14). Again $|\det(\partial_{R,F})| = |H_{d-1}(F, R)|$ ([61], Lemma 17). So
we have $|H_{d-1}(F, R)| \neq 0$. Hence, by Corollary 16 we get that $F$ has at least one discrete morse function with the prescribed properties. 

Now we are in a position to prove Theorem 11.

Proof. By Corollary 17, there exists at least one acyclic pairing $\phi$ of $(F, R)$. Let $\tilde{H}$ be the modified Hasse diagram of the simplices of $F$ associated to the pairing $\phi$. Hence by Theorem 14, $\tilde{H}$ is acyclic. Let $\tilde{H}_1$ be the subgraph of $\tilde{H}$, induced by the vertices of the face poset of $F$ corresponding to all $d$–faces of $F$ and all $(d - 1)$–faces of $\bar{R}$. Since $\tilde{H}$ is acyclic, $\tilde{H}_1$ is also acyclic. Hence $\tilde{H}_1$ has a vertex with no inward-oriented arrows. Since each $d$–face is paired, the vertex should be a $(d - 1)$–face $r_1$ of $\bar{R}$, which implies that the coface $f_1$ with which $r_1$ is paired is the only matched face of $F$ containing $r_1$, i.e. $r_1$ is a 'free face' of $f_1$. Hence $F$ collapses onto $F \setminus (\text{int}(r_1) \cup \text{int}(f_1)) = F_1$. We now consider $\tilde{H}_2 = \tilde{H}_1 \setminus \{r_1, f_1\}$(also removing the corresponding edges). Again $\tilde{H}_2$ is acyclic and so we can apply the above process to the simplicial complex $F_1$. Continuing this process, we eventually end up when all the $d$–faces and all $(d - 1)$–faces of $\bar{R}$ are removed. So what is left is precisely the simplicial complex $R$. This completes the proof. 

Theorem 11 is interesting in the sense that this does not hold in general for arbitrary rooted forests. In fact, if $F$ is a closed, connected, non-orientable, triangulated $d$–manifold, $F$ is automatically a forest. Let $R$ be a root of $F$. Then $F$ and $R$ are not homotopy equivalent, as the codimension one homology of $F$, $H_{d-1}(F, \mathbb{Z})$ contains $\mathbb{Z}_2$ as a summand, whereas $H_{d-1}(R, \mathbb{Z})$, being the top homology group of $R$, can not contain any torsion. In section 6.4, we develop a general theory of rooted-forests in arbitrary triangulated closed manifolds. As a recipe to do so, we prove a "rooted-labeled matrix pseudo forest Theorem" in the next section.
6.3 A rooted-labeled matrix pseudo forest Theorem

Let $G$ be a graph (or a one dimensional simplicial complex). Let $E$ be a subset of $E(G)$ (edge set of $G$) and $V$ be a subset of $V(G)$ (vertex set of $G$) such that $|E| = |V|$. A discrete vector-field of $G$, with critical edges $E(G) \setminus E$ and critical vertices $V(G) \setminus V$ is defined to be a bijection $\phi$ between $V$ and $E$ such that each vertex $v \in V$ is mapped to an edge $e \in E$ containing the vertex $v$. A flow line of $\phi$ is a sequence of vertices and edges:

$$v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n, v_{n+1},$$

where $e_i = \phi(v_i) \forall i = 1, 2, \ldots, n$ and each $v_{i+1} \in e_i \forall i = 1, 2, \ldots, n$ and $v_{n+1}$ not necessarily belongs to the set $V$.

For a graph $G$, select a subset $L$ of edges $E(G)$ and call them “labeled edges”.
Now, define the vertex-edge Pseudo Incidence Matrix $\tilde{\partial}(G)$ as the same as the Incidence Matrix $\partial(G)$ except, for the columns corresponding to the edges in $L$, where for each edge $e$ in $L$, each non-zero entry of the corresponding column to $e$ is 1. Note that, as already stated in Section 2, the columns of $\partial(G)$ are ordered in the lexicographic order. Similarly define the Pseudo Laplacian Matrix of $G$, denoted by $\tilde{L}(G)$ is defined to be $(\tilde{\partial})(\tilde{\partial})^T$. Let us define $\tilde{\partial}_{V,E}$ to be the submatrix of $\tilde{\partial}$, with rows and columns of $\tilde{\partial}_{V,E}$ are $V$ and $E$ respectively.

Before stating the next very crucial lemma, let us recall that, a pseudo forest is a graph, each of whose component is either a unicyclic graph (i.e a connected graph with exactly one cycle) or a tree.

**Lemma 18.** Suppose that $G$ is a graph. Let $E \subseteq E(G)$ and $V \subseteq V(G)$ such that $|E| = |V|$. Then $\tilde{\partial}_{V,E}$ is nonsingular if and only if the subgraph $G_E$ of $G$, formed by the edges in $E$ is a pseudo forest, having at least $|V(G)| - |V|$ components, some consisting of trees and each of the remaining components, consisting of the unicyclic graph has an odd number labeled edges in its cycle. Call this pseudo forest, a rooted-labeled pseudo forest, where the roots correspond to the vertices in $V(G) \setminus V$. Moreover, $|\det(\tilde{\partial}_{V,E})| = 2^m$, where $m$ is the number of the components, consisting of the unicyclic graphs.
Proof. First assume that, $\partial_{V,E}$ is nonsingular. Also assume that $V(G)\setminus V \neq \emptyset$. So, let $v \in V(G)\setminus V$. Now, since $\partial_{V,E}$ is nonsingular, $\exists$ at least a discrete vector field $\phi$ of $G$, with critical edges $E(G)\setminus E$ and critical vertices $V(G)\setminus V$. Otherwise $|\text{det}(\partial_{V,E})| = 0$. Clearly, $v$ is a critical vertex of $\phi$. Consider the set of all vertices $v'$, which, following a flow line of the vector field $\phi$, eventually reach the vertex $v$. See the following the figure.

Denote this set of vertices by $\hat{V}$. Clearly the vertex set $\hat{V} \cup \{v\}$ and the edges involved in the aforesaid flow lines form a tree and the tree is the connected component of $G_E$, containing the vertex $v$. This tree can be thought to be a rooted-tree with $v$ as the root vertex. So, for each $v \in V(G)\setminus V$, we have a connected component of $G_E$, containing a rooted-tree.

Now consider a component (if at all exists) of $G_E$, containing no vertices of $V(G)\setminus V$. Since no vertices and edges in this component are critical, they are equal
in number. and hence, this component is a unicyclic graph. Clearly this component has precisely two distinct discrete vector fields, namely the restriction of \( \phi \) on this component and the other, denoted by \( \dot{\phi} \) is same as \( \phi \), but the arrows reversed in the cycle. See the following figure.

Consider the cycle \( C \) in the unicyclic graph. Let the vertex and edge set of the cycle be \( \{v_1, v_2, \ldots, v_k\} \) and \( \{e_1, e_2, \ldots, e_k\} \) respectively. Without loss of generality assume that, among the the edges \( \{e_1, e_2, \ldots, e_k\} \), \( e_1, e_2, \ldots, e_t \) are the labeled edges. Now, to the discrete vector field \( \phi \) restricted to \( C \), we can associate a unique permutation \( \pi_{\phi|C} \in S_k \) (in short \( \pi_{\phi} \)), such that \( \phi \) maps the \( i \)th vertex in \( \{v_1, v_2, \ldots, v_k\} \) to the \( \pi_{\phi}(i) \)th edge in \( \{e_1, e_2, \ldots, e_k\} \) in the lexicographic order and denote the product \( sgn(\pi_{\phi})\Pi_{i=1}^{k} \tilde{\partial}(v_i, \phi(v_i)) \) by \( \Lambda(\phi, C) \). Now we have

\[
\Lambda(\phi, C) = sgn(\pi_{\phi})\Pi_{i=1}^{k} \tilde{\partial}(v_i, \phi(v_i)) \\
= sgn(\pi_{\phi})\Pi_{i=1}^{k} \tilde{\partial}(\phi^{-1}(e_i), e_i) \\
= sgn(\pi_{\phi})\Pi_{i=1}^{t} \tilde{\partial}(\phi^{-1}(e_i), e_i)\Pi_{i=t+1}^{k} \tilde{\partial}(\phi^{-1}(e_i), e_i) \\
= sgn(\pi_{\phi})\Pi_{i=1}^{t} \tilde{\partial}(\phi^{-1}(e_i), e_i)\Pi_{i=t+1}^{k}(-\tilde{\partial}(\phi^{-1}(e_i), e_i))
\]
\[= sgn(c_k \circ \pi_{\hat{\phi}}) \Pi_{i=1}^{t+1} \tilde{\partial}(\hat{\phi}^{-1}(e_i), e_i) \Pi_{i=t+1}^k (-\tilde{\partial}(\hat{\phi}^{-1}(e_i), e_i)) \] [\(c_k\) is a permutation-cycle of length \(k\)]

\[= (-1)^{(k-1)+(k-t)} sgn(\pi_{\hat{\phi}}) \Pi_{i=1}^k \tilde{\partial}(\hat{\phi}^{-1}(e_i), e_i) \]

\[= (-1)^{t-1} sgn(\pi_{\hat{\phi}}) \Pi_{i=1}^k \tilde{\partial}(v_i, \dot{\phi}(v_i)) \]

\[= (-1)^{t-1} \Lambda(\dot{\phi}, C). \]

See the following figure.

In the above figure, the bold edges are labeled edges.

Now remembering that, for each \(v \in V\), \(\tilde{\partial}(v_i, \psi(v_i)) = \pm 1\), for any general vector field \(\psi\) on \(G\), we easily have

\[|\det(\tilde{\partial}_{V,E})| = \Pi_{i=1}^m |\Lambda(\phi, C_i) + \Lambda(\dot{\phi}, C_i)|, \text{ where } \{C_1, C_2, \cdots, C_m\} \text{ is the set of cycles in } G_E. \]

Now we have

\[|\Lambda(\phi, C_i) + \Lambda(\dot{\phi}, C_i)| \]
\[ = |\Lambda(\phi, C_i)|1 + (-1)^{t_{C_i}}| \quad \text{[where } t_{C_i} \text{ is the number of labeled edges in } C_i] \]

\[
= |1 + (-1)^{t_{C_i}}|.
\]

But as \( \det(\tilde{\partial}_{V,E}) \neq 0 \), it follows that \( t_{C_i} \) is odd for all \( i = 1, 2, 3, \cdots, m \) and also we have \( |\det(\tilde{\partial}_{V,E})| = 2^m \). This proves one part of the lemma. For the converse part, the computation of the determinant is exactly similar. \( \square \)

Now let us recall the “Generalized Cauchy-Binet theorem” due to Oliver Knill [67].

**Theorem 19.** Let \( A \) and \( B \) be two \( n \times m \) matrices. Then
\[
\det(x \text{Id} + AB^T) = \sum_{k=0}^{n} \sum_{|P|=k} \det(A_P)\det(B_P), \quad \text{where, } \det(H_P) \text{ is the minor in } H, \text{ masked by a square pattern } P = I \times J.
\]

Combining Lemma 18 and Theorem 19 and remembering that \( \tilde{\mathcal{L}}(\tilde{G}) = (\tilde{\partial})(\tilde{\partial})^T \), we have the following *Rooted-Labeled Matrix Pseudo-Forest Theorem*:

**Theorem 20.** \( \det(\tilde{\mathcal{L}}(\tilde{G}) + x \text{Id}) = \sum_{\Gamma} 4^{m_{\Gamma}} x^{|R_{\Gamma}|} \), where the summation runs over all rooted labeled pseudo forests \( \Gamma \), \( R_{\Gamma} \) (a subset of vertices) is the root set of \( \Gamma \) and \( m_{\Gamma} \) is the number of cycles in \( \Gamma \).

### 6.4 Rooted forests in a triangulated closed manifold

Let \( M \) be a triangulated, closed, \( d \)-dimensional manifold. Now the *Poincare Dual Graph*, denoted by \( PD(M) \) is a graph, whose vertices are the \( d \)-simplices of \( M \) and two vertices are adjacent if and only if their corresponding \( d \)-simplices \( (f_1, f_2 \text{ (say)}) \) share a common \( (d - 1) \)-face \( r \), say). More over, if \( \partial_d[r, f_1] = \partial_d[r, f_2] \) (\( \partial_d \) is the \( d \)-th incidence matrix of \( M \)), then declare the edge in \( PD(M) \), corresponding to \( r \) to be a \textit{labeled edge} in the sense of the previous section. Note that, if \( M \) is non-orientable, its Poincare Dual graph \( PD(M) \) always contains labeled edges (vaguely speaking, traversing along a labeled edge reverses the local orientation of \( M \)).
So we see that, there are two bijections, one from the set of \(d-\)simplices of \(M\) to \(V(PD(M))\) and the other from the set of \((d-1)-\)simplices of \(M\) to \(E(PD(M))\).

As there would be no additional confusion, let us denote these two bijections by the same notation, \(D\), say.

Now, a pseudo-incidence matrix \(\tilde{\partial}(PD(M))\) of \(PD(M)\) is defined as follows:

\[
\tilde{\partial}[v,e] := \partial_d[D^{-1}(e), D^{-1}(v)], \text{ if } e \text{ is not a labeled edge of } PD(M), \text{ otherwise } \tilde{\partial}[v,e] := |\partial_d[D^{-1}(e), D^{-1}(v)]|.
\]

We observe that, if \(F\) is a subset of the set of all \(d-\)faces of \(M\) and \(R\) is a subset of \((d-1)-\)faces of \(M\), such that \(|F| = |\overline{R}|\), then there is a natural one to one correspondence between the set of all fitting orientations of \((F,R)\) and the set of discrete vector fields of \(PD(M)\), with the critical edges \(E(PD(M)) \setminus D(\overline{R})\) and critical vertices \(V(PD(M)) \setminus D(F)\).

Now suppose that \((F,R)\) is a rooted forest of \(M\). It was shown in [61] that, \(|H_{d-1}(F,R)| = |\text{det}(\tilde{D}_{\overline{R},F})|\). Again we can easily see that \(|\text{det}(\tilde{D}_{R,F})| = |\text{det}(\tilde{D}_{(F),D(\overline{R})})|\).

Hence from Lemma 18, we have the following theorem:

**Theorem 21.** Let \((F,R)\) be a rooted forest of \(M\). Then \(PD(M)_{D(\overline{R})}\) is a rooted-labeled pseudo forest, where the roots correspond to the vertices in \(V(PD(M)) \setminus D(F)\).

Moreover, \(|H_{d-1}(F,R)| = 2^m\), where \(m\) is the number of components of \(PD(M)_{D(\overline{R})}\), consisting of the unicyclic graphs.

**Remark:** In fact, modifying the argument of Lemma 18 and using the fact that the Smith Normal forms of a matrix and its transpose are the same, one can easily show that the group \(H_{d-1}(F,R)\) is an elementary abelian group.

As a corollary of this theorem, we get the following interesting result:

**Theorem 22.** Let \(M\) be an orientable manifold and \((F,R)\) be any rooted forest of \(M\). Then \(H_{d-1}(F,R)\) is trivial.

**Proof.** Since \(M\) is orientable, we can always orient its faces, so that there would be no labeled edges in \(PD(M)\). Hence no component of \(PD(M)_{D(\overline{R})}\) consists of unicyclic
Now we prove the following theorem, which is one of the main results of this chapter.

**Theorem 23.** Let \((F, R)\) be a rooted forest of a manifold \(M\). Then \(F\) simplicially collapses onto \(R\) if and only if \(H_{d-1}(F, R)\) is trivial.

**Proof.** One direction is easy, i.e. if \(F\) simplicially collapses onto \(R\), then \(H_{d-1}(F, R)\) is trivial. One can use standard topological arguments (say, for example homology long exact sequence).

The interesting part is the converse, i.e. suppose that \(H_{d-1}(F, R)\) is trivial. By Theorem 21, we can say that, each component of \(PD(M)_{D(R)}\) is a tree and hence \(PD(M)\) has a unique discrete vector field with the critical edges \(E(PD(M))\setminus D(R)\) and critical vertices \(V(PD(M))\setminus D(F)\). More over this vector field is acyclic. So, \((F, R)\) has a unique fitting orientation \(\phi\), which is also a discrete gradient vector field, whose set of all critical simplices is precisely the root complex \(R\). Hence mimicking the proof of theorem 11, we get the desired result.

Combining Theorems 22 and 23 we can reestablish theorem 11.

**Remark:** Theorem 23 does not hold for general simplicial complexes. To see exactly what happens for a general simplicial complex seems to be a challenging problem. This problem seems to have a direct connection to the enumeration of perfect matchings in bipartite graphs and the famous *Zeeman’s conjecture*.

### 6.5 An application to the enumeration of rooted-forests in surfaces

In this section we apply the theorems of the previous sections to obtain an interesting new result on closed triangulated surfaces. Before stating the next theorem we fix some notations. From now on, till the end of this chapter we will use
the notation $\Sigma$ to denote a general closed surface. If $\Sigma$ is non-orientable, then the maximum number of pairwise disjoint Möbius bands in $\Sigma$ is denoted by $g(\Sigma)$. This number is independent of the triangulation of $\Sigma$. We also use the notation $|(F, R)|_\Sigma$ to denote the number of rooted forests $(F, R)$ in $\Sigma$.

**Theorem 24.** Let $\Sigma$ be a closed surface and $L_\Sigma$ be its Laplacian matrix. Then:

1. If $\Sigma$ is orientable, then $\det(I + L_\Sigma) = |(F, R)|_\Sigma$.

2. If $\Sigma$ is non-orientable, then $1 \leq \frac{\det(I + L_\Sigma)}{|(F, R)|_\Sigma} \leq 4g(\Sigma)$.

**Proof. Proof of 1:** This immediately follows from Theorem 22 and the Bernardi-Klivans forest theorem.

**Proof of 2:** Suppose that $(F, R)$ is a rooted forest in $\Sigma$. Then Theorem 21 implies that $|H_1(F, R)| = 2^m$, where $m$ is the number of components of $PD(\Sigma)_{D(\overline{R})}$, consisting of the unicyclic graphs. Now the crucial observation is that the cycle in the unicyclic graph corresponds to a Möbius band in $\Sigma$. Hence $|H_1(F, R)| = 2^m \leq 2g(\Sigma)$. Now putting $x = 1$ in the Bernardi-Klivans forest theorem we get the desired inequality. \qed