Chapter 3

On the power graph of the direct product of two groups

Here we investigate relationship of $\mathcal{P}(G_1 \times G_2)$ with $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$ for any two groups $G_1$ and $G_2$. There are three fundamental products of graphs, namely, cartesian product, direct product and strong product of graphs [58]. Here we show that, in general, $\mathcal{P}(G_1 \times G_2)$ is not isomorphic to either of these products of $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$. So we introduce a new product of two graphs $\Gamma_1$ and $\Gamma_2$, which we call generalized product of $\Gamma_1$ and $\Gamma_2$. Reason behind such naming is that each of the cartesian, direct and strong products is a special case of the generalized product of two graphs.

Our main theorem states that $\mathcal{P}(G_1 \times G_2)$ is isomorphic to a generalized product of the power graphs $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$.

3.1 Main results

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. For any two integers $a$ and $b$, $AP(a, b)$ is the arithmetic progression with initial term $a$ and common difference $b$.

Let $\Gamma$ be a graph. Then we define a generalization on $\Gamma$ to be a function $W : A(\Gamma) \cup \Delta \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$, where $A(\Gamma)$ is the arc set of $\Gamma$ and $\Delta = \{(v, v) : \text{visavedrtextof}\Gamma\}$. 

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**Definition:** Let \((G_1, W_1)\) and \((G_2, W_2)\) be two graphs equipped with two generalizations \(W_1, W_2\) respectively. Then the generalized product \(G_1 \times_W G_2\) is a graph with vertex set \(V(G_1) \times V(G_2)\) and \((g_1, g_2) \sim (g_1', g_2')\) if and only if the following two conditions hold simultaneously:

(i) \((g_1, g_2) \neq (g_1', g_2')\)

(ii) \(AP(W_1(g_1, g_1')) \cap AP(W_2(g_2, g_2')) \cap \mathbb{N} \neq \emptyset\) or \(AP(W_1(g_1, g_1')) \cap AP(W_2(g_2, g_2')) \cap \mathbb{N} \neq \emptyset\)

We shall show below that the generalized product of two graphs \(G_1\) and \(G_2\) generalizes the aforesaid products on graphs, namely the direct product, cartesian product and normal product. First, we recall the definitions of these products.

**Definition:** Let \(G_1, G_2\) be two graphs. Then:

(i) The direct product \(G_1 \times G_2\) is defined as follows:

\[V(G_1 \times G_2) = V(G_1) \times V(G_2)\]

and \((g_1, g_2) \sim (g_1', g_2')\) if and only if \(g_1 \sim g_1'\) and \(g_2 \sim g_2'\)

(ii) The cartesian product \(G_1 \Box G_2\) is defined follows:

\[V(G_1 \Box G_2) = V(G_1) \times V(G_2)\]

and \((g_1, g_2) \sim (g_1', g_2')\) if and only if either \(g_1 = g_1'\) and \(g_2 \sim g_2'\) or \(g_1 \sim g_1'\) and \(g_2 = g_2'\)

(iii) The normal product \(G_1 * G_2\) is defined follows:

\[V(G_1 * G_2) = V(G_1) \times V(G_2)\]

and \((g_1, g_2) \sim (g_1', g_2')\) if and only if either \(g_1 \sim g_1'\) and \(g_2 = g_2'\) or \(g_1 = g_1'\) and \(g_2 \sim g_2'\) or \(g_1 \sim g_1'\) and \(g_2 = g_2'\)

**Theorem 6.** Each of the products, defined above is a particular generalized product.

**Proof.** Let \(G_1\) and \(G_2\) be two graphs.

(i) Take \(W_1 : A(G_1) \cup \Delta \rightarrow \mathbb{Z}_+ \times \mathbb{Z}_+\) to be \(W_1(x, y) = (1, 1)\) for \(x \neq y\) and \(W_1(x, x) = (0, 0)\) for all \(x \in V(G_1)\) and take \(W_2\) similarly. Then it is not difficult to verify that \(G_1 \times_W G_2 = G_1 \times G_2\).

(ii) Take \(W_1 : A(G_1) \cup \Delta \rightarrow \mathbb{Z}_+ \times \mathbb{Z}_+\) to be \(W_1(x, y) = (1, 0)\) for \(x \neq y\) and \(W_1(x, x) = (1, 1)\) for all \(x \in V(G_1)\) and take \(W_2(x, y) = (2, 0)\) for \(x \neq y\) and \(W_2(x, x) = (1, 1)\) for all \(x \in V(G_2)\). Then \(G_1 \times_W G_2 = G_1 \Box G_2\).
(iii) Take $W_1 : A(G_1) \cup \Delta \rightarrow \mathbb{Z}^2 \times \mathbb{Z}^2$ to be $W_1(x, y) = (1, 0)$ for $x \neq y$ and $W_1(x, x) = (1, 1)$ for all $x \in V(G_1)$ and take $W_2(x, y) = (1, 0)$ for $x \neq y$ and $W_2(x, x) = (1, 1)$ for all $x \in V(G_2)$. Then $G_1 \times_W G_2 = G_1 \ast G_2$. 

Now we try to uncover the relationship of the power graphs $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$ with $\mathcal{P}(G_1 \times G_2)$ for any two groups $G_1$ and $G_2$.

**Theorem 7.** Let $G_1$ and $G_2$ be two proper finite groups. Then $\mathcal{P}(G_1 \times G_2)$ is never isomorphic to $\mathcal{P}(G_1) \boxtimes \mathcal{P}(G_2)$.

**Proof.** Suppose, on the contrary that the two graphs, stated in the theorem are isomorphic. Let $e_{G_i}$ be the identity of the group $G_i$. Let $g_1, g_2$ be two elements of $G_1$ and $G_2$ respectively, such that $g_i \neq e_{G_i}$ for $i = 1, 2$. Then by the definition of the cartesian product of graphs, we see that $(e_{G_1}, e_{G_2})$ is not adjacent to $(g_1, g_2)$ in $\mathcal{P}(G_1) \boxtimes \mathcal{P}(G_2)$, whereas $(e_{G_1}, e_{G_2}) \sim (g_1, g_2)$ in $\mathcal{P}(G_1 \times G_2)$. A contradiction! 

**Example :** In this example we show that $\mathcal{P}(G_1 \times G_2)$ is not generally isomorphic to either of the direct or normal products of $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$. Take, for example $G_1 = G_2 = \mathbb{Z}_2$. Then $\mathcal{P}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has precisely three edges, each edge emanating from the identity of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and connects the remaining three vertices, whereas $\mathcal{P}(\mathbb{Z}_2) \ast \mathcal{P}(\mathbb{Z}_2)$ is the complete graph $K_4$ and $\mathcal{P}(\mathbb{Z}_2) \times \mathcal{P}(\mathbb{Z}_2)$ is a graph with precisely two edges.

Now, we prove the main theorem of this chapter. But before that, we state the following simple lemma.

**Lemma 8.** Let $G$ be a group and $a, b \in G$. Let $n$ be the smallest positive integer such that $a^n = b$. Then $\{m \in \mathbb{N} : a^m = b\} = AP(n, o(a))$.

**Theorem 9.** For two groups $G_1$ and $G_2$, $\mathcal{P}(G_1 \times G_2) = \mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$ for some choice of generalizations $W_1$ and $W_2$ on $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$ respectively.

**Proof.** For a group $G$, first we specify the choice of the generalization $W$ as $W(a, b) = (m, o(a))$, where $m$ is the smallest positive integer for which $a^m = b$; provided such
an $m$ exists, and otherwise set $W(a,b) = (0,0)$. Now we show that with respect to this particular generalization $W$, $\mathcal{P}(G_1 \times G_2) = \mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$. Let $(g_1,g_2) \sim (\hat{g}_1,\hat{g}_2)$ in $\mathcal{P}(G_1 \times G_2)$. This implies that $(g_1,g_2)^m = (\hat{g}_1,\hat{g}_2)$ or $(g_1,g_2) = (\hat{g}_1,\hat{g}_2)^n$ for some $m,n > 0$. Consider the first case i.e. $(g_1,g_2)^m = (\hat{g}_1,\hat{g}_2)$. This implies that $g_1^m = \hat{g}_1$ and $g_2^m = \hat{g}_2$. Combining this and the previous lemma we get that $AP(W_1(g_1,\hat{g}_1)) \cap AP(W_2(g_2,\hat{g}_2)) \cap \mathbb{N} \neq \emptyset$ and as a consequence $(g_1,g_2) \sim (\hat{g}_1,\hat{g}_2)$ in $\mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$. Similarly we can show the converse i.e. if $(g_1,g_2) \sim (\hat{g}_1,\hat{g}_2)$ in $\mathcal{P}(G_1) \times_W \mathcal{P}(G_2)$, $(g_1,g_2) \sim (\hat{g}_1,\hat{g}_2)$ in $\mathcal{P}(G_1 \times G_2)$.

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