Chapter V

DEVELOPING PRODUCT-TYPE SYNTHETIC ESTIMATORS FOR SMALL AREA ESTIMATION AND THEIR STUDY

5.1 INTRODUCTION

5.1.1 Product Method of Estimation

In the attempt to develop some improved and extended generalized synthetic estimators for tackling the problem of small area estimation (SAE), we have developed efficient families of synthetic estimators in previous three chapters, assuming that the information on an auxiliary variable is available. All the families were, therefore, ratio-type synthetic estimators in the sense that these classes converged to either ratio estimator or certain ratio-type estimators as the value of the parameter increases indefinitely. Due to this reason, we applied them for illustrating their properties on empirical data which exhibited a positive and high correlation between the study and auxiliary variable. Needless to mention, that all these families included either usual (classical) product estimator or product-type estimators as a particular cases, but these could not be used for application due to the nature of the data used.

Contrary to the situation of positive correlation in the population, if someone observes negative correlation between the variables under consideration, a ratio-type estimator seems not to be appropriate for the estimation purpose for obvious reasons mentioned by Murthy (1964, 1967). He showed that, to cope up with the estimation problem utilizing the information on auxiliary variable which has high negative correlation with the study
variable, product method of estimation be widely used which has smaller variance than the conventional mean per unit estimator if $\rho_{CY/CX} < -\frac{1}{2}$. In this sense, product method of estimation may be considered as a complementary problem to the ratio method of estimation, as the latter is preferred if $\rho_{CY/CX} > \frac{1}{2}$.

Quite often, in practice, survey sampling practitioners come across with variables which are negatively correlated, though, there are some misconceptions about the occurrence of negative correlation in practice. It is true that the negative correlations are often induced through inverse transformations or through appropriate linear transformation in one of the variables. A number of empirical investigations from agricultural, medical, biological and economical fields also exhibit such inverse relationship between study and auxiliary variables. For example, number of eggs laid down by a specific variety of hen and weights of each egg; average demand of goods in future with present amount of supply as auxiliary variable; risk of coronary heart disease and content of serum high-density lipoprotein cholesterol, etc., exhibit negative correlations.

The product method of estimation was first proposed by Robson (1957). Goodman (1960) proposed product estimators for population mean and total of the study variable as $t_p = \frac{\bar{y}}{\bar{X}}$ and $\hat{t}_p = \frac{N\bar{y}}{\bar{X}}$; $\bar{X} \neq 0$ and considered the question of obtaining the variance and variance estimators of product of estimators. Murthy (1964) discussed the use of product method of estimation and provided a technique of obtaining unbiased product estimators based on inter-penetrating sub-samples. Later on, Srivastava (1966) made a study of similar type of estimator.

Although, several authors developed design-based product estimators or product-type estimators, based upon certain unknown parameters for population mean, but some of the reasonable works in this direction are due to Shah and Shah (1979), Gupta and Adhvaryu (1982), Chaudhary and Arnab (1982), Ray and Sahai (1978, 1980), Ray et al (1979), Singh (1984), Srivenkataramana and Tracy (1981), Shukla (1988) and Singh and
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5.1.2 Predictive Approach Product Estimator (PAPE)

Although the conventional product estimator $t_p$ is complementary to ratio estimator and easy to define and compute, unlike the usual ratio estimator and regression estimator, it does not have an intuitive basis, that is, if one adopts the prediction approach for predicting the mean $\bar{Y}_U$ of the unobserved part of the population and uses the product estimator as the predictor, the resulting estimator of overall population mean $\bar{Y}$ is not the customary product estimator. The result is due to Basu (1971). Following Srivastava (1983) the result could be shown as below:

The population mean $\bar{Y}$ of a finite population with size $N$ can be written as

$$\bar{Y} = \frac{1}{N} \left[ \sum_s y_i + \sum_{\bar{s}} y_i \right] = \frac{n}{N} \bar{y}_s + \frac{N-n}{N} \bar{y}_{\bar{s}},$$

(5.1)

where $s$ denotes a sample of size $n$ selected from the population and $\bar{s}$ denotes that part of the population which is unobserved and is complementary to $s$. 
Therefore, an estimator $\hat{y}$ of $\bar{Y}$ can be written as

$$\hat{y} = \frac{n}{N} \bar{y}_s + \frac{N-n}{N} T,$$

(5.2)

where $T$ is considered to be a predictor of $\bar{y}_s$. If the information on the variable $X$ is used in the form of a product estimator $\bar{y}_s \bar{x}_s$ and $T$ is replaced by it in (5.2), then we have

$$\hat{y} = \frac{n}{N} \bar{y}_s + \frac{N-n}{Nn} \bar{x}_s \bar{X}_s.$$

(5.3)

Since $\bar{X}_s = \frac{(N\bar{X} - n\bar{x}_s)}{(N-n)}$, the resulting estimator $\hat{y}$ would be

$$\hat{y} = \bar{y}_s \left[ \frac{n\bar{X} + (N-2n)\bar{x}_s}{NX - n\bar{x}_s} \right] = \hat{y}_{PAPE} \text{ (say)},$$

(5.4)

which is different from $t_p$. Thus $\hat{y}$ as obtained in (5.4) is not the conventional product estimator. Let us call the estimator $\hat{y}_{PAPE}$ in (5.4) as **Predictive Approach Product Estimator (PAPE)**. Singh (1992) and Shukla (2010) studied the salient properties of PAPE under a special form of superpopulation, namely polynomial regression model.

In previous chapter, it was observed that while conventional product estimator was one of the members of the families developed, no class included PAPE as special case. It was, therefore, felt that in an attempt to suggest classes of product-type synthetic estimators for SAE, the two estimators $t_p$ and $\hat{y}$ of (5.4) should be treated separately. It can be seen that $\hat{y}_{PAPE}$ is expressible in the form

$$\hat{y}_{PAPE} = \bar{y} \left[ \frac{w_1 \bar{X} + w_2 \bar{x}}{w_3 \bar{X} + w_4 \bar{x}} \right],$$

(5.5)

where $w_1 = n$, $w_2 = N - 2n$, $w_3 = N$, $w_4 = -n$ and $w_1 + w_2 = w_3 + w_4$.

5.1.3. In the present chapter, our aim is to suggest two families of product-type synthetic estimators for estimating domain-specific mean of the study variable $Y$ which are,
in fact, factor-type estimators based on a parameter and the known information on an auxiliary variable $X$. While the first family includes the usual product estimator as a special case and does not include PAPE as a member, the second family includes both the usual product estimator and PAPE as special cases. Both the families are exclusively product-type estimators as they converge asymptotically to a product-type estimator. Some of the special cases of both the families have been obtained. Important properties of both the families have been discussed. Using a computer-generated data set showing negative correlation, the comparison of the families has been done. Finally, a simulation study has been made of both the families.

5.2 FAMILY OF PRODUCT-TYPE SYNTHETIC ESTIMATORS AND SOME OF ITS PARTICULAR CASES

5.2.1. In this section, we suggest a class of synthetic estimators which is basically product-type synthetic estimator. With the same notations and aim of devising synthetic estimator as mentioned in chapter II, we define the following estimator:

$$T_{P,I}^a(\alpha) = \bar{y} \left[ \frac{(A + 2fB) \bar{X}_a + (C - fB) \bar{x}}{(A + fB + C) X_a} \right].$$

(5.6)

**Remark 1.** The estimator $T_{P,I}^a(\alpha)$ is a synthetic version of the factor-type estimator, developed by Shukla (1988) and Singh and Shukla (1995). The estimator is a product-type estimator and includes some product-type estimators as members. Moreover, it is shown in the coming paragraphs that it converges asymptotically to the usual (conventional) synthetic product estimator $\bar{y}_{PS,a} = T_{2,a} = \frac{\bar{y} \bar{x}}{\bar{X}_a}$, given under the expression (2.3).

**Remark 2.** Asymptotic convergence of $T_{P,I}^a(\alpha)$

For finding out the limiting value of the estimator $T_{P,I}^a(\alpha)$, we divide the numerator and denominator of the right hand side of (5.6) by $\alpha^3$ and take the limit as $\alpha \to \infty$. Then we have

$$\lim_{\alpha \to \infty} T_{P,I}^a(\alpha) = \bar{y} \left( \frac{\bar{x}}{\bar{X}_a} \right) = \bar{y}_{PS,a},$$

(5.7)
which indicates that the estimator do exists even if one selects an indefinitely large value of the parameter \( \alpha \). We have seen earlier that estimator \( T_{\alpha,a} \) and \( T_{\alpha,r} \) defined in chapters II and III respectively are also convergent as these are also factor-type estimators.

### 5.2.2 Particular Cases of \( T_{P,I}^\alpha (\alpha) \)

Let us now consider the proposed family for some of its particular cases and observe their relevance with some product-type estimators.

(i) Let \( \alpha = 1 \), then \( T_{P,I}^\alpha (1) = \bar{y}_{PS,a} \), which is product-type synthetic estimator, complementary to the ratio-type synthetic estimator \( \bar{y}_{RS,a} = \bar{y} \left( \frac{\bar{X}}{\bar{x}} \right) \), defined in Rao (2003).

(ii) Further, let \( \alpha = 2 \). Then \( T_{P,I}^\alpha (2) = t_{SR} (1) \), where \( t_{SR} (\lambda) = \bar{y} \left[ 2 - \left( \frac{\bar{x}}{\bar{X}} \right)^{\lambda} \right] \), \( \lambda \) being a constant is the product type estimator suggested by Sahai and Ray (1980).

(iii) For \( \alpha = 3 \), the proposed estimator is \( T_{P,I}^\alpha (3) = t_v (1 - g) = t_{RS} (-g) \), \( g = n (N - n)^{-1} \), where \( t_v (w) = \bar{y} \left[ w + (1 - w) \frac{\bar{x}}{\bar{X}} \right] \) and \( t_{RS} (\gamma) = \bar{y} \left[ (1 + \gamma) - \frac{\bar{x}}{\bar{X}} \right] \) are product-type estimators, proposed by Vos (1980) and Ray et al (1979); \( w \) and \( \gamma \) are some arbitrarily chosen constants.

(iv) For \( \alpha = 4 \), the estimator becomes the simple synthetic estimator \( \bar{y}_{SS,a} = \bar{y} \).

Thus, it is apparent that the suggest class of synthetic estimators covers some product-type synthetic estimators which may be considered as the synthetic versions of some well-known product type estimators existing in the literature.
5.3 DESIGN-BIAS AND MSE OF $T^a_{P,I}(\alpha)$

5.3.1 Bias of the Estimator

Theorem 1. The design-bias of the estimator $T^a_{P,I}(\alpha)$ is given by

$$B \left[ T^a_{P,I}(\alpha) \right] = \frac{\bar{Y}}{\bar{X}_a} \left[ RX_a + (1 - R) \bar{X} \{1 + V_{11}\} \right] - \bar{Y}_a, \quad (5.8)$$

where $R = \frac{(A+2fB)}{(A+fB+C)}$.

Proof: As before, we consider the large sample approximation

$$\bar{y} = \bar{Y} (1 + e_0), \quad \bar{x} = \bar{X} (1 + e_1)$$

so that $E(e_0) = E(e_1) = 0$ and $E(e^i_0 e^j_1) = V_{ij}; i, j = 1, 2, ...$.

We then have

$$T^a_{P,I}(\alpha) = \frac{\bar{Y}}{\bar{X}_a} \left[ \frac{A + 2fB}{A+fB+C} (1 + e_0) + \frac{C - fB}{A+fB+C} \frac{\bar{X}}{\bar{X}_a} (1 + e_0 + e_1 + e_0 e_1) \right]$$

$$= \frac{\bar{Y}}{\bar{X}_a} \left[ R(1 + e_0) + (1 - R)(1 + e_0 + e_1 + e_0 e_1) \right].$$

Taking expectation of both the sides and applying the value of $E(e_i), i = 1, 2$ and $E(e^i_0 e^j_1); i, j = 1, 2, ...$, we have

$$E \left[ T^a_{P,I}(\alpha) \right] = \frac{\bar{Y}}{\bar{X}_a} \left[ RX_a + (1 - R) \bar{X} \{1 + V_{11}\} \right].$$

Hence

$$B \left[ T^a_{P,I}(\alpha) \right] = E \left[ T^a_{P,I}(\alpha) \right] - \bar{Y}_a,$$

provides the expression (5.8)

Remark 3. The bias expression of the particular members, as discussed in the section (5.2.2) can be deduced from (5.8) by choosing appropriate value of $\alpha$. For instance, if $\alpha = 1$ then $R = 0$ and hence

$$B[\bar{y}_{PS,a}] = \left( \frac{\bar{Y}}{\bar{X}_a} \frac{\bar{X}}{\bar{X}_a} - \bar{Y}_a \right) + \frac{\bar{Y}}{\bar{X}_a} \bar{X} V_{11}, \quad (5.9)$$
and if $\alpha = 4$, then $R = 1$ and hence

$$B[\bar{y}_{SS,a}] = (\bar{Y} - \bar{Y}_a). \quad (5.10)$$

**Remark 4.** From the expression (5.8), it can be seen that

$$\lim_{\alpha \to \infty} B[T_{P,I}^\alpha (\alpha)] = B[\bar{y}_{PS,a}],$$

which is a trivial result in the light of **Remark 2**.

### 5.3.2 MSE of the Estimator

**Theorem 2.** The MSE of the estimator to the order $O(n^{-1})$ is given by

$$M[T_{P,I}^\alpha (\alpha)] = \left[ \bar{Y} \left( R + \frac{L \bar{X}}{X_a} \right) - \bar{Y}_a \right]^2 + \left( R + \frac{L \bar{X}}{X_a} \right)^2 V_{20}$$

$$+ \left( \frac{L \bar{X}}{X_a} \right)^2 V_{02} + 4 \left( \frac{L \bar{X}}{X_a} \right) \left\{ \left( R + \frac{L \bar{X}}{X_a} \right) V_{11} \right\}$$

$$- 2 \bar{Y} \bar{Y}_a \left( \frac{L \bar{X}}{X_a} \right) V_{11}, \quad (5.11)$$

where $L = (1 - R)$.

**Proof:** We have

$$M[T_{P,I}^\alpha (\alpha)] = E[T_{P,I}^\alpha (\alpha)]^2 - 2 \bar{Y}_a E[T_{P,I}^\alpha (\alpha)] + \bar{Y}_a^2. \quad (5.12)$$

Writing $T_{P,I}^\alpha (\alpha)$ in terms of $e_0$ and $e_1$, we have

$$T_{P,I}^\alpha (\alpha) = \bar{Y} \left[ R (1 + e_0) + \left( \frac{L \bar{X}}{X_a} \right) (1 + e_0 + e_1 + e_0 e_1) \right]. \quad (5.13)$$
Therefore, squaring it, retaining terms of $e_0$ and $e_1$ upto the second power and taking expectation we have

\[
E \left[ T_{P,I}^a (\alpha) \right]^2 = \bar{Y}^2 R^2 (1 + V_{20}) + \bar{Y}^2 \left( \frac{L \bar{X}}{X_a} \right)^2 (1 + V_{02} + 4V_{11} + V_{20}) \\
+ 2\bar{Y}^2 R \left( \frac{L \bar{X}}{X_a} \right) (1 + 2V_{11} + V_{20}). \tag{5.14}
\]

Now taking expectation of (5.13) and using it and expression (5.14) in (5.12), we have the expression (5.11).

**Remark 5.** As a trivial consequence of the Remark 2, it can be seen that as $\alpha \to \infty$, \[ \lim_{\alpha \to \infty} M \left[ T_{P,I}^a (\alpha) \right] = M \left[ \bar{y}_{PS,a} \right], \] where

\[
M \left[ \bar{y}_{PS,a} \right] = \left( \frac{\bar{Y} \bar{X}}{X_a} - \bar{Y}_a \right)^2 + \left( \frac{\bar{Y} \bar{X}}{X_a} \right)^2 [V_{20} + V_{02} + 4V_{11}] - 2\bar{Y}_a \left( \frac{\bar{Y} \bar{X}}{X_a} \right) V_{11}. \tag{5.15}
\]

**Remark 6.** The expression of MSEs of some of the particular cases of the suggested family $T_{P,I}^a (\alpha)$ can be obtained from (5.11). For example, if $\alpha = 1$, then $R = 0$ and $L = 1$. Hence we have the expression (5.15). Further, if $\alpha = 4$, then we have

\[
M \left[ \bar{y}_{SS,a} \right] = \bar{Y}^2 V_{20} + (\bar{Y} - \bar{Y}_a)^2. \tag{5.16}
\]

### 5.4 OPTIMUM CHOICE OF THE PARAMETER $\alpha \left( = \alpha_0, \text{say} \right)$

The MSE expression of $T_{P,I}^a (\alpha)$ is a function of the parameter $\alpha$ and for different choices of $\alpha$, the family generates its member, some of which have been discussed under the section 5.2.2. However, there might be some choice of $\alpha$, for which the MSE would be minimum. Say such a value of $\alpha$ is $\alpha_0$. Therefore, the problem would be to minimize $M \left[ T_{P,I}^a (\alpha) \right]$ with respect to $\alpha$ and to obtain $\alpha_0$ and $M \left[ T_{P,I}^a (\alpha) \right]$. It is, therefore, necessary to solve the equation \[ \frac{\partial M \left[ T_{P,I}^a (\alpha) \right]}{\partial \alpha} = 0 \] for $\alpha$. The roots of this equation will provide optimum value(s) of $\alpha$. 
It is evident from the expression (5.11) that \( L \) and \( R \) are the functions of \( \alpha \) and rest of the values are fixed for a given population and specific small domain. Further, the expression (5.11) is a complex function of \( L \) and \( R \), therefore, an explicit expression of \( \frac{\partial M[T_{P,I}^\alpha(\alpha)]}{\partial \alpha} = 0 \) would not be easy to obtain, though could be attempted by substituting the values of \( R' = \frac{\partial R}{\partial \alpha} \) and \( L' = -R' \). Thus, we get

\[
\frac{\partial M[T_{P,I}^\alpha(\alpha)]}{\partial \alpha} = 2\bar{Y}R' \left[ \bar{Y} \left( R + \frac{L\bar{X}}{\bar{X}_a} \right) - \bar{Y}_a \right] \left\{ \frac{\bar{X}_a - \bar{X}}{\bar{X}_a} \right\} \\
+ 2R' \left( R + \frac{L\bar{X}}{\bar{X}_a} \right) \left( \frac{\bar{X}_a - \bar{X}}{\bar{X}_a} \right) V_{20} - 2R' \left( \frac{L\bar{X}}{\bar{X}_a} \right) \left( \frac{\bar{X}}{\bar{X}_a} \right) V_{02} \\
- 4R' \left( \frac{\bar{X}}{\bar{X}_a} \right) \left( R + \frac{L\bar{X}}{\bar{X}_a} \right) V_{11} + 4R' \left( \frac{L\bar{X}}{\bar{X}_a} \right) \left( \frac{\bar{X}_a - \bar{X}}{\bar{X}_a} \right) V_{11} \\
+ 2R'\bar{Y}\bar{Y}_a \left( \frac{\bar{X}}{\bar{X}_a} \right) V_{11}. \tag{5.17}
\]

For a given set of data, therefore, the expression (5.17) can be used for finding the optimum value of \( \alpha \), by equating it to zero.

**Remark 7.** The expression (5.17) would be a function of \( \alpha \) of order more than two. It is, therefore, expected to get more than one \( \alpha_0 \) for which \( M[T_{P,I}^\alpha(\alpha)] \) would be minimum. Some values of \( \alpha_0 \) might be negative and/or imaginary depending upon the population values. Obviously, because of the condition \( \alpha > 0 \), one should consider only the real and positive \( \alpha_0 \).

**Remark 8.** Since MSE of the estimator would be same for all \( \alpha_0 \), the criterion mentioned under the section 2.4.3. should be followed.

### 5.5 Family of Predictive Approach Product-Type Synthetic Estimators and Some of its Particular Cases

#### 5.5.1. In section 5.1.2. it has been shown that the usual product-type synthetic estimator does not have an intuitive basis, so a predictive approach product estimator (PAPE) is to be obtained. Further, as the family \( T_{P,I}^\alpha(\alpha) \) does not include such a synthetic estimator,
there is a need to develop a separate class of product-type synthetic estimators which might include PAP synthetic estimator. With this view, we now define another family of product-type synthetic estimator below:

\[ T_{PAPE}^{a}(\alpha) = \bar{y} \left[ \frac{(A + 2fB) \bar{x} + (C - fB) \bar{X}_a}{(A + C) \bar{X}_a + fB \bar{x}} \right]. \]  

(5.18)

Clearly, \( T_{PAPE}^{a}(\alpha) \) is a FTE in the form of (5.5).

**Remark 9.** Re-writing (5.18) in terms of \( \alpha \), dividing the numerator and denominator by \( \alpha^3 \) and taking limit as \( \alpha \rightarrow \infty \), it can be seen that

\[ \lim_{\alpha \to \infty} T_{PAPE}^{a}(\alpha) = \bar{y} = \bar{y}_{SS,a}. \]  

(5.19)

Thus, the estimator \( T_{PAPE}^{a}(\alpha) \) also is a convergent estimator.

### 5.5.2 Particular Cases of \( T_{PAPE}^{a}(\alpha) \)

The family characterized by \( T_{PAPE}^{a}(\alpha) \) includes some of the product-type synthetic estimators which are described below:

(i) Let \( \alpha = 1 \) then

\[ T_{PAPE}^{a}(1) = \bar{y} = \bar{y}_{SS,a}, \]  

(5.20)

whose bias ans MSE are presented in (5.10) and (5.16) respectively. \( \bar{y}_{SS,a} \) is also the limiting estimator of \( T_{PAPE}^{a}(\alpha) \) as \( \alpha \rightarrow \infty \).

(ii) For \( \alpha = 2 \), we observe that
\[ T_{PAPE}^a (2) = \bar{y} \left[ 2 - \frac{\bar{X}_a}{\bar{x}} \right] = t_{SR} (-1). \] (5.21)

(iii) For \( \alpha = 3 \), we have
\[ T_{PAPE}^a (3) = \bar{y} \left[ \frac{n \bar{X}_a + (N - 2n) \bar{x}}{N \bar{X}_a - n \bar{x}} \right] = t_{PAPE} (\text{say}). \] (5.22)

Obviously, \( t_{PAPE} \) is the synthetic estimator version of PAPE obtained in (5.4).

(iv) If \( \alpha = 4 \), then
\[ T_{PAPE}^a (4) = \bar{y} \left( \frac{\bar{x}}{X_a} \right) = \bar{y}_{PS,a} \] (5.23)

which is usual product-type synthetic estimator, whose bias and MSE are given under (5.9) and (5.15) respectively.

**Remark 10.** It is now evident that the family characterized by \( T_{PAPE}^a (\alpha) \) generates the estimators \( \bar{y}_{SS,a}, \bar{y}_{PS,a} \) and \( t_{PAPE} \), the PAP type synthetic estimators whereas the family generated by \( T_{P,I}^a (\alpha) \) does not include \( t_{PAPE} \) but includes \( \bar{y}_{SS,a}, \bar{y}_{PS,a} \) along with other estimators.

### 5.6 DESIGN-BIAS AND MSE OF \( T_{PAPE}^a (\alpha) \)

#### 5.6.1 Bias of the Estimator

**Theorem 3.** The bias of \( T_{PAPE}^a (\alpha) \) up to the order \( O (n^{-1}) \) is given by

\[ B \left[ T_{PAPE}^a (\alpha) \right] = \left( \frac{YG_1}{G_2} - Y_a \right) + \left( \frac{YG_1}{G_2} \right) (F_2 - F_1) \{ F_2 V_{02} - V_{11} \}, \] (5.24)

where \( G_1 = (A + 2fB) \bar{X} + (C - fB) \bar{X}_a \), \( G_2 = fB \bar{X} + (A + C) \bar{X}_a \),
\[ F_1 = (A + 2fB) \frac{\bar{x}}{G_1}, \quad F_2 = \frac{fB \bar{X}}{G_2}. \]
The proof of the expression (5.24) cab be obtained on the similar lines as presented under section 5.3.1.

**Remark 11.** If we select $\alpha = 1$, then $G_1 = -6\bar{X}_a$, $G_2 = -6\bar{X}_a$, $F_1 = F_2 = 0$. Hence the expression (5.24) yields

$$B \left[ T_{\text{PAPE}}^\alpha (1) \right] = B \left[ \bar{y}_{SS,a} \right] = (\bar{Y} - \bar{Y}_a),$$

which tallies with the expression (5.10).

**Remark 12** For $\alpha = 4$, we have $G_1 = 6\bar{X}$, $G_2 = 6\bar{X}_a$, $F_1 = 1$, $F_2 = 0$ and hence

$$B \left[ T_{\text{PAPE}}^\alpha (4) \right] = B \left[ \bar{y}_{PS,a} \right] = \left( \frac{\bar{Y} \bar{X}}{\bar{X}_a} - \bar{Y}_a \right) + \frac{\bar{Y} \bar{X}}{\bar{X}_a} V_{11},$$

which is same as the expression (5.9).

**Remark 13.** Letting $\alpha = 3$, we see that $G_1 = 2 \left\{ (1-2f) \bar{X} + f \bar{X}_a \right\}$, $G_2 = 2 \left\{ \bar{X}_a - f \bar{X} \right\}$, $F_1 = \frac{(1-2f)X}{(1-2f)X + fX_a}$ and $F_2 = \frac{-f \bar{X}}{X_a - f \bar{X}}$. Therefore, substituting these values in the expression (5.24), the bias of PAP type synthetic estimator $t_{\text{PAPE}}$ can be obtained. Further, it can be seen that the biases of usual product-type synthetic estimator, $\bar{y}_{PS,a}$ and $t_{\text{PAPE}}$ are different.

### 5.6.2 MSE of the Estimator

**Theorem 4.** Upto the order $O(n^{-1})$, the MSE of $T_{\text{PAPE}}^\alpha (\alpha)$ is given by

$$M \left[ T_{\text{PAPE}}^\alpha (\alpha) \right] = \left( \frac{\bar{Y} G_1}{G_2} - \bar{Y}_a \right)^2 + \left( \frac{\bar{Y} G_1}{G_2} \right)^2 V_{20}$$
\[
+ \left( \frac{\bar{Y}G_1}{G_2} \right) (F_2 - F_1) \left\{ (3F_2 - F_1) \left( \frac{\bar{Y}G_1}{G_2} \right) - 2F_2\bar{Y}_a \right\} V_{02}
- 2 \left( \frac{\bar{Y}G_1}{G_2} \right) (F_2 - F_1) \left\{ 2 \left( \frac{\bar{Y}G_1}{G_2} \right) - \bar{Y}_a \right\} V_{11}
\]

\[(5.27)\]

The proof of the expression (5.27) can be obtained on the similar lines as presented under section 5.3.2.

Remark 14. Letting \( \alpha = 1 \) and 4 and using the corresponding values of \( G_1, G_2, F_1 \) and \( F_2 \) in (5.27), it is easy to observe that \( M[T_{PAPE}^\alpha(\alpha)] \) reduces to \( M[\bar{y}_{SS,a}] \) and \( M[\bar{y}_{PS,a}] \) respectively.

5.7 Optimizing \( M[T_{PAPE}^\alpha(\alpha)] \)

The MSE of \( T_{PAPE}^\alpha(\alpha) \) is a function of the parameter \( \alpha \) and hence we can minimize MSE with respect to \( \alpha \) so as to locate the optimum estimator within the class. However, since the expression (5.27) is a complex function of \( \alpha \) and we obtain four different functions of \( \alpha \), namely \( G_1, G_2, F_1 \) and \( F_2 \), an explicit expression for minimum \( M[T_{PAPE}^\alpha(\alpha)] \) is not easy to obtain. The value of optimum \( \alpha (= \alpha_0) \) can be obtained from the equation \( \frac{\partial M[T_{PAPE}^\alpha(\alpha)]}{\partial \alpha} = 0 \), using high-speed computers. Accordingly, minimum MSE can be obtained by putting \( \alpha = \alpha_0 \) in (5.27).

5.8 Empirical Study

5.8.1. It is reasonable to compare the efficiency of the two classes under corresponding optimality conditions and also between the different members of the two families. Since \( \bar{y}_{SS,a} \) and \( \bar{y}_{PS,a} \) are members of both the classes, there is no need to compare the two families for \( \alpha = 1 \) and 4. However, for \( \alpha = 3 \) in the family \( T_{PAPE}^\alpha(\alpha) \), we have predictive approach product-type synthetic estimator, it should be compared with \( \bar{y}_{PS,a} \) to see the superiority of one over other. Further, the minimum MSE of both the families may be compared for their performance.
5.8.2 The Data

Since a data set showing negative correlation between the study and auxiliary variables was seldom available in the literature of sampling theory for our purpose of SAE, we used a computer-generated data set for this purpose. On the basis of a computer programme, a data set consisting of 300 units was generated which yielded negative correlation between the variables. Further, from this data set, which was treated as population, we selected three independent random samples of size 25, 30 and 40, which were treated as small domains 1, 2 and 3 respectively. The population and domain-specific values are shown in Table 5.1

### Population and domain values

<table>
<thead>
<tr>
<th>Domain value</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$N_a$</td>
<td>25</td>
</tr>
<tr>
<td>$\bar{Y}_a$</td>
<td>120.46</td>
</tr>
<tr>
<td>$\bar{X}_a$</td>
<td>63.3</td>
</tr>
<tr>
<td>$S^2_{Y_a}$</td>
<td>226.09</td>
</tr>
<tr>
<td>$S^2_{X_a}$</td>
<td>307.87</td>
</tr>
<tr>
<td>$S_{Y_aX_a}$</td>
<td>-232.6</td>
</tr>
<tr>
<td>$\rho_{Y_aX_a}$</td>
<td>-0.8816</td>
</tr>
</tbody>
</table>
5.8.3. Utilizing the above data, we computed the MSEs of the estimators $\bar{y}_{SS,a}$, $\bar{y}_{PS,a}$, $T_{PAPE}^a$ (3) and minimum MSEs of the estimators $T_{P,I}^a$ and $T_{PAPE}^a$ along with the corresponding $\alpha_0$. The values are depicted in Table 5.2.

### Bias and MSE of different estimators

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$T_{P,I}^a (1)$</td>
<td>Bias</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
</tr>
<tr>
<td>$T_{P,I}^a (4)$</td>
<td>Bias</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
</tr>
<tr>
<td>$T_{PAPE}^a$ (3)</td>
<td>Bias</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
</tr>
<tr>
<td>$T_{P,I}^a (\alpha_0)$</td>
<td>MSE</td>
</tr>
<tr>
<td></td>
<td>$\alpha_0$</td>
</tr>
<tr>
<td>$T_{PAPE}^a (\alpha_0)$</td>
<td>MSE</td>
</tr>
<tr>
<td></td>
<td>$\alpha_0$</td>
</tr>
</tbody>
</table>

From the table, the following facts can be observed:

(i) In general, the PAP type synthetic estimator does not perform better than the usual product-type synthetic estimator $\bar{y}_{PS,a}$ ($T_{P,I}^a (1)$). Perhaps this is the reason that even having intuitive ground PAPEs are not used over usual product estimator.

(ii) $T_{PAPE}^a (\alpha)$ is not preferable over $T_{P,I}^a (\alpha)$ estimator even under corresponding optimality conditions.

However, these observations are subjected to criticism since these depend upon only one sample and population and domain-specific values, which are generally not known. In the next section, therefore, we have presented a simulation study of the two families of estimators based on a large number of samples selected from the population.
5.9 A SIMULATION STUDY

As before, we have presented here a simulation study of the estimators $T_{P,I}^a(\alpha)$ and $T_{P,\text{PAP}}^a(\alpha)$ for their optimum values of $\alpha$.

Taking 500 independent samples of size 15 from the population, we calculated ARB and SRSE of the two estimators. The results are presented in the following table:

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Domain</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{P,I}^a(\alpha_0)$</td>
<td>ARB</td>
<td>1.36</td>
<td>0.28</td>
<td>0.99</td>
</tr>
<tr>
<td></td>
<td>SRSE</td>
<td>30.57</td>
<td>6.32</td>
<td>22.3</td>
</tr>
<tr>
<td>$T_{P,\text{PAP}}^a(\alpha_0)$</td>
<td>ARB</td>
<td>1.85</td>
<td>0.17</td>
<td>1.68</td>
</tr>
<tr>
<td></td>
<td>SRSE</td>
<td>31.4</td>
<td>4.82</td>
<td>27.61</td>
</tr>
</tbody>
</table>

The table makes it clear that both the optimum estimators perform almost in similar manner if the performance was looked on the basis of a large number of samples of equal size. Hence, it is sufficient to utilize usual product-type synthetic estimator for SAE if the correlation in the population is expected to be negative rather than utilizing predictive approach product-type synthetic estimator.

5.10 CONCLUDING REMARKS

Recapitulating what have been discussed in this chapter, we arrive at some conclusion. First, we developed two families of product-type synthetic estimators separately and observed some of their salient features. Secondly, while comparing the two families for their performances, it was observed that virtually the average performance of both of them is almost same and any one of them do suffice the purpose of efficient estimation of small domains. However, the usual product synthetic estimator, which is complementary to usual ratio synthetic estimator, performs slightly better than the PAP type synthetic estimator, though the latter has some intuitive ground as compared to previous one.