CHAPTER 4

MAGNETOHYDRODYNAMIC FLOW OF VISCOUS FLUID BETWEEN TWO PARALLEL POROUS PLATES

4.1 INTRODUCTION

The Magnetohydrodynamic flow between two parallel porous plates is a classical problem in Fluid dynamics and it is known as the Hartmann flow. The solution of the above problem has many applications in magnetohydrodynamic pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and sprays.

The influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite, parallel, stationary and insulated plates was studied Hartmann (1937). The problem of steady flow of an incompressible viscous fluid through a porous channel with rectangular cross section, when the Reynolds number is low was studied and a perturbation solution assuming normal wall velocities to be equal was obtained Berman (1953).

A detailed analysis of forced convection heat transfer to an electrically conducting liquid flowing in a channel with transverse magnetic field was studied Perlmutter and Siegel (1961). The Hall effect on the steady motion of electrically conducting and viscous fluids in channels was studied Tani (1962).
The effect of the Hall currents on the steady magnetohydrodynamic couette flow with heat transfer was studied Soundalgekar, Vighnesam and Takhar (1979, 1986). The temperatures of the two plates were assumed either to be constant Soundalgekar, Vighnesam and Takhar (1979) or to vary linearly along the plates in the direction of the flow Soundalgekar and Uplekar (1986).

The effect of Hall current on the steady Hartmann flow subjected to a uniform suction and injection at the boundary plates was studied Abo-El–Dahab (1993). The effect of temperature dependent viscosity on the flow in a channel has been studied in the hydromagnetic case Attia and Kotb (1996) and Attia (1999).

In this chapter the steady two dimensional laminar flow of an incompressible viscous fluid between two parallel porous plates in the presence of a transverse magnetic field is considered by assuming the normal wall velocities to be equal. The perturbation solution obtained for this problem reduces to the results of Berman when the Hartmann number is zero Berman (1953).

4.2 MATHEMATICAL FORMULATION

The steady laminar flow of an incompressible viscous fluid between two parallel porous plates is considered in the presence of a transverse magnetic field of strength Ho applied perpendicular to the walls. The origin is taken at the centre of the channel and let $x$ and $y$ be the coordinate axes parallel and perpendicular to the channel walls.
The length of the channel is assumed to be \( L \) and \( 2h \) is the distance between the two plates. Let \( u \) and \( v \) be the velocity components in the \( x \) and \( y \) directions respectively.

The equation of continuity is

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(4.1)

The equations of momentum are

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\sigma B^2 u}{\rho}
\]

(4.2)

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

(4.3)

where \( \sigma \) is the electrical conductivity and \( B = \mu_e H_0 \) = the electromagnetic induction, \( \mu_e \) being the magnetic permeability. The boundary conditions for the above problem are \( u(x, h) = 0 \), \( u(x, -h) = 0 \), \( v(x, h) = V \) and \( v(x, -h) = -V \) where \( V \) is the velocity of suction at the walls of the channel.

Let \( \eta = \frac{y}{h} \) = dimensionless distance and let \( v = \frac{\mu}{\rho} \) = kinematic viscosity and the equations (4.1) (4.2) and (4.3) become

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} = 0
\]

i.e., \( \frac{\partial u}{\partial x} + \frac{1}{h} \frac{\partial u}{\partial \eta} = 0 \)

(4.4)
where \( \rho \) is the density of the fluid, \( \mu \) the coefficient of viscosity and \( p \) the pressure.

The boundary conditions are converted into

\[
\begin{align*}
&u(\,x, \, 1) = 0, \, u(\,x, \, -1) = 0 \quad \text{and} \\
&v(\,x, \, 1) = V, \, v(\,x, \, -1) = -V
\end{align*}
\tag{4.7}
\]

Since at \( y=h \), \( \eta = \frac{y}{h} = 1 \) and at \( y = -h \), \( \eta = \frac{y}{h} = -1 \)

Let \( \psi \) be the stream function such that

\[
\begin{align*}
&u = \frac{\partial \psi}{\partial y} = \frac{1}{h} \frac{\partial \psi}{\partial \eta} \\
&v = -\frac{\partial \psi}{\partial x}
\end{align*}
\tag{4.9}
\]

The equation of continuity can be satisfied by a stream function of the form

\[
\psi(x, \, y) = [h \, U(0) - Vx]f(\eta) \quad (G.M. \, Shrestha \, 1999)
\tag{4.11}
\]
where $U(0)$ is the average entrance velocity at $x = 0$. From equation (4.11), the velocity components (4.9) and (4.10) are given by

$$u = \frac{1}{h} \left[ hU(0) - Vx \right] f'(\eta)$$  \hspace{1cm} (4.12)

$$v = Vf(\eta)$$  \hspace{1cm} (4.13)

where the prime denotes the differentiation with respect to the dimensionless variable $\eta = \frac{y}{h}$. Since the fluid is being withdrawn at constant rate from both the walls, $v$ is independent of $x$. Using (4.12) and (4.13) in (4.5) and (4.6), the equations of momentum reduce to

$$- \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{h} \left( hU(0) - Vx \right) f'(\eta) \left( -\frac{V}{h} \right) f''(\eta) + \frac{Vf(\eta)}{h} \frac{1}{h} (hU(0) - Vx) f''(\eta)$$

$$- V \frac{1}{h^2} \frac{1}{h} (hU(0) - Vx) f'''(\eta) + \frac{\sigma B^2}{\rho} \frac{1}{h} (hU(0) - Vx) f'(\eta)$$

$$\Rightarrow - \frac{1}{\rho} \frac{\partial p}{\partial x} = \left[ U(0) - \frac{Vx}{h} \right] \left( -\frac{V}{h} (f'' - f'^2) + \frac{Vf}{h^2} f'' - \frac{\nu}{h^2} f''' + \frac{\sigma B^2}{\rho} f' \right)$$

i.e.,

$$- \frac{1}{\rho} \frac{\partial p}{\partial x} = \left[ U(0) - \frac{Vx}{h} \right] \left( \frac{V}{h} \left( f'' - f'^2 \right) - \frac{\nu}{h^2} f''' + \frac{\sigma B^2}{\rho} f' \right)$$  \hspace{1cm} (4.14)

and

$$- \frac{1}{hp} \frac{\partial p}{\partial \eta} = \frac{Vf}{h} f' - \frac{\nu}{h^2} f''$$

i.e.,

$$- \frac{1}{hp} \frac{\partial p}{\partial \eta} = \frac{V^2}{h} f' - \frac{\nu}{h^2} f''$$
\[
- \frac{1}{h \rho} \frac{\partial^2 p}{\partial \eta^2} = \frac{V^2}{h} f' f'' - \frac{V V}{h^2} f'''
\]  
(4.15)

Now differentiating (4.15) w.r.t. \(x\), it is seen that \(\frac{\partial^2 p}{\partial x \partial \eta} = 0\)  
(4.16)

Differentiating (4.14) w.r.t. \(\eta\), it can be proved that

\[- \frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial \eta} = \left( U(0) - \frac{V x}{h} \right) \frac{d}{d \eta} \left( \frac{V}{h} \left( f f' - f' f' \right) - \frac{V}{h^2} f' + \frac{\sigma B^2 f'}{\rho} \right) \]
(4.17)

From (4.16), equation (4.17) can be written as

\[
\frac{d}{d \eta} \left[ \frac{V}{h} \left( f f' - f' f' \right) - \frac{V}{h^2} f'' - \frac{\sigma B^2 f'}{\rho} \right] = 0
\]

i.e.,

\[
\frac{d}{d \eta} \left[ -\frac{V}{h} \frac{h^2}{\nu} (f f' - f' f') + f''' - \frac{h^2}{\nu} \frac{\sigma B^2}{\rho} f' \right] = 0
\]

i.e.,

\[
\frac{d}{d \eta} \left[ f''' + R (f'^2 - f f') - aR f' \right] = 0
\]
(4.18)

which is true for all \(x\).

Let \(R\) = Suction Reynolds number = \(\frac{h V}{V}\)

\(M =\) Hartmann number = \(B h \left( \frac{\sigma}{\sqrt{\nu \rho}} \right)^2\)

Integrating (4.18) w.r.t. \(\eta\) and substituting the above expressions w.r.t. it is seen that

\[
f''' + R (f'^2 - f f') - aR f' = K
\]
(4.19)

where \(aR = M^2\) and if \(M=0\) then the above differential equation reduces to the result of Berman(1953).
where \( a = \frac{H^2_s \mu^2 \sigma h}{\rho V} \) and \( K \) is an arbitrary constant.

Boundary conditions on \( f(\eta) \) are \( u(x,1) = \frac{1}{h}(hU(0) - Vx)f'(1) = 0 \)
\[ \Rightarrow f'(1) = 0, \]
\[ u(x,-1) = \frac{1}{h}(hU(0) - Vx)f'(-1) = 0 \quad \Rightarrow f'(-1) = 0, \]
\[ v(x,1) = Vf(1) = V \quad \Rightarrow f(1) = 1 \quad \text{and} \]
\[ v(x,-1) = Vf(-1) = -V \quad \Rightarrow f(-1) = -1 \]

Hence the boundary conditions are \( f(1) = 1, \quad f(-1) = -1, \quad f'(1) = 0 \)
and \( f'(-1) = 0 \) \quad (4.20)

Hence the solution of the equations of motion and continuity is given by a nonlinear third order differential equation (4.19) subject to the boundary conditions (4.20).

4.3 APPROXIMATE ANALYTIC SOLUTION

The nonlinear ordinary differential equation (4.19) subject to boundary conditions (4.20) must in general be integrated numerically. However for the special case when ‘\( R \)’ and ‘\( a \)’ are small, approximate analytic results can be obtained by use of a regular perturbation approach. In this situation \( f \) and \( k \) may be expanded in the form.

\[
f = \sum_{n=0}^{\infty} R^n f_n(\eta) \quad (4.21)
\]
\[
\sum_{n=0}^{\infty} K_n R^n = k
\]

(4.22)

(4.21) \Rightarrow f(\eta) = f_0(\eta) + Rf_1(\eta) + R^2 f_2(\eta) + \ldots \ldots

when \( \eta = -1 \),

\[f(-1) = f_0(-1) + Rf_1(-1) + R^2 f_2(-1) + \ldots \ldots\]

\[-1 = f_0(-1) + Rf_1(-1) + R^2 f_2(-1) + \ldots \ldots \quad (\text{since } f(-1) = -1)\]

Comparing, it is seen that \( f_0(-1) = -1,\ f_1(-1) = f_2(-1) = \ldots = 0 \).

when \( \eta = 1 \),

\[f(1) = f_0(1) + Rf_1(1) + R^2 f_2(1) + \ldots \]

\[1 = f_0(1) + Rf_1(1) + R^2 f_2(1) + \ldots \quad (\text{since } f(1) = 1)\]

Comparing, it is evident that \( f_0(1) = 1,\ f_1(1) = f_2(1) = \ldots = 0 \)

\[\Rightarrow f'(\eta) = f_0'(\eta) + Rf_1'(\eta) + R^2 f_2'(\eta) + \ldots \ldots\]

when \( \eta = -1 \),

\[f'(-1) = 0 = f_0'(-1) + Rf_1'(-1) + R^2 f_2'(-1) + \ldots \ldots \quad (\text{since } f'(-1) = 0)\]

Comparing, it follows that \( f_0'(-1) = 0,\ f_1'(-1) = f_2'(-1) = \ldots = 0 \)

when \( \eta = 1 \),

\[f'(1) = 0 = f_0'(1) + Rf_1'(1) + R^2 f_2'(1) + \ldots \ldots \quad (\text{since } f'(1) = 0)\]

Comparing, it is clear that \( f_0'(1) = 0,\ f_1'(1) = f_2'(1) = \ldots = 0 \)
Therefore $f_n$ satisfies the conditions

$$f_0(-1) = -1, \ f_0(1) = 1, \ f_0'(-1) = 0 = f_0'(1) \quad (4.23)$$

and $f_n(-1) = f_n'(-1) = 0$ and $f_n(1) = f_n'(1) = 0$ when $n > 0 \quad (4.24)$

Substituting (4.21) and (4.22) in (4.19),

$$(f'' + Rf'' + R^2f'' \ldots \ldots) + R[(f'' + Rf'' + R^2f'' \ldots \ldots)^2 -$$

$$((f'' + Rf'' + R^2f'' \ldots \ldots)(f'' + Rf'' + R^2f'' \ldots \ldots)) -$$

$$a (f'' + Rf'' + R^2f'' \ldots \ldots)] = K_0 + K_1R + K_2R^2 + \ldots \ldots$$

Here $f_n$'s and $k_n$'s are independent of $R$.

Equating the coefficients of $R^0, R^1, R^2$ it follows that

$$f'' = K_0 \quad (4.25)$$

$$f'' + f'' - f_0f_0'' - a f_0' = K_1 \quad (4.26)$$

$$f'' + 2f_0'f_1' - f_0f_1'' - f_0''f_1 - a f_1' = K_2 \quad (4.27)$$

From (4.25), $f''(\eta) = K_0$

Integrating w.r.t. ‘$\eta$’, $f''(\eta) = K_0\eta + A$

Again integrating w.r.t. $\eta$, it is seen that

$$f'_0(\eta) = \frac{K_0\eta^2}{2} + A\eta + B$$

Now integrating w.r.t. ‘$\eta$’ it follows that
\[ f_0(\eta) = \frac{K_0}{6} \eta^3 + \frac{A}{2} \eta^2 + B \eta + C \]  

(4.28)

Applying the boundary conditions (4.23)

\[ f_0(-1) = -1, \quad f_0(1) = 1, \quad f'_0(-1) = 0 \quad \text{and} \quad f'_0(1) = 0 \]

to the above equations, it is clear that

\[ -1 = -\frac{K_0}{6} + \frac{A}{2} - B + C \]  

(4.29)

\[ 1 = \frac{K_0}{6} + \frac{A}{2} + B + C \]  

(4.30)

Subtracting equation (4.29) from equation (4.30) it follows that

\[ K_0 = 6(1-B) \]

Adding the above two equations (4.29) and (4.30), it is clear that

\[ A = -2C. \]

Also

\[ 0 = \frac{K_0}{2} - A + B \]  

(4.31)

and

\[ 0 = \frac{K_0}{2} + A + B \]  

(4.32)

Adding the above two equations (4.31) and (4.32), it follows that

\[ K_0 = -2B. \]

Subtracting equation (4.31) from (4.32), it is clear that \( A = 0 \)

\[ \Rightarrow C = 0, \quad B = \frac{3}{2} \quad \text{and} \quad K_0 = -3 \]
\[ f_0(\eta) = -\frac{3}{6} \eta^3 + \frac{3}{2} \eta = \frac{\eta}{2} (3 - \eta^2) \]  
(4.33)

From equation (4.26),

\[ f''_1 + f'_0^2 - f_0 f'' - af'_0 = K_1 \]

\[ \Rightarrow f''_1 + \left( \frac{K_0}{2} \eta^2 + A \eta + B \right)^2 - \left( \frac{\eta}{2} (3 - \eta^2) \right) (K_0 \eta + A) \]

\[-a \left( \frac{K_0}{2} \eta^2 + A \eta + B \right) = K_1 \]

\[ \Rightarrow f''_1(\eta) + \left( -\frac{3}{2} \eta^2 + \frac{3}{2} \right)^2 - \left( \frac{3\eta}{2} - \frac{\eta^3}{2} \right) (-3\eta) - a \left( -\frac{3\eta^2}{2} + \frac{3}{2} \right) = K_1 \]

\[ \Rightarrow f''_1(\eta) + \left( \frac{9}{4} \eta^4 + \frac{9}{4} - \frac{18}{4} \eta^2 + \frac{9}{2} \eta^2 - \frac{3}{2} \eta^4 \right) + \frac{3a}{2} (\eta^2 - 1) = K_1 \]

\[ \Rightarrow f''_1(\eta) = K_1 - \left( \frac{3}{4} \eta^4 + \frac{9}{4} \right) - \frac{3}{2} a (\eta^2 - 1) \]

\[ \Rightarrow f''_1(\eta) = \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) - \frac{3}{2} a \eta^2 - \frac{3}{4} \eta^4. \]

Integrating w.r.t. ‘\eta’, it follows that

\[ f'''(\eta) = \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) \eta - \frac{3a}{2} \eta^3 - \frac{3}{4} \eta^5 + A \]

Integrating the above equation w.r.t. ‘\eta’, it is seen that

\[ f'_1(\eta) = \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) \eta^2 - \frac{a \eta^4}{8} - \frac{\eta^6}{40} + A \eta + B \]

Integrating the above equation w.r.t. ‘\eta’, it is clear that
\[ f_1(\eta) = \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) \eta^3 + \frac{an^5}{40} - \frac{\eta^7}{280} + \frac{A\eta^2}{2} + B\eta + C \] (4.34)

Since

\[ f_1(1) = 0, \quad f_1(-1) = 0, \quad f_1'(1) = 0 \quad \text{and} \quad f_1'(-1) = 0 \]

\[ \Rightarrow 0 = \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) \frac{1}{6} - \frac{a}{40} - \frac{1}{280} + \frac{A}{2} + B + C \] (4.35)

and

\[ 0 = \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) \left( \frac{-1}{6} \right) + \frac{a}{40} + \frac{1}{280} + \frac{A}{2} - B + C \] (4.36)

Adding the above equations (4.35) and (4.36), it is clear that

\[ A = -2C \]

Subtracting equation (4.36) from (4.35), it follows that

\[ 0 = \frac{1}{3} \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) - \frac{a}{20} - \frac{1}{140} + 2B \]

\[ \Rightarrow 2B = \frac{a}{20} + \frac{1}{140} + \frac{1}{3} \left( -K_1 - \frac{3a}{2} + \frac{9}{4} \right) \]

\[ \Rightarrow 2B = \frac{-9a}{20} + \frac{53}{70} - \frac{K_1}{3} \] (4.37)

Now \( f_1'(1) = 0 \)

\[ \Rightarrow \frac{1}{2} \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) - \frac{a}{8} - \frac{1}{40} + A + B = 0 \] (4.38)
Now \( f'_1(-1) = 0 \)

\[
\Rightarrow \frac{1}{2}\left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) - \frac{a}{8} - \frac{1}{40} - A + B = 0 \quad (4.39)
\]

Subtracting the equation (4.39) from (4.38), it is seen that

\[ A = 0 \Rightarrow C = 0 \quad (\therefore A = -2C) \]

Adding (4.38) and (4.39), it is evident that

\[
\left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) - \frac{a}{4} - \frac{1}{20} + 2B = 0 \quad (4.40)
\]

\[
\Rightarrow 2B = \left( K_1 + \frac{3a}{2} - \frac{9}{4} \right) + \frac{a}{4} + \frac{1}{20} = -K_1 - \frac{3a}{2} + \frac{9}{4} + \frac{a}{4} + \frac{1}{20}
\]

\[
\Rightarrow 2B = -K_1 - \frac{5a}{4} + \frac{23}{10} \quad (4.41)
\]

Since \( 2B = -K_1 - \frac{5a}{4} + \frac{23}{10} \),

From equations (4.37) and (4.41), it follows that

\[
\Rightarrow -K_1 - \frac{5a}{4} + \frac{23}{10} = \frac{-9a}{20} + \frac{53}{70} - \frac{K_1}{3}
\]

\[
\Rightarrow \frac{-2K_1}{3} - \frac{16a}{20} + \frac{108}{70} = 0 \quad \Rightarrow K_1 = \frac{81}{35} - \frac{6a}{5} \quad (4.42)
\]

Now, \( 2B = \frac{-9a}{20} + \frac{53}{70} - \frac{1}{3} \left( \frac{-6a}{5} + \frac{81}{35} \right) = \frac{-9a}{20} + \frac{53}{70} + \frac{2a}{5} - \frac{27}{35} \)

\[
\Rightarrow B = -\frac{a}{40} - \frac{1}{140} \quad (4.43)
\]
\[ f_i(\eta) = \left[ \left( -\frac{6a}{5} + \frac{81}{35} \right) + \frac{3a}{2} - \frac{9}{4} - \frac{a^3}{2} \right] \eta^3 - \frac{a}{40} \eta^5 - \frac{\eta^7}{140} \eta \]

\[ \Rightarrow f_i(\eta) = -\frac{a}{5} \eta^3 + \frac{27}{70} \eta^3 + \frac{3a}{3} \eta^3 - \frac{3\eta^3}{8} - \frac{a}{40} \eta^5 - \frac{\eta^7}{140} \]

\[ = -\frac{\eta^7}{140} + \eta^3 \left( \frac{27}{3} - \frac{3}{8} \right) + \eta^3 \left( -\frac{1}{5} + \frac{1}{4} \right) - \frac{a}{40} \eta^5 - \frac{a}{40} \eta - \frac{\eta}{140} \]

(4.44)

Since \( f_0(\eta) = \frac{1}{2} (3\eta - \eta^3) \)

\[ \Rightarrow f_0'(\eta) = \frac{1}{2} \left( 3 - 3\eta^2 \right) \] and

\[ f_0''(\eta) = \frac{1}{2} (-6\eta) = -3\eta \]

Also \( f_1(\eta) = \frac{1}{280} \left( -\eta^7 + 3\eta^3 - 2\eta \right) + \frac{a}{40} (-\eta^5 + 2\eta^3 - \eta) \)

\[ \Rightarrow f_1'(\eta) = \frac{1}{280} \left( -7\eta^6 + 9\eta^2 - 2 \right) + \frac{a}{40} (-5\eta^4 + 6\eta^2 - 1) \]

\[ f_1''(\eta) = \frac{1}{280} \left( -42\eta^5 + 18\eta \right) + \frac{a}{40} (-20\eta^3 + 12\eta) \]

From equation (4.27), it is clear that

\[ K_2 = f_2'' + 2f_0'f_1' - f_0f_1'' - f_1f_0'' + af_1' \]

\[ f_2''(\eta) = K_2 - 2f_0'f_1' + f_0f_1'' + f_1f_0'' + af_1' \]

\[ = K_2 - 2 \left[ \frac{1}{2} (3 - 3\eta^2) \left( \frac{1}{280} (-7\eta^6 + 9\eta^2 - 2) + \frac{a}{40} (-5\eta^4 + 6\eta^2 - 1) \right) \right] \]
\[\frac{1}{2}(3\eta - \eta^3) \left( \frac{-42\eta^5 + 18\eta}{280} + \frac{a}{40} (-20\eta^3 + 12\eta) \right)\]

\[+ \left[ \frac{-3\eta^7 + 3\eta^3 - 2\eta}{280} + \frac{a}{40} (-\eta^5 + 2\eta^3 - \eta) \right]\]

\[+ a \left( \frac{-7\eta^6 + 9\eta^2 - 2}{280} \right) + \frac{a}{40} (-5\eta^4 + 6\eta^2 - 1)\]

\[\Rightarrow f_2'(\eta) = K_2 + \frac{1}{280} \left[ \frac{3\eta^2 - 3}{-7\eta^6 - 35a\eta^4 + 42a\eta^2 + 9\eta^2 - 2 - 7a} \right]\]

\[+ (3\eta - \eta^3)(-2\eta^5 - 70a\eta^3 + 9\eta + 42a\eta)\]

\[+ (3\eta^8 + 21a\eta^6 - 9\eta^4 - 42a\eta^4 + 6\eta^2 + 21a\eta^2) +\]

\[a(-7\eta^6 - 35a\eta^4 + 9\eta^2 + 42a\eta - 2 - 7a)\]

\[f_2''(\eta) = K_2 + \frac{1}{280} \left[ \frac{-2\eta^8 - 105a\eta^6 + 126a\eta^4 + 27\eta^4 - 6\eta^2 - 21a\eta^2}{+ 2\eta^6 + 105a\eta^4 - 126a\eta^2 - 27\eta^2 + 6 + 2a} +\right.\]

\[(-63\eta^6 - 210a\eta^4 + 27\eta^2 + 126a\eta^2 + 2\eta^8 + 70a\eta^6 - 9\eta^4 - 42a\eta^4)\]

\[+ (3\eta^8 + 21a\eta^6 - 9\eta^4 - 42a\eta^4 + 21a\eta^2 + 6\eta^2)\]

\[+ (-7a\eta^6 - 35a^2\eta^4 + 9a\eta^2 + 42a^2\eta^2 - 2a - 7a^2)\]

\[\Rightarrow f_2''(\eta) = K_2 + \frac{1}{280} \left[ \frac{3\eta^8 - 21a\eta^6 - 42a\eta^6 - 63a\eta^4 - 35a^2\eta^4 + 9\eta^4}{+ 9a\eta^2 + 42a^2\eta^2 + 19a + 6 - 7a^2} \right]\]

Integrating w.r.t. \(\eta\), it follows that

\[f_2'''(\eta) = K_2\eta + \frac{1}{280} \left[ \frac{3\eta^9 - 21a\eta^7 - 42a\eta^7 - 63a\eta^5 - 35a^2\eta^5 + 9\eta^5}{9 + \frac{9a\eta^3}{3} + \frac{42a^2\eta^3}{3} + 19a\eta + 6\eta - 7a^2\eta} \right] + A\]
\[ f''_2(\eta) = K_2 \eta + \frac{1}{280} \left[ \frac{\eta^9}{3} - 3a \eta^7 - 6a \eta^7 - \frac{63a \eta^5}{5} - 7a^2 \eta^5 + \frac{9a \eta^5}{5} + 3a \eta^5 \right] + A \]

Integrating the above equation w.r.t. \( \eta \), it is seen that

\[ f'_2(\eta) = \frac{K_2}{2} \eta^2 + \frac{1}{280} \left[ \frac{\eta^{10}}{30} - \frac{3a \eta^8}{8} - \frac{6a \eta^8}{8} - \frac{63a \eta^6}{30} - \frac{7a^2 \eta^6}{6} + \frac{9a \eta^6}{30} \right] + A \eta + B \]

Integrating the above equation w.r.t. \( \eta \), it follows that

\[ f_2(\eta) = \frac{K_2}{6} \eta^3 + \frac{1}{280} \left[ \frac{\eta^11}{330} - \frac{a \eta^9}{24} - \frac{a \eta^9}{12} - \frac{3a \eta^7}{10} - \frac{a^2 \eta^7}{6} + \frac{3a \eta^7}{70} \right] + \frac{A \eta^2}{2} + B \eta + C \]

(4.45)

Given \( f_2(1)=0 \) and \( f_2(-1)=0 \)

\[ 0 = \frac{K_2}{6} + \frac{1}{280} \left[ \frac{1}{330} - \frac{a}{24} - \frac{1}{12} - \frac{3a^2}{6} + \frac{3a}{70} + \frac{3a}{20} \right] + \frac{A}{2} + B + C \]

\[ \Rightarrow 0 = \frac{A}{2} + B + C + \frac{K_2}{6} + \frac{1}{280} \left[ \frac{1}{330} - \frac{1}{12} + \frac{3}{70} + 1 \right] + a \left[ \frac{1}{24} - \frac{3}{10} + \frac{3}{20} + \frac{19}{6} \right] + \frac{a^2}{6} \left[ \frac{-1}{6} + \frac{7}{10} - \frac{7}{6} \right] \]

\[ \Rightarrow 0 = \frac{A}{2} + B + C + \frac{K_2}{6} + 0.00344 + 0.01063a - 0.00226a^2 \]

\[ \Rightarrow \frac{A}{2} + B + C + \frac{K_2}{6} + 0.00344 + 0.01063a - 0.00226a^2 = 0 \]  

(4.46)
Now \( f_2(-1)=0 \)

\[
0 = -\frac{K_2}{6} + \frac{1}{280} \left[ \begin{array}{c}
-\frac{1}{330} + \frac{a}{24} + \frac{1}{12} + \frac{3}{10} a + \frac{a^2}{6} - \frac{3}{70} - \frac{3a}{20} \\
-\frac{7a^2}{10} + \frac{19a}{6} - \frac{1}{6} + \frac{7a^2}{6}
\end{array} \right] + \frac{A}{2} - B + C
\]

\[
0 = \frac{A}{2} - B + C - \frac{K_2}{6} + \frac{1}{280} \left[ \begin{array}{c}
-\frac{1}{330} + \frac{1}{12} - \frac{3}{70} - 1 \\
a \left[ \frac{1}{24} + \frac{3}{10} - \frac{3}{20} - \frac{19}{6} \right] + a^2 \left[ \frac{1}{6} - \frac{7}{10} + \frac{7}{6} \right]
\end{array} \right]
\]

\[
\Rightarrow \frac{A}{2} - B + C - \frac{K_2}{6} - 0.00344 - 0.01063a + 0.00226a^2 = 0 \quad (4.47)
\]

\[(4.46) + (4.47) \Rightarrow 0 = A + 2C \Rightarrow A = -2C \quad (4.48)\]

\[(4.46) - (4.47) \Rightarrow 2B + \frac{K_2}{3} + 0.00688 + 0.02126a - 0.00452a^2 = 0 \quad (4.49)\]

Since \( f_2'(1) = 0, f_2'(-1) = 0 \)

\[
0 = \frac{K_2}{2} + \frac{1}{280} \left[ \begin{array}{c}
\frac{3a}{30} - \frac{3}{8} + \frac{63}{4} a - \frac{7a^2}{6}
\frac{3}{10} + \frac{3a}{4} + \frac{7a^2}{2} + \frac{19a}{2} + 3 - \frac{7a^2}{6}
\end{array} \right] + A + B
\]

\[
\Rightarrow A + B + \frac{K_2}{2} + \frac{1}{280} \left[ \begin{array}{c}
\left[ \frac{1}{30} - \frac{3}{4} + \frac{3}{10} + 3 \right]
\left[ \frac{3}{8} - \frac{63}{30} + \frac{3}{4} + \frac{19}{2} \right] + a^2 \left[ \frac{7}{6} + \frac{7}{2} - \frac{7}{2} \right]
\end{array} \right] = 0
\]

\[
\Rightarrow A + B + \frac{K_2}{2} + \frac{1}{280} \left[ 2.5833 + 7.775a - 1.1667a^2 \right] = 0
\]
\[ A + B + \frac{K_2}{2} + 0.00923 + 0.0277a - 0.00417a^2 = 0 \]  \hspace{1cm} (4.50)

Now \( f_2'(-1)=0 \)

\[ 0 = \frac{K_2}{2} + \frac{1}{280} \left[ \frac{1}{30} - \frac{3a}{8} - \frac{3}{4} \cdot \frac{63}{30} a - \frac{7a^2}{6} + \frac{3}{10} + \frac{3a}{4} + \frac{7a^2}{2} + \frac{19a}{2} + 3 - \frac{7a^2}{6} \right] - A + B \]

\[ -A + B + \frac{K_2}{2} + 0.00923 + 0.0277a - 0.00417a^2 = 0 \]  \hspace{1cm} (4.51)

Subtracting equation (4.50) from (4.51), it is clear that \( A=0 \).

Since \( A=-2C \), it follows that \( C=0 \).

\[ \therefore (4.46) \Rightarrow B + \frac{K_2}{6} + 0.00344 + 0.01063a - 0.00226a^2 = 0 \]

and (4.51) \( \Rightarrow B + \frac{K_2}{2} + 0.00923 + 0.0277a - 0.00417a^2 = 0 \)

(4.46) - (4.51) \( \Rightarrow - \frac{K_2}{3} - 0.00579 - 0.01707a + 0.00191a^2 = 0 \)

\[ \Rightarrow K_2 = -3(0.00579 + 0.01707a - 0.00191a^2) \]

\[ \Rightarrow K_2 = -(0.0174 + 0.05121a - 0.00573a^2) \]

\[ \therefore B = -\frac{K_2}{6} - 0.00344 - 0.01063a + 0.00226a^2 \]

\[ \Rightarrow B = 0.0029 + 0.00854a - 0.00096a^2 - 0.00344 - 0.01063a + 0.00226a^2 \]

\[ \Rightarrow B = -0.00054 - 0.00209a + 0.0013a^2 \]
\[ f_2(\eta) = \eta^3(-0.0029 - 0.00854a + 0.000955a^2) \]

\[ + \frac{1}{280} \left( \frac{\eta^{11}}{330} - \frac{a\eta^9}{24} - \frac{\eta^9}{12} - \frac{3a\eta^7}{10} - \frac{a^2\eta^7}{6} + \frac{3\eta^7}{70} + \frac{3a\eta^5}{20} + \frac{7a^2\eta^5}{10} \right) \]

\[ + \eta\left(-0.00054 - 0.00209a + 0.0013a^2\right) \]

i.e., \[ f_2(\eta) = \frac{1}{280} \left( \frac{\eta^{11}}{330} - \frac{a\eta^9}{24} - \frac{\eta^9}{12} - \frac{3a\eta^7}{10} - \frac{a^2\eta^7}{6} + \frac{3\eta^7}{70} + \frac{3a\eta^5}{20} + \frac{7a^2\eta^5}{10} \right) \]

\[ + \eta\left(-0.00054 - 0.00209a + 0.0013a^2\right) \] (4.52)

Also \[ K_0 = -3 \] (4.53)

\[ K_1 = -\frac{6a}{5} + \frac{81}{35} \] (4.54)

\[ K_2 = (-0.0174 - 0.05121a + 0.00573a^2) \] (4.55)

Hence the first order perturbation solutions for \( f(\eta) \) and \( K \) are

\[ f^{(1)}(\eta) = f_0(\eta) + Rf_1(\eta) \text{ and} \]

\[ K^{(1)} = K_0 + K_1R = -3 + \left(-\frac{6a}{5} + \frac{81}{35}\right)R \] (4.56)

i.e. \[ f^{(1)}(\eta) = \frac{\eta}{2}(3 - \eta^2) + \frac{R}{280}\left(-\eta^7 + 3\eta^5 - 2\eta\right) + \frac{M^2}{40}\left(-\eta^5 + 2\eta^3 - \eta\right) \] (4.57)

where \( aR = M^2 \)
The Second order perturbation solutions for $f(\eta)$ and $K$ are

$$f^{(2)}(\eta) = f_0(\eta) + R f_1(\eta) + R^2 f_2(\eta)$$

$$K^{(2)} = K_0 + K_1 R + K_2 R^2$$

i.e. $f^{(2)}(\eta) = \frac{\eta}{2} (3 - \eta^2) + \frac{R}{280} (-3\eta^2 + 3\eta^3 - 2\eta) + \frac{M^2}{40} (-\eta^5 + 2\eta^3 - \eta)$

$$+ \frac{R^2 \eta^{11}}{92400} - \frac{M^2 R \eta^9}{6720} - \frac{R^2 \eta^9}{3360} - \frac{3M^2 R \eta^7}{2800} - \frac{M^4 \eta^7}{1680} + \frac{3R^2 \eta^7}{1960} + \frac{3M^2 R \eta^5}{5600} + \frac{7M^4 \eta^5}{2800}$$

$$+ \eta^3 (0.000671 R^2 + 0.0027695 M^2 R - 0.0032116 M^4)$$

$$+ \eta (-0.0054 R^2 - 0.00209 M^2 R + 0.0013 M^4)$$

$\therefore$ (a $R = M^2$, a $R^2 = aR R = M^2 R$, a$^2 R^2 = M^4$)

$$K^{(2)} = -3 + \left( -\frac{6a}{5} + \frac{81}{35} \right) R - 0.0174 R^2 - 0.05121 M^2 R + 0.00573 M^4$$

The above results reduce to the results of Berman when $M = 0$ (Berman 1953).

Hence the first order expressions for the velocity components are

$$u(x, \eta) = \left( U(0) - \frac{Vx}{h} \right) f'(\eta)$$
(4.61)

$$v(\eta) = V f(\eta)$$

(4.62)

$$= V \left[ \frac{1}{2} \eta(3-\eta^2) + \frac{R}{280}(3\eta^3 - 2\eta - \eta^7) + \frac{M^2}{40}(2\eta^3 - \eta - \eta^5) \right]$$

4.4 PRESSURE DISTRIBUTION

From equation (4.14), it is seen that

$$\frac{-1}{\rho \left( \frac{h^2}{V} \right)} \frac{\partial p}{\partial x} = \left[ -\frac{Vh}{Vh} \left( ff'' - f' \right) + f''' + \frac{\sigma B^2}{\rho} \left( \frac{-h^2}{V} \right) f' \right] \left( U(0) - \frac{Vx}{h} \right)$$

\[ \Rightarrow \frac{h^2}{\rho V} \frac{\partial p}{\partial x} = \left( U(0) - \frac{Vx}{h} \right) \left( -\frac{Vh}{Vh} \left( ff'' - f' \right) + f''' + \frac{h^2 \sigma B^2 f'}{\rho V} \right) \]

\[ = \left( U(0) - \frac{Vx}{h} \right) \left( f'''(\eta) + R(f'(\eta) - f(\eta)f''(\eta)) - M^2 f'(\eta) \right) \]

\[ \Rightarrow \frac{h^2}{\rho V} \frac{\partial p}{\partial x} = \left( U(0) - \frac{Vx}{h} \right) \left( f'''(\eta) + R(f'(\eta) - f(\eta)f''(\eta)) - M^2 f'(\eta) \right) \]

Since \( f''' + R(f'^2 - ff'' - af') = K \) (from (4.20))

it is seen that \( \frac{h^2}{\rho V} \frac{\partial p}{\partial x} = K U(0) - \frac{Vx}{h} \)
and since \( f'''(\eta) + R \left( f''(\eta) - f(\eta)f''(\eta) \right) - M^2 f'(\eta) = K \) (from 4.19) it is clear that

\[
\frac{\partial p}{\partial x} = \frac{K \rho V}{h^2} \left( U(0) - \frac{V_x}{h} \right) = \frac{K \mu}{h^2} \left( U(0) - \frac{V_x}{h} \right) \Rightarrow \left\{ V = \frac{\mu}{\rho} \right\} \tag{4.63}
\]

Now, from equation (4.15), it is seen that

\[
\frac{\partial p}{\partial \eta} = \frac{\mu V}{h} f''(\eta) - \rho V^2 \frac{f(\eta) f'(\eta)}{h} \tag{4.64}
\]

\[
\frac{\partial p}{\partial \eta} = -\frac{\nu^2}{h} \frac{f f'}{h^2} + \frac{V V}{h^2} \frac{f''}{h^2}
\]

\[
= \frac{\mu \nu}{\rho^2} \frac{f''}{h^2} + \frac{\nu}{h^2} \frac{f''}{h^2}
\]

Since \( dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \), it follows that

\[
\Rightarrow dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial \eta} d\eta \left( \therefore \eta = \frac{y}{h} \right)
\]

\[
\Rightarrow dp = \frac{K \mu}{h^2} \left( U(0) - \frac{V_x}{h} \right) dx + \left( \frac{\mu V}{h} f''(\eta) - \rho V^2 \frac{f(\eta) f'(\eta)}{h} \right) d\eta \tag{4.65}
\]

Integrating (4.65), it is seen that

\[
p(x, \eta) = \frac{K \mu}{h^2} \left( U(0)x - \frac{V_x^2}{2h} \right) + \frac{\mu V}{h} \int f''(\eta) d\eta
\]

\[
- \rho V^2 \int f(\eta) f'(\eta) d\eta + K_i
\]
\[ \begin{align*}
\frac{K\mu}{h^2} & \left( U(0)x - \frac{Vx^2}{2h} \right) + \frac{\mu V}{h} f'(\eta) - \rho v^2 \int f(\eta) f'(\eta) d\eta + K_1 \\
& \quad + \frac{K\mu}{h^2} \left( U(0)x - \frac{Vx^2}{2h} \right) + \left( \frac{\mu V}{h} f'(\eta) - \frac{\rho v^2 f^2(\eta)}{2} \right) + K_1 \\
& \quad + \frac{K\mu}{h^2} \left( U(0)x - \frac{Vx^2}{2h} \right) + \frac{\mu V}{h} \left( f'(\eta) - f'(0) \right)
\end{align*} \] 

\( (4.66) \)

\[ p(x, \eta) = p(0, 0) - \frac{\rho V^2}{2} f^2(\eta) + \frac{K\mu}{h^2} \left[ U(0)x - \frac{Vx^2}{2h} \right] + \frac{\mu V}{h} \left( f'(\eta) - f'(0) \right) \]  

\( (4.67) \)

\[ : \text{The pressure drop in the major flow direction is given by} \]

\[ p(0, \eta) - p(x, \eta) = \frac{K\mu}{h^2} \left( \frac{Vx^2}{2h} - xU(0) \right) \]  

\( (4.68) \)

4.5 \hspace{1cm} \textbf{NUMERICAL SOLUTION}

The approximate results of the previous section are not reliable when the Reynolds number is not small (Shrestha 1999). To obtain the detailed information on the nature of the flow for different values of R and M, a numerical solution to the governing equations is necessary. For different ranges of the parameters R and M, the two point boundary value problem expressed by equations (4.19) and (4.20) has been integrated by using R–K Gill’s method (Jeffrey Winicour 1997) and the graphs have been drawn.
4.6 DISCUSSION

The axial velocity and the radial velocity profiles have been drawn for values of R in the range of $0 \leq R \leq 10$ and different values of M. These are shown in Figures 4.1-4.6. Figure 4.1 represents the axial velocity profiles for $M = 0$ when R takes the values 0.5, 1.0 and 5.0. It is seen that the axial velocity decreases in the central region and increases near the walls with the increase of R. This correlates well with the result of Berman (1953).

Figure 4.2 shows the axial velocity profiles for $M = 0.5, 0.707, 1.581$ for the values of R = 0.5, 1.0 and 5.0 respectively. As M increases the axial velocity decreases in the central region and increases near the walls.

In Figures 4.3 and 4.4 it is seen that as M increases the axial velocity profiles become flat in the central portion and steep near the walls. This gives that for large M, the fluid moves like a block which shows some sort of rigidity. This confirms the idea that in conducting fluids, magnetic field brings rigidity in the fluid. Hence it is observed that $f'(\eta)$ decreases with increase in the values of R and the profile is parabolic. This is in good agreement and correlates well with the results of Berman (1953).

The function $f(\eta)$ (velocity profiles) is plotted against $\eta$ for various values of R in Figures 4.5 and 4.6 respectively. It is observed that for $R > 0$ and for different values of M in the region $-1 \leq \eta \leq 0$, $f$ decreases with increase of R while in the region $0 \leq \eta \leq 1$, $f$ increases with increase of R where $0 \leq R \leq 10$. 
Figure 4.1 Velocity profile when $M=0$ for different values of $R$

Figure 4.2 Velocity profiles when $M=0.5$, $M=0.707$, $M=1.1581$ for different values of $R$
Figure 4.3  Velocity profiles when $M=0.707$, $M=1.0$, $M=2.236$, $M=2.828$, $M=3.162$ for different values of $R$

Figure 4.4  Velocity profiles when $M=0.866$, $M=1.225$, $M=2.739$, $M=3.464$ for different values of $R$
Figure 4.5  Radial velocity profiles when \( M=1.225, M=2.739, M=3.464 \) for different values of \( R \)

Figure 4.6  Radial velocity profiles when \( M=1.0, M=2.236, M=2.828, M=3.162 \) for different values of \( R \)
4.7 CONCLUSION

In the above analysis a class of solutions of the magnetohydrodynamic flow of viscous fluid between two parallel porous plates is presented, in the presence of a transverse magnetic field when the fluid is being withdrawn through both the walls of a channel at the same rate. The result obtained for this problem reduces to the result of Berman (1953) when the Hartmann number is Zero (i.e. when $M = 0$).