CHAPTER 1

Introduction and Preliminaries

In this thesis, we consider three problems, two are related to theta series associated with certain quadratic forms and the third one is about the construction of Shimura and Shintani liftings. The first problem is about finding explicit formulas for the number of representations of these quadratic forms, which extends various works done by many authors ([1, 2, 3, 5, 6, 7, 8, 40, 41]), the second one is about obtaining Ramanujan-Mordell type formulas, which extends the works of Ramanujan [66], Mordell [50], Cooper et. al [22], Ye [82]. The third problem is about the construction of explicit Shimura and Shintani maps between certain subspaces of modular forms of half-integral and integral weights. This work generalizes the result of Choi-Kim [18] for odd square-free levels. These three problems are discussed in three different chapters, and each chapter’s introduction gives a detailed account of the history of the problem and the results obtained in the respective chapter.

In this chapter, we give some preliminary facts on modular forms of integral and half-integral weights.
1.1 Notation

Let \( \mathbb{N} \), \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \) be the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively. For a complex number \( z \), \( \text{Re}(z) \) and \( \text{Im}(z) \) denote the real and imaginary parts of \( z \) respectively. For \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z} \setminus \{0\} \), we write \( a|b \) when \( b \) is divisible by \( a \) and \( a \mod b \) means that \( a \) varies over a complete set of residue classes modulo \( b \). Let \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) be the complex upper half plane. Unless stated otherwise we denote by \( q = e^{2\pi i z}, i = \sqrt{-1} \) for \( z \in \mathbb{H} \). For a commutative ring \( R \), we write the set of all \( n \times n \) matrices with entries in \( R \) by \( M_n(R) \). Next we mention few special group of matrices. The general linear group over \( \mathbb{Q} \);

\[
GL_2^+(\mathbb{Q}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}) : ad - bc > 0 \right\}.
\]

Special linear group over \( \mathbb{Z} \) (full modular group);

\[
SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}.
\]

It is known that \( GL_2^+(\mathbb{Q}) \) acts on the Poincaré upper half-plane \( \mathbb{H} \) by the fractional linear transformation as follows. For any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}) \) and \( z \in \mathbb{H} \), we let

\[
\gamma z := \frac{az + b}{cz + d} \in \mathbb{H}.
\]  

Then for any integer \( k \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}) \), the slash operator on functions \( f : \mathbb{H} \to \mathbb{C} \) is defined by

\[
(f |_k \gamma)(z) := (\det \gamma)^{k/2}(cz + d)^{-k} f(\gamma z).
\]
§1.1. Notation

For a positive integer $N$, the principal congruence subgroup of level $N$ is denoted by $\Gamma(N)$ and defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv 1(N), b \equiv c \equiv 0(N) \right\},$$

$$= \text{Ker} \left( SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}) \right).$$

A subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ is said to be a congruence subgroup of $SL_2(\mathbb{Z})$ if it contains a principal congruence subgroup of level $N$, for some $N \in \mathbb{Z}, N > 0$. The smallest such $N$ is called the level of $\Gamma$. Besides $\Gamma(N)$, the following two congruence subgroups are equally important. We list them below:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0(N) \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1(N), c \equiv 0(N) \right\}.$$}

The inclusion relation amongst these congruence subgroups is given by $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$. Unless stated otherwise, we always let $k \in \mathbb{Z}$ and $\Gamma$ denotes a congruence subgroup of level $N$.

**Cusps:** Let $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. We then extend the action of $SL_2(\mathbb{Z})$ on $\hat{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, the extended upper half-plane as follows. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \hat{\mathbb{H}}$, define

$$\gamma z := \begin{cases} 
\infty & \text{if } z = -d/c, \\
\frac{a}{c} & \text{if } z = \infty, \\
\frac{az + b}{cz + d} & \text{otherwise}.
\end{cases}$$

A cusp of $\Gamma$ is a $\Gamma$-equivalent class of elements in $\mathbb{P}^1(\mathbb{Q})$ under the action of $\Gamma$. Note that
the group $SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, hence there is only one cusp of $SL_2(\mathbb{Z})$. Since the index of every congruence subgroup $\Gamma$ of $SL_2(\mathbb{Z})$ is finite, it follows that there are only finitely many cusps of $\Gamma$.

**Holomorphicity at the cusps:** Assume that $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function which satisfies the **modular transformation property** for $\Gamma$, namely $f \mid_k \gamma = f$ for all $\gamma \in \Gamma$ (such an $f$ is said to be **weakly holomorphic modular form** of weight $k$ with respect to $\Gamma$). Let $D'$ be the open unit disk in $\mathbb{C}$ with the origin removed. Then $z \mapsto q_N := e^{2\pi i z/N}$ defines a map from $\mathbb{H}$ into $D'$. Since $f \mid_k \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} = f$, it follows that $f$ is periodic with period $N$ and hence there exists a function $F : D' \to \mathbb{C}$ such that $F(q_N) = f(z)$. If for all $q_N \in D'$, we have the Laurent series expansion of the form

\[
f(z) = F(q_N) = \sum_{n \geq 0} a(n) q_N^n,
\]

then $f$ is said to be holomorphic at $\infty$. Moreover, if $a(0) = 0$, we say that $f$ vanishes at $\infty$. Eq. (1.2) is called the Fourier expansion of $f$ at $\infty$ or the $q$-expansion of $f$ about $\infty$, and the numbers $a(n) \in \mathbb{C}$ are called the **Fourier coefficients** of $f$. Since any cusp $s \in \mathbb{P}^1(\mathbb{Q})$ can be written as $s = \gamma_0 \infty$, for some $\gamma_0 \in SL_2(\mathbb{Z})$ and therefore holomorphy at $s$ is naturally defined in terms of holomorphy at $\infty$ via the slash operator. More precisely, $f$ is said to be holomorphic (or vanishes) at the cusp $s$ if $f \mid \gamma_0$ is holomorphic (or vanishes) at $\infty$.

### 1.2 Modular forms

#### 1.2.1 Modular forms of integer weight

**Definition 1.2.1** A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is said to be a **modular form** of weight $k$ for $\Gamma_0(N) \subset SL_2(\mathbb{Z})$ with a Dirichlet character $\chi$ modulo $N$, if $f$ satisfies the following conditions:
§1.2. Modular forms

1. For all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \) and \( z \in \mathbb{H} \),

\[
(cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right) =: f |_k \gamma(z) = \chi(d) f(z),
\]

2. \( f \) is holomorphic at all the cusps of \( \Gamma_0(N) \).

We denote the \( \mathbb{C} \)-vector space of all such modular forms of weight \( k \), level \( N \) with a Dirichlet character \( \chi \) modulo \( N \) by \( M_k(N, \chi) \). We denote the subspace of \( M_k(N, \chi) \) consisting of all cusp forms by \( S_k(N, \chi) \). If the associated Dirichlet character \( \chi \) (mod \( N \)) is trivial, then we denote these spaces simply by \( M_k(N) \) and \( S_k(N) \) respectively. Further, in the case of full modular group \( \text{SL}_2(\mathbb{Z}) \), we denote the respective spaces by \( M_k \) and \( S_k \) respectively.

**Hecke Operators**: Let \( n \) be a positive integer. For \( (n, N) = 1 \), the \( n \)-th Hecke operator, denoted by \( T_n \), is defined in terms of the Fourier coefficients as follows.

\[
f|_{T_n}(z) = \sum_{m=0}^{\infty} \sum_{d \mid (m,n)} \chi(d) d^{k-1} a \left( \frac{mn}{d^2} \right) e^{2\pi i mz}.
\]

The family \( \{ T_n : (n, N) = 1 \} \) form a commuting family of Hecke operators on the space \( M_k(N, \chi) \). For every positive integer \( n \), we define the \( U(n) \) operator on formal sums as follows:

\[
U(n) : \sum_{m \geq 0} a(m) e^{2\pi imz} \to \sum_{m \geq 0} a(mn) e^{2\pi imz}.
\]  \hfill (1.3)

For a prime \( p \mid N \), \( U(p) \) denotes the \( p \)-th Hecke operator on \( M_k(N, \chi) \). It is a fact that \( \{ T_p, p \notmid N; U(p), p \mid N \} \) generate the Hecke algebra on \( M_k(N, \chi) \). Moreover, the operators \( T_p, p \notmid N, U(p), p \mid N \) preserve the space of cusp forms. Further, it is a fact that the Hecke operators \( T_n, (n, N) = 1 \) satisfy the following commuting property on \( M_k(N, \chi) \):

\[
T_m T_n = \sum_{d \mid \gcd(m,n)} \chi(d) d^{k-1} T_{mn} \frac{d^2}{d^2}.
\]
Definition 1.2.2 (Petersson inner product) Let \( f, g \in M_k(N, \chi) \) be such that at least one of them is a cusp form. Write \( z = x + iy \), then the Petersson inner product of \( f \) and \( g \) is defined as:
\[
\langle f, g \rangle := i_N^{-1} \int_{\Gamma_0(N) \setminus \mathbb{H}} f(z)\overline{g(z)}y^k \frac{dx dy}{y^2},
\]
(1.4)
where \( \Gamma_0(N) \setminus \mathbb{H} \) is a fundamental domain, \( \frac{dx dy}{y^2} \) is an invariant measure under the action of \( SL_2(\mathbb{Z}) \) on \( \mathbb{H} \) and \( i_N \) denotes the index of \( \Gamma_0(N) \) in \( SL_2(\mathbb{Z}) \).

Remark 1.2.1 It is well-known that \( S_k(N, \chi) \) is a finite-dimensional Hilbert space with respect to the inner product defined by (1.4). The Hecke operators \( T_p \) for all primes \( p \mid N \) are Hermitian with respect to the Petersson scalar product. Further, as they form a commuting family of operators, it follows from linear algebra that the space \( S_k(N, \chi) \) has a basis of eigenforms with respect to all \( T_p \), \( p \mid N \). However, by the theory of newforms developed by Atkin and Lehner [10] and W. W. Li [42], there exists a subspace of \( S_k(N, \chi) \), denoted by \( S_k^{\text{new}}(N, \chi) \), called the space of newforms which has an orthogonal basis of eigenforms with respect to all \( T_p \), \( p \mid N \), \( U(p) \), \( W_p \), \( p \mid N \). These basis elements, which are eigenforms with respect to all the Hecke operators, are called Hecke eigenforms. In the above, the Atkin-Lehner \( W \)-opreators \( W_p \) for \( p \mid N \) are defined as follows. Let \( p \mid N \), with \( p^\alpha \mid N \) (i.e., \( p^\alpha \mid N \), \( p^\alpha+1 \mid N \)), then
\[
W_p = \begin{pmatrix} p^\alpha a & b \\ Nc & p^\alpha d \end{pmatrix},
\]
where \( a, b, c, d \) are integers satisfying the properties, \( p^{2\alpha} ad - Nbc = p^\alpha \) and \( b \equiv 1 \pmod{N'} \), \( N' \) being the conductor of \( \chi \). The operators \( W_p \) are independent of the choice of the representatives \( a, b, c, d \) and satisfy the property that \( W_p^2 = \chi(N/p^\alpha) \). In particular, \( W_p^2 = \text{Identity} \), if \( \chi \) is the principal character.
1.2.2 Modular forms of half-integral weight

For complex numbers \( z \) and \( x \in \mathbb{C} \setminus \{0\} \), we let \( e^x = e^{\log z}, \log z = \log |z| + i \arg(z), -\pi < \arg(z) < \pi \). Let \( \zeta \) be a fourth root of unity. Let \( G \) denote the four-sheeted covering of \( GL^+_2(\mathbb{Q}) \), defined as the set of all ordered pairs \((\alpha, \phi(z))\), where \( \phi(z) \) is a holomorphic function on \( \mathbb{H} \) such that \( \phi^2(z) = \xi^2 \frac{cz+d}{\sqrt{\Delta} \delta} \) and \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+_2(\mathbb{Q}) \). Then \( G \) is a group with the multiplicative rule: \( (\alpha, \phi(z))(\beta, \psi(z)) = (\alpha \beta, \phi(\beta z) \psi(z)) \). Let \( k \geq 2 \) be a natural number. For a function \( f : \mathbb{H} \to \mathbb{C} \) defined on the upper half-plane \( \mathbb{H} \) and an element \((\alpha, \phi(z)) \in G\), define the stroke operator by

\[
| f|_{k+1/2}(\alpha, \phi(z))(z) = \phi(z)^{-2k-1} f(\alpha z).
\]

We omit the subscript \( k+1/2 \) wherever there is no ambiguity. For the congruence subgroup \( \Gamma_0(4) \) and its subgroups, we take the lifting \( \Gamma_0(4) \to G \) as the collection \( \{ (\alpha, j(\alpha, z)) \} \), where \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \) and

\[
 j(\alpha, z) = \left( \frac{c}{d} \right) \left( \frac{-4}{d} \right)^{-1/2} (cz + d)^{1/2}.
\]

Here \( \left( \frac{c}{d} \right) \) denotes the generalized quadratic residue symbol and \( \left( \frac{-4}{d} \right)^{-1/2} \) is equal to either 1 or \( i \) according as \( d \) is 1 or 3 modulo 4.

**Definition 1.2.3** Let \( M \) be a natural number. A holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) is called a modular form of weight \( k+1/2 \) for \( \Gamma_0(4M) \) with character \( \chi \) (modulo 4M), if

\[
f|_{k+1/2}(\gamma, j(\gamma, z))(z) = \chi(d) f(z), \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4M),
\]

and \( f \) is holomorphic at all the cusps of \( \Gamma_0(4M) \). If further, \( f \) vanishes at all the cusps of \( \Gamma_0(4M) \), then it is called a cusp form.

The set of modular forms (resp. cusp forms) defined as above becomes a complex vector space denoted by \( M_{k+1/2}(4M, \chi) \) (resp. \( S_{k+1/2}(4M, \chi) \)). If \( \chi \) is trivial character, then the
space is denoted by $M_{k+1/2}(4M)$ (resp. $S_{k+1/2}(4M)$).

The Fourier expansion of a modular form $f$ at the cusp infinity is usually written as

$$f(z) = \sum_{n \geq 0} a_f(n)q^n, \quad q = e^{2\pi i z}.$$  

For every positive integer $n$, the operator $U(n)$ is defined as in (1.3). We define the duplicating operator $B(n)$, which is also defined on formal sums, by

$$B(n) : \sum_{m \geq 0} a(m)q^m \to \sum_{m \geq 0} a(m)q^m.$$  

For any $f \in M_{k+1/2}(4M, \chi)$ and a prime $p \nmid 2M$, we define the Hecke operator $T(p^2)$ by

$$f \mid T(p^2) = \sum_{n \geq 0} \left\{ a_f(np^2) + \chi(p) \left( \frac{(-1)^k n}{p} \right) p^{k-1} a_f(n) + \chi(p^2) p^{2k-1} a_f(n/p^2) \right\} q^n,$$

using the recurrence relation and the commutation relations

$$T(p^{2(n+1)}) = T(p^2)T(p^{2n}) - p^{2k-1}T(p^{2(n-1)}) \quad (n \geq 1)$$

and

$$T(n^2m^2) = T(n^2)T(m^2) \quad (n,m) = (mn,2M) = 1,$$

one can extend the definition of $T(n^2)$ for $n \in \mathbb{N}$, $(n,2M) = 1$. The operators $T(n^2)$ for $n \in \mathbb{N}$, $(n,2M) = 1$ and $U(n^2)$ for $n \in \mathbb{N}, n|2M$ are the Hecke operators on $M_{k+1/2}(4M, \chi)$.

Further, $B(n)$ maps $M_{k+1/2}(4M, \chi)$ into $M_{k+1/2}(4Mn, \chi \chi_n)$, where $\chi_n$ is the quadratic character $\left( \frac{\cdot}{n} \right)$. Finally for $f, g \in M_{k+1/2}(4M, \chi)$ with $f$ or $g$ is a cusp form, the Petersson inner product is defined by

$$\langle f, g \rangle = \frac{i_{4M}^{-1}}{\mathcal{F}_{4M}} \int_{\mathcal{F}_{4M}} f(z)\overline{g(z)}v^{k-\frac{3}{2}}du dv,$$

where $\mathcal{F}_{4M}$ is a fundamental domain for the action of $\Gamma_0(4M)$ on $\mathbb{H}$, $i_{4M}$ is the index of $\Gamma_0(4M)$ in $SL_2(\mathbb{Z})$ and $z = u + iv$. For more details on the theory of modular forms of half-integral weight, we refer to Koblitz's book [35] and the work of Shimura [73].
1.2.3 Examples of modular forms

**Eisenstein series:** Let $k \geq 2$ be an even integer. The normalized Eisenstein series $E_k(z)$ of weight $k$ for $SL_2(\mathbb{Z})$ is defined as:

$$E_k(z) := \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mz+n)^k}. \quad (1.5)$$

It is known that $E_k$ is a modular form of weight $k$ for $SL_2(\mathbb{Z})$ for $k \geq 4$ and it has the Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n)q^n, \quad (1.6)$$

where $\sigma_r(n) = \sum_{d|n} d^r$, for any positive integer $r$ and $B_k$'s are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!}.$$ 

Because we use Eisenstein series in our work, we give the Fourier expansion of the first few Eisenstein series:

$$E_4(z) = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n, \quad E_6(z) = 1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n,$$

$$E_8(z) = 1 + 480 \sum_{n \geq 1} \sigma_7(n)q^n, \quad E_{10}(z) = 1 - 24 \sum_{n \geq 1} \sigma_9(n)q^n,$$

$$E_{12}(z) = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n)q^n, \quad E_{14}(z) = 1 - 24 \sum_{n \geq 1} \sigma_{13}(n)q^n.$$ 

When $k = 2$, we have $E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma(n)q^n$ and it is the fundamental quasimodular form of weight 2 for $SL_2(\mathbb{Z})$.

**Generalized Eisenstein series:** Suppose that $\chi$ and $\psi$ are primitive Dirichlet characters with conductors $M$ and $N$ respectively. For a positive integer $k$, let

$$E_{k,\chi,\psi}(z) := -\frac{B_k}{2k} \delta_{M,1} + \sum_{n \geq 1} \sigma_{k-1;\chi,\psi}(n) q^n, \quad (1.7)$$
where $B_{k,\psi}$ denotes generalized Bernoulli number with respect to the character $\psi$ and

$$
\sigma_{k-1;\chi,\psi}(n) := \sum_{d\mid n} \psi(d) \chi(n/d) d^{k-1}.
$$

Then, $E_{k,\chi,\psi}(z)$ belongs to the space $M_k(\Gamma_0(MN), \chi/\psi)$, provided $\chi(-1)\psi(-1) = (-1)^k$ and $MN \neq 1$. When $\chi = \psi = 1$ (i.e., when $M = N = 1$) and $k \geq 4$, we have $E_{k,\chi,\psi}(z) = -\frac{B_k}{2k}E_k(z)$, where $E_k$ is the normalized Eisenstein series of weight $k$ as defined before. For more details we refer to the book of Miyake [49, Chapter 7] and the book of Stein [77, Section 5.3].

**Ramanujan Delta function:** The Ramanujan delta function is defined as

$$
\Delta(z) := \frac{1}{1728} \left( E_4(z)^3 - E_6(z)^2 \right)
$$

and it is the unique cusp form of weight 12 for $SL_2(\mathbb{Z})$, with the Fourier expansion

$$
\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n)q^n.
$$

Here $\tau(n)$ is the Ramanujan tau function.

**The Dedekind eta function and eta quotients:** The Dedekind eta function $\eta(z)$ is defined by

$$
\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n). \tag{1.8}
$$

Note that $\eta^{24}(z) = \Delta(z)$. An eta-quotient is a finite product of integer powers of $\eta(z)$ and we denote it as

$$
\prod_{i=1}^{s} \eta^{r_i}(d_iz) := d_1^{r_1}d_2^{r_2} \cdots d_s^{r_s}, \tag{1.9}
$$

where $d_i$’s are positive integers and $r_i$’s are non-zero integers.

**Poincaré series:** Let $k, n, N$ be positive integers. The $n$-th Poincaré series of weight $k$ for
\( \Gamma_0(N) \) is defined by
\[
P_{k,N;n}(z) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} (cz + d)^{-k} e^{2\pi i n y_c},
\]
where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( \Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{Z} \right\} \). It is known that \( P_{k,N;n} \in S_k(N) \) for \( k \geq 2 \) and it is characterized by the following property.

**Lemma 1.2.4** Let \( f \in S_k(N) \) with Fourier expansion \( f(z) = \sum_{m \geq 1} a_f(m)q^m, \quad q = e^{2\pi iz}. \) Then
\[
\langle f, P_{k,N;n} \rangle = \frac{\Gamma(k-1)}{(4\pi N)^{k-1}} a_f(n).
\]

The following familiar result tells about the growth of the Fourier coefficients of a modular form in which the first statement can be easily obtained and the second is due to P. Deligne [24].

**Proposition 1.2.5** Let \( a_f(n) \) be the \( n \)-th Fourier coefficient of a modular form \( f \in M_k(N) \). Then for any \( \varepsilon > 0 \), we have
\[
a_f(n) \ll \varepsilon n^{k-1+\varepsilon},
\]
and when \( f \in S_k^{\text{new}}(N) \) is a (normalized) Hecke eigenform, then
\[
a_f(n) \ll \varepsilon n^{\frac{k-1}{2}+\varepsilon}.
\]