CHAPTER 5

NUMERICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS FOR UNSTEADY MAGNETOHYDRODYNAMIC FLOW BETWEEN TWO PARALLEL POROUS PLATES

5.1 INTRODUCTION

The Unsteady Magnetohydrodynamic flow between two parallel porous plates is a classical problem in fluid dynamics whose solution has many applications in magnetohydrodynamic (MHD) power generators, magnetohydrodynamic pumps, accelerators, aerodynamic heating, polymer technology, petroleum industry, centrifugal separation of matter from fluid, purification of crude oil and fluid droplets and sprays.

Hartmann and Lazarus (1937) studied the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite parallel, stationary and insulated plates. Hassanien and Mansour (1990) discussed the unsteady magnetic flow through a porous medium between two infinite parallel plates. There has been a renewed interest shown by Makinde and Sibanda (1998) in studying Magnetohydrodynamic (MHD) flow and heat transfer in porous media because of the effect on magnetic fields on the performance of many systems. Aboul-Hassan and Attia (2002) discussed the flow of a conducting Visco elastic fluid between two horizontal porous plates in the presence of a transverse-magnetic field.

Hazeem Ali Attia (2005) studied the unsteady laminar flow of an incompressible viscous fluid and heat transfer between two parallel plates in the presence of a uniform suction and injection considering variable properties.


The objective of this study is to analyse the unsteady Magnetohydrodynamic flow of viscous fluid between two parallel porous plates when the fluid is being withdrawn through both the walls of the channel at the same rate. The solution of this problem reduces to the results of Ganesh and Krishnambal (2006) when $\alpha = 0$.

5.2 FORMULATION OF THE PROBLEM

The unsteady laminar flow of an incompressible viscous fluid between two parallel porous plates is considered in the presence of a transverse magnetic field of strength $H_0$ applied perpendicular to the walls. The origin is taken at the centre of the channel and let $x$ and $y$ be the coordinate axes parallel and perpendicular to the channel walls. The length of
the channel is assumed to be L and 2h is the distance between the two plates. Let \( u \) and \( v \) be the velocity components in the \( x \) and \( y \) directions respectively.

The equation of continuity is
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  
(5.1)

Equations of momentum are
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\sigma_e B_0^2 u}{\rho}
\]  
(5.2)

and
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]  
(5.3)

where \( \sigma_e \) is the electrical conductivity and \( B_0 = \mu_e H_0 \), \( \mu_e \) being the magnetic permeability.

The boundary conditions are \( u = 0 \) on \( y = h \) and \( y = -h \); \( v = v_0 e^{i\omega t} \) on \( y = h \) and \( v = -v_0 e^{i\omega t} \) on \( y = -h \), where \( v_0 \) is the velocity of suction at the walls of the channel.

Let \( \eta = \frac{y}{h} \), \( u = u(x,y) e^{i\omega t} \), \( v = v(x,y) e^{i\omega t} \), \( p = p(x,y) e^{i\omega t} \) where \( \omega \) is the frequency and the equations (5.1), (5.2) and (5.3) become
\[
\frac{\partial u}{\partial x} + \frac{1}{h} \frac{\partial v}{\partial \eta} = 0
\]  
(5.4)

\[
\frac{\partial u}{\partial x} + \frac{v}{h} \frac{\partial u}{\partial \eta} + i\omega u = - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 u}{\partial \eta^2} \right) - \frac{\sigma_e B_0^2}{\rho} u
\]  
(5.5)
\[
\frac{u}{h} \frac{\partial v}{\partial \eta} + \frac{v}{h} \frac{\partial u}{\partial \eta} + i\omega v = -\frac{1}{\rho h} \frac{\partial p}{\partial \eta} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 v}{\partial \eta^2} \right)
\]  
(5.6)

where \( \nu = \frac{\mu}{\rho} \) = Kinematic Viscosity, \( \rho \) the density of the fluid, \( \mu \) the coefficient of viscosity and \( p \) the pressure.

The boundary conditions are converted into

\[ u(x,1) = 0, \; u(x,-1) = 0, \]  
(5.7)

and \( v(x,1) = v_0, \; v(x,-1) = -v_0 \)  
(5.8)

Let \( \psi \) be the stream function such that

\[ u = \frac{1}{h} \frac{\partial \psi}{\partial \eta} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \]  
(5.9)

(5.10)

The equation of continuity can be satisfied by a stream function of the form

\[ \Psi(x,\eta) = (hu(0) - v_0 x) f(\eta) \]  
(5.11)

where \( u(0) \) is the average entrance velocity at \( x = 0 \). From equation (5.11), the velocity components (5.9) and (5.10) are given by

\[ u = \frac{1}{h} \left( hu(0) - v_0 x \right) f'(\eta) \]  
(5.12)

\[ v = v_0 f(\eta) \]  
(5.13)

where the prime denotes the differentiation with respect to the dimensionless variable \( \eta = \frac{y}{h} \). Since the fluid is being withdrawn at constant rate from both the walls, \( v_0 \) is independent of \( x \).

Using (5.12) and (5.13) in (5.5) and (5.6), the equations of momentum reduce to
\[-\frac{1}{\rho} \frac{\partial \rho}{\partial x} = \frac{i \omega}{h} (hu(0) - v_0 x) f'(\eta) + \frac{1}{h} (hu(0) - v_0 x) f''(\eta) + \frac{v_0}{h} \left(1 - \frac{v_0}{h} \right) \frac{f'}{f'}(\eta)\]

\[+ \left( \frac{v_0 f(\eta)}{h} \right) \frac{1}{h} (hu(0) - v_0 x) f''(\eta) - v_0 \frac{1}{h} \left(1 - \frac{v_0}{h} \right) \frac{f'}{f'}(\eta) + \frac{\sigma_v B_0^2}{\rho} \frac{1}{h} (hu(0) - v_0 x) f''(\eta)\]

i.e., \[-\frac{1}{\rho} \frac{\partial \rho}{\partial x} = \left( u(0) - \frac{v_0 x}{h} \right) \left( i \omega f' - \frac{v_0 f''}{h^2} + \frac{\sigma_v B_0^2}{\rho} f' + \frac{v_0 (ff'' - f'^2)}{h^2} \right) \] (5.14)

and \[-\frac{1}{h \rho} \frac{\partial \rho}{\partial \eta} = -i \omega (-v_0) f(\eta) + \left( \frac{v_0 f(\eta)}{h} \right) v_0 f'(\eta) - \frac{v \cdot v_0 f''(\eta)}{h^2} \]

i.e., \[-\frac{1}{h \rho} \frac{\partial \rho}{\partial \eta} = i \omega v_0 f - \frac{v_0^2}{h^2} f''(\eta) - \frac{v \cdot v_0 f''(\eta)}{h^2} \]

\[-\frac{1}{h \rho} \frac{\partial \rho}{\partial \eta} = i \omega v_0 f - \frac{v \cdot v_0 f''}{h^2} + \frac{v_0^2 ff'}{h} \] (5.15)

Now differentiating (5.15) w.r.t. ‘x’, it is clear that

\[\frac{\partial^2 \rho}{\partial x \partial \eta} = 0 \] (5.16)

Differentiating (5.14) w.r.t. ‘\eta’ , it follows that

\[\frac{\partial^2 \rho}{\partial x \partial \eta} = \left( u(0) - \frac{v_0 x}{h} \right) \frac{d}{d \eta} \left( i \omega f' - \frac{v_0 f''}{h^2} + \frac{\sigma_v B_0^2}{\rho} f' + \frac{v_0 (ff'' - f'^2)}{h^2} \right) \] (5.17)

From (5.16), equation (5.17) can be written as

\[\frac{d}{d \eta} \left( i \omega f' - \frac{v_0 f''}{h^2} + \frac{\sigma_v B_0^2}{\rho} f' + \frac{v_0 (ff'' - f'^2)}{h^2} \right) = 0 \] (5.18)

which is true for all x.

Let \[R = \text{Suction Reynolds number} = \frac{hv_0}{\nu} \]
\[ M = \text{Hartmann number} = B_0 h \left( \frac{\sigma_e}{\nu \rho} \right)^{\frac{1}{2}} \]

Integrating (5.18) w.r.t. \( \eta \) and substituting the above expressions it follows that

\[ -\frac{\nu}{h^2} f'' + \frac{\nu_0 (ff'' - f'H^2)}{h} + i\omega f' + \frac{\sigma_e B_0^2 f'}{\rho} = K \quad (5.19) \]

where \( K \) is an arbitrary constant.

Dividing by \(-\nu / h^2\), it follows that

\[ f'' - \frac{i h^2 \omega}{\nu} f' - \frac{\sigma_e B_0^2 h^2 f'}{\rho \nu} - \frac{\nu_0 h^2}{\nu h} (ff'' - f'H^2) = K \]

i.e.,

\[ f'' - \alpha^2 h^2 f' - aR f' - R (ff'' - f'H^2) = K \]

\[ \therefore \quad \text{Equation (5.19) can be rewritten as} \]

\[ f'' + R(f'H^2 - ff'') - aRf' - \alpha^2 h^2 f' = K \]

where

\[ a = \frac{H_0^2 \mu_e^2 \sigma_e h}{\rho \nu_0}, \quad aR = \frac{h^2}{\nu} \cdot \frac{\sigma_e B_0^2}{\rho} = M^2, \quad B_0 = H_0 \mu_e, \quad \alpha^2 = \frac{i \rho \omega}{\mu} \]

i.e.,

\[ f'' - (\alpha^2 h^2 + M^2) f' + R(f'H^2 - ff'') = K \quad (5.20) \]

Let \( A = \alpha^2 h^2 + M^2 \)

\[ f'' - Af' + R(f'H^2 - ff'') = K \quad (5.21) \]
Differentiating (5.21) w.r.t ‘η’, it is seen that

\[ f'''' + R(f 'f'' - f f''') - Af'' = 0 \]  \hspace{1cm} (5.22)

Boundary conditions on \( f(\eta) \) are

\[ f(1) = 1, \ f(-1) = -1, \ f '(1) = 0 \ \text{and} \ f '(-1) = 0 \]  \hspace{1cm} (5.23)

Hence the solution of the equations of motion and continuity is given by a nonlinear fourth order differential equation (5.22) subject to the boundary conditions (5.23).

5.3 NUMERICAL SOLUTION

To obtain the detailed information on the nature of the flow for different values of R and A, a numerical solution to the governing equations is necessary. For different ranges of parameters R and A, the two point boundary value problem expressed by equations (5.22) and (5.23) has been integrated by using R-K Gill’s method (Jeffrey Winicour 1997) and the graphs have been drawn for the dimensionless function \( f ' \) and \( f \). These are shown in Figures 5.1-5.8.

Figures 5.1–5.4 represent the variation of the dimensionless function \( f ' \) for different values of A namely A= 1,3,5 and 10 and for R>0. These profiles decrease in the central region and increase near the walls of the channel with the decrease of R. It is also seen that as A increases considerably the velocity profiles tend to become flat in the central region and steep near the walls as seen in Figures 5.3 and 5.4. This shows that for the larger values of A, the fluid moves like a block with some sort of rigidity. This confirms the idea that in conducting fluids, magnetic field brings rigidity in the fluid. Hence it is observed that \( f(\eta) \) decreases with the decrease in the values of R and the profile is parabolic which is different when compared to the steady case.
In Figure 5.5, we see that graphs for variation of the dimensionless function $f'$ is drawn for $A=1$ and for different values of $R$ namely $R<0$. From the above graph we see that $f'$ decreases as $R$ decreases.

The function $f(\eta)$ ($\eta$- velocity profiles) is plotted against $\eta$ for various values of $R$ in Figure 5.6. It is observed that if $R>0$ and for different values of $A$ in the region $-1 \leq \eta \leq 0$, $f$ decreases with increase of $R$ while in the region $0 \leq \eta \leq 1$, $f$ increases with increase of $R$. If $\alpha = 0$, the equation (5.20) reduces to $f'' - M^2 f' + R (f^2 - f f'') = K$ which is same as the equation obtained in the paper “Magnetohydrodynamic flow of viscous fluid between two parallel porous plates” by Ganesh and Krishnambal (2006) and the solution obtained here reduces to the result of the above paper when $\alpha = 0$.

![Figure 5.1](image_url)  
**Figure 5.1** Variation of the dimensionless function $f'$ when $A=1$ and for $R>0$
Figure 5.2 Variation of the dimensionless function $f'$ when $A=3$ and for $R > 0$

Figure 5.3 Variation of the dimensionless function $f'$ when $A=5$ and for $R>0$
Figure 5.4  Variation of the dimensionless function \( f' \) when \( A=10 \) and for \( R > 0 \)

Figure 5.5  Variation of the dimensionless function \( f' \) when \( A=1 \) and for \( R < 0 \)
Figure 5.6  Variation of the dimensionless function $f$ when $A =0.75$ and for $R > 0$

5.4  CONCLUSION

In the above analysis, it is seen that in the presence of a transverse magnetic field the class of solutions of unsteady magnetohydrodynamic flow of viscous fluid between two parallel porous plates is presented when the fluid is being withdrawn through both the walls of a channel at the same rate. The result obtained here reduces to the results of Ganesh and Krishnambal (2006) when $\alpha = 0$. 