Part II

Some problems in combinatorial algebraic topology
Chapter 5

Introduction to part-II

In this chapter, we recall some topological preliminaries needed for the rest of this thesis. We conclude this chapter with a brief summary of the results obtained in this part of the thesis.

5.1 Topological preliminaries

The most fundamental object of study in combinatorial algebraic topology is abstract simplicial complex. An abstract simplicial complex (or simply a simplicial complex) $\Delta$ is defined to be an ordered pair $(V,F)$, where $V$ is any finite set (we only consider finite case here) and $F \subset \mathcal{P}(V)$, the power set of $V$ satisfying the condition: if $\sigma \in F$ and $\tau \subset \sigma$, then $\tau \in F$.

The non-empty elements of $F$ are called the faces of the simplicial complex $\Delta$. Dimension of the face $\sigma$, denoted by $\dim(\sigma)$ is defined to be $|\sigma| - 1$, where $|\sigma|$ is the cardinality of $\sigma$. The zero-dimensional faces of $\Delta$ are called vertices of $\Delta$. A $k$-dimensional face of $\Delta$ is also called a $k-$ simplex of $\Delta$. The dimension of a simplicial complex is defined to be the maximum of the dimensions of its simplices. Note that, graphs are one dimensional simplicial complexes. The Euler Characteristic of a simplicial complex $\Delta$, denoted by $\chi(\Delta)$ is defined to be the summation $\sum_{\sigma \in \Delta} (-1)^{\dim(\sigma)}$.

Suppose that $\Delta$ is an abstract simplicial complex. Let $\tau, \sigma$ are two faces of $\Delta$, such that $\tau \subset \sigma$ and $\sigma$ is a maximal face of $\Delta$ and no other maximal face of $\Delta$ contains $\tau$. then $\tau$ is called a free face of $\Delta$. A simplicial collapse of $\Delta$ is the removal of all simplices $\gamma$ such that $\tau \subset \gamma \subset \sigma$, where $\tau$ is a free face. If additionally we have
dim(τ) = dim(σ) − 1, then this is called an elementary collapse. A simplicial complex that has a sequence of collapses leading to a point is called collapsible.

Let σ be a k−simplex of Δ. Two orderings of its vertex set are equivalent if they differ by an even permutation. If dim(σ) > 0 then the orderings of the vertices of σ fall into two equivalence classes. Each class is called an orientation of σ. An oriented simplex is a simplex σ together with an orientation of σ. Let a₀, a₁, · · · , aₖ be an ordering of the vertices of σ. then we shall use the symbol [a₀, a₁, · · · , aₖ] to denote the oriented simplex. Suppose that b₀, b₁, · · · , bₖ be another vertex-ordering, which differs from the previous ordering by an odd permutation, then we write [a₀, a₁, · · · , aₖ] = −[b₀, b₁, · · · , bₖ]. Note that vertices have only one orientation. Let Cₖ(Δ) be the free Z−Module generated by all oriented k−simplices of Δ. Now for k > 0, we define a homomorphism ∂ₖ : Cₖ(Δ) → Cₖ−1(Δ), called the boundary operator as follows:

Let σ = [v₀, v₁, · · · , vₖ] be an oriented simplex. Then ∂ₖ(σ) = ∑ᵢ₌₀⁽ⁿ⁻¹⁾(−1)ᵢ[v₀, v₁, · · · , ̂vᵢ, · · · , vₖ], where ̂vᵢ means that the vertex vᵢ has been omitted.

Now it is well known and can be checked easily that ∂ₖ−1 ◦ ∂ₖ ≡ 0 for all k ≥ 1. This in fact implies that Im(∂ₖ+1) ⊂ ker(∂ₖ). Then the k−th homology group of Δ, denoted by Hₖ(Δ), is defined to be ker(∂ₖ)/Im(∂ₖ+1).

Suppose that X be a simplicial complex and S be a subcomplex of X. Then ∂ₖ induces a map ̂∂ₖ : Cₖ(X)/Cₖ(S) → Cₖ−1(X)/Cₖ−1(S). We also have ̂∂ₖ−1 ◦ ̂∂ₖ ≡ 0.

And similarly as before, the k−th relative homology group of the pair (X, S), denoted by Hₖ(X, S) is defined to be ker(̂∂ₖ)/Im(̂∂ₖ+1).

A combinatorial d-manifold (or sometimes called a pseudo manifold) is a d−dimensional simplicial complex M, such that the following conditions hold:

1. M is homogeneous, i.e. each simplex of M is a face of some d−simplex;

2. M is unramified, i.e. each (d − 1)−simplex of M is a face of at most two d−simplices;

3. M is strongly connected, i.e. for any two d−simplices Δ₁ and Δ₂, there exists a
sequence of $d$–simplices starting and ending with $\Delta_1$ and $\Delta_2$ respectively with two neighboring simplices in the sequence share a common $(d - 1)$–face.

A combinatorial manifold $M$ is said to be orientable, if its top dimensional homology group is nontrivial, otherwise $M$ is said to be non-orientable.

5.2 Objective of part-II

The notion of forests and their roots in higher dimensions was introduced by Olivier Bernardi and Caroline Klivans in [61] to obtain a generalization of the “Matrix Forest Theorem” in higher dimension. In fact, they proved that, for a $d$–dimensional simplicial complex $G$,

$$\sum_{(F,R)} |H_{d-1}(F, R)|^2 x^{|R|} = det(L_G + xId),$$

where the summation runs over all rooted forests of $G$ and $Id$ is the identity matrix of dimension $|G_{d-1}|$, and $L_G$ is the Laplacian matrix. A combinatorial interpretation of the factor $|H_{d-1}(F, R)|$ for a general simplicial complex $G$ is a challenging problem. (See [61]). In Chapter 6 we prove that the relative homology $H_{d-1}(F, R)$ is a 2–group for any rooted forest $(F, R)$ in a triangulated closed $d$–manifold $M^d$. We also give a combinatorial interpretation of the cardinality of the group $H_{d-1}(F, R)$, which answers an open question of Olivier Bernardi and Caroline Klivans for any closed manifold. We have shown that for any rooted forest $(F, R)$ in a closed manifold $M^d$, the forest $F$ simplicially collapses onto its root $R$ iff $H_{d-1}(F, R)$ is trivial.

In Chapter 7, we give a purely combinatorial proof of a Ryser-type formula for determinants inspired by multivariate finite operator calculus. This argument also includes a combinatorial proof of an interesting identity about Stirling number of second kind. Also we give a topological proof of Ryser’s formula for permanents.