Chapter 4

Qualitative Study of Anisotropic Cosmological Models with Dark Sector Coupling

4.1 Introduction

The recent evidences of SNe Ia [Perlmutter et al. (1997, 1998, 1999); Riess et al. (1998); Garnavich et al. (1998); Schmidt et al. (1998); Tonry et al. (2003); Clocchiatti et al. (2006)], CMBR [de Bernardis et al. (2000); Hanany et al. (2000)], LSS [Spergel et al. (2003); Tegmark et al. (2004)], SDSS [Seljak et al. (2005); Adelman-McCarthy et al. (2006)], WMAP Bennett et al. (2003) and Chandra X-ray observatory [Allen et al. (2004)] have suggested that the Universe is expanding and the dark energy is responsible for driving the late-time acceleration of the Universe. The recent Hubble’s diagram suggested that the more than 71.4% of the energy of the Universe is in the form of dark energy and dark matter is believed to about 24% of the Universe. One of the simplest scalar field models of time-evolving dark energy is quintessence. Without violating the observational constraints, dark energy may interact with dark matter by means of energy transfer between each other. In the past history of the Universe, structure formation is one of the example of the interaction between dark components of the Universe. In the several literatures [Spergel et al. (2003); Liddle & Lyth (2003); Magana & Matos (2012); Zimdahl & Pavón (2004); Zimdahl et al. (2005); Pavón et al. (2004); Tsujikawa & Sami (2004); Gumjudpai et al. (2005); Barrow & Clifton (2006)] authors are discussed the behaviour of different models with energy exchange. Recently some authors [Amendola & Tsujikawa (2010); Wei (2010); Zimdahl et al. (2001); Guo et al. (2007); Cabral et al. (2009)] have suggested that the energy density of dark energy and dark matter is of the same order and there is a possibility of coupling between them. An expression for general interaction between a scalar field \( \phi \) (that contains
dark energy) and dark matter is defined as the covariant derivative of the energy momentum tensor of the scalar field $\phi$ and covariant derivative of non-relativistic matter with opposite sign of interaction term respectively.

Some authors [Ellis & Elst (1999); Ellis (1993); Kolb & Turner (1990); Misner (1968); Misner et al. (1973); Hu & Parker (1978); Hawking & Ellis (1973); Belinskii et al. (1970); MacCallum (1979); Ellis & Williams (2000); Ellis & Wainwright (2005); Belinskii & Khatatnikov (1976)] have stressed that the Bianchi Universes provide a generalisation of the Friedmann-Lemaître model (based on homogeneous and isotropic Robertson-Walker model). Therefore, Bianchi Universe have played an important role in observational cosmology and these models are more realistic than FRW model. Some authors [Coley & Dunn (1992); Burd & Coley (1994); Goliath & Ellis (1999)] have used dynamical systems methods to study the evolution of Bianchi cosmological models. A series of papers [Bohmer & Chan (2014); Salcedo et al. (2015); Roy & Banerjee (2014, 2015); Rudra et al. (2012); Cabral et al. (2009); Bohmer et al. (2010); Bohmer & Harko (2010); Toribio (2006); Tamanini (2014); Kofinas (2016); Faraoni & Protheroe (2010); Nojiri et al. (2015)] have written on dynamical system analysis under the background of FRW metric. Recently several authors [Copeland et al. (1998); Bohmer et al. (2010); Salcedo et al. (2015); Paliathanasis et al. (2015); Jesus et al. (2016); Carloni et al. (2015); Avagyam et al. (2016); Biswas & Chakraborty (2015a); Dutta & Zonunmawia (2016); Gosenca & Coles (2015)] have studied phase-plane analysis of cosmological models in modified gravity. The dynamical systems analysis of anisotropic cosmological models with scalar fields are studied by [Chaubey & Raushan (2016a,b)]. In this chapter, we use suitable transformation of variables, which makes the connection between the curvature and dynamics more explicit.

This chapter has the following structure. In section 4.2, we will explore the model and basic equations for LRS BI cosmological Universe. In section 4.3, we will discuss the dynamical evolution of the BI model with linear interaction between dark sector of the Universe by considering $Q = \sqrt{\frac{2}{3} \kappa \beta \rho_\gamma \dot{\phi}}$. The stability and viability of this model for linear coupling between both dark components are presented and discussed. In section 4.4 and 4.5, we will explore the dynamics of the model by considering $Q = \alpha H \rho_\gamma$ and $Q = \Gamma \rho_\gamma$ respectively. We will also work out the phase plane analysis, local and classical stability analysis of critical points in respective sections for each models. Finally conclusions will be given in last section 4.6.

### 4.2 Model and Basic Equations

For LRS Bianchi type-I (LRS BI) cosmological model, the metric is given by
Here the metric \( a_1(t), a_2(t) \) are expansion in the respective directions. The directional Hubble parameters are defined as

\[
H_i = \frac{\dot{a}_i}{a_i}, \quad i = 1, 2. \tag{4.2.2}
\]

Here dot denotes differential with respect to cosmic time ‘\( t \)’. By using equation (4.2.2), the Hubble parameter \( H \) is given by

\[
H = \frac{\dot{a}}{a} = \frac{1}{3} \left( \frac{\dot{a}_1}{a_1} + 2 \frac{\dot{a}_2}{a_2} \right) = \frac{1}{3} (H_1 + 2H_2) \tag{4.2.3}
\]

Here we take an ansatz \( a_1 \propto a_2^n, n > 0 \), LRS Bianchi type I model reduces to flat FRW cosmological model when \( n = 1 \), whereas the model becomes anisotropic when \( n \neq 1 \).

From above assumption and equation (4.2.3), we can find a relation between \( H_i (i = 1, 2) \) and \( H \) as

\[
H_1 = nH_2 = \left( \frac{3n}{n+2} \right) H. \tag{4.2.4}
\]

The Einstein’s field equations are written as

\[
G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = - \frac{8\pi G}{c^4} T_{ij} \tag{4.2.5}
\]

Here \( R_{ij} \), \( R \) and \( g_{ij} \) are the Ricci tensor, Ricci scalar and metric tensor respectively. \( T_{ij} \) is the energy momentum tensor consisting of dark sector and scalar field.

From equations (4.2.1) and (4.2.5) the Einstein’s field equations (4.2.5) are written as

\[
2 \frac{\dot{a}_1^2}{a_1 a_2} - \left( \frac{\dot{a}_2}{a_2} \right)^2 = (\rho_\gamma + \rho_\phi) \tag{4.2.6}
\]

\[
2 \frac{\dot{a}_2^2}{a_2} + \left( \frac{\dot{a}_1}{a_1} \right)^2 = -(p_\gamma + p_\phi) \tag{4.2.7}
\]

\[
\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} = -(p_\gamma + p_\phi) \tag{4.2.8}
\]

Here \( 8\pi G = c = 1 \) and we do not consider the baryons (which are not coupled to dark energy) and radiation (since we are mainly interested in the late Universe). Here \( \rho_\gamma \) and \( p_\gamma \) are the energy density and pressure of dark sector of the Universe respectively. The energy density and pressure of the scalar field are connected by an equation of state \( p_\phi = \gamma \rho_\phi \), defined as
4.2. Model and Basic Equations

\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad (4.2.9) \]

\[ p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (4.2.10) \]

By varying the action of the quintessence scalar field with respect to \( \phi \) and dark sector coupling \( Q \) is defined as

\[ \ddot{\phi} + (H_1 + 2H_2) \dot{\phi} + \frac{dV}{d\phi} = \frac{Q}{\dot{\phi}} \quad (4.2.11) \]

Here \( Q \) is the rate of energy exchange in the dark sector. When \( Q > 0 \) the energy transfer from dark matter to dark energy whereas the energy transfer is in reverse direction when \( Q < 0 \).

A general coupling between a quintessence field \( \phi \) and dark matter (with density \( \rho_\gamma \)) may be described in the background by the balance equations

\[ \dot{\rho}_\gamma = -(H_1 + 2H_2)\rho_\gamma - Q \quad (4.2.12) \]

\[ \dot{\rho}_\phi = -(H_1 + 2H_2)(1 + \gamma_\phi)\rho_\phi + Q \quad (4.2.13) \]

From equations (4.2.4)-(4.2.10), the Friedmann and Raychaudhuri equations are written as

\[ H^2 = \frac{(n + 2)^2}{9(2n + 1)} \left( \rho_\gamma + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \quad (4.2.14) \]

\[ \dot{H} = -\frac{(n + 2)^2}{6(2n + 1)} \left( \rho_\gamma + \dot{\phi}^2 \right) \quad (4.2.15) \]

Here the equation of state for dark energy is defined as \( p_\gamma = \gamma \rho_\gamma \), where \( \gamma \) is a constant.

We introduce two new dimensionless variables \( x \) and \( y \) as

\[ x^2 = \frac{(n + 2)^2}{18(2n + 1)} \left\{ \frac{\dot{\phi}^2}{H^2} \right\} ; y^2 = \frac{(n + 2)^2}{9(2n + 1)} \left\{ \frac{V(\phi)}{H^2} \right\} \quad (4.2.16) \]

There is no as yet basis in fundamental theory for a specific coupling in the dark sector and therefore any coupling model will necessarily be phenomenological, although some models will have a more physical justification than others. Various models of energy exchange have been considered by Cabral et al. (2009). A satisfactory model requires at least that \( Q \) should be expressed in terms of the energy density and other covariant quantities. Here we will discuss three physically viable coupling model.
4.3 Coupling Model (I): $Q = \sqrt{\frac{2}{3}} \kappa \beta \rho \gamma \dot{\phi}$

Here $\beta$ is a dimensionless constant whose sign determines the direction of energy transfer. When $\beta > 0$, the energy transfer from dark matter to dark energy whereas when $\beta < 0$, the energy transfer from dark energy to dark matter. Here we are considering exponential potential $V(\phi) = V_0 e^{-\lambda \phi}$, where $\phi$ is the quintessence scalar field. Now we differentiate the above equation (4.2.16) with respect to the number of e-folding $N = \frac{1}{3} \ln(a_1 a_2^2)$ then the system of autonomous equations are written as

$$\frac{dx}{dN} \equiv x' = -3x + \frac{3\sqrt{(2n+1)}}{\sqrt{2(n+2)}} \lambda y^2 + \frac{3}{2} x(1 + x^2 - y^2) + \frac{\sqrt{3}\sqrt{(2n+1)}}{n+2} \beta (1 + x^2 - y^2)$$

(4.3.1)

$$\frac{dy}{dN} \equiv y' = -\lambda \frac{3\sqrt{(2n+1)}}{\sqrt{2(n+2)}} xy + \frac{3}{2} y(1 + x^2 - y^2)$$

(4.3.2)

From equations (4.2.14) and (4.2.16), one can obtain

$$1 - x^2 - y^2 = \frac{(n+2)^2}{9(2n+1)} \{ \frac{\rho \gamma}{H^2} \}$$

(4.3.3)

Using constraint equation in above equation (4.3.3), we get

$$\Omega_\phi \equiv \frac{\rho_\phi(n+2)^2}{9H^2(2n+1)} = x^2 + y^2$$

(4.3.4)

For scalar field, the effective equation of state is defined as

$$\gamma_\phi = \frac{p_\phi}{\rho_\phi}$$

(4.3.5)

Using equations (4.2.9), (4.2.10), (4.2.16) in equation (4.3.5), we get

$$\gamma_\phi = \frac{x^2 - y^2}{x^2 + y^2}$$

(4.3.6)

The deceleration parameter ‘$q$’ is given as

$$q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{3}{2} (1 + x^2 - y^2)$$

(4.3.7)

4.3.1 Phase-space analysis of model I

To study the dynamical system [equations (4.3.1) and (4.3.2)], we firstly find the fixed/ critical points of the system. The critical points of the system are the solution of the following equations
4.3. Coupling Model (I): \( Q = \sqrt{\frac{3}{2}} k \beta \rho \phi \)

<table>
<thead>
<tr>
<th>Point</th>
<th>( x_* )</th>
<th>( y_* )</th>
<th>( \Omega_\phi )</th>
<th>( \gamma_\phi )</th>
<th>( q )</th>
<th>Existence</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( \forall \lambda, \gamma )</td>
</tr>
<tr>
<td>B</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( \forall \lambda, \gamma )</td>
</tr>
<tr>
<td>C</td>
<td>( \frac{2 \beta k'}{3} )</td>
<td>0</td>
<td>( \frac{4 \beta^2 k'^2}{9} )</td>
<td>1</td>
<td>( \frac{1}{2} + \frac{3}{4} \beta^2 k'^2 )</td>
<td>( \beta \leq \frac{\sqrt{3(n+2)}}{2 \sqrt{2n+1}} )</td>
</tr>
<tr>
<td>D</td>
<td>( \frac{\lambda\lambda}{3} )</td>
<td>( \left( 1 - \frac{\lambda^2 \lambda^2}{9} \right)^{1/2} )</td>
<td>1</td>
<td>( \frac{2 \beta^2 \lambda^2}{9} - 1 )</td>
<td>( -1 + \frac{\lambda^2 \lambda^2}{3} )</td>
<td>( \lambda \leq \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} )</td>
</tr>
<tr>
<td>E</td>
<td>( \frac{3}{2(\lambda' - \beta k')} \sqrt{\frac{4 \beta^2 k'^2 - 4 \beta \lambda k' + 9}{2(\beta k' - \lambda')} + \frac{\beta k'(\lambda' - \beta k')}{2(\beta k' - \lambda')}} )</td>
<td>( \frac{4 \beta^2 k'^2 - 4 \beta \lambda k' + 9}{2(\beta k' - \lambda')} + \frac{\beta k'(\lambda' - \beta k')}{2(\beta k' - \lambda')} )</td>
<td>( \frac{1}{2} ) ( \lambda' - \beta k' )</td>
<td>( \lambda' ) ( \beta k' )</td>
<td>( \lambda \geq \frac{\sqrt{3(n+2)}}{\sqrt{2n+1}} )</td>
<td>( + \frac{2}{\sqrt{3 \beta}} )</td>
</tr>
</tbody>
</table>

Table 4.1: Physical parameters at fixed points for model I.

\[
\frac{dx}{dN} = 0, \quad \frac{dy}{dN} = 0.
\]  
(4.3.8)

To study the phase space behaviour of the system, we analyse the stability of fixed points by finding eigenvalues of the matrix, evaluated at the fixed point \((x_*, y_*)\):

\[
M = \begin{bmatrix}
\frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \\
\frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y}
\end{bmatrix}
\]  
(4.3.9)

where \( f(x, y) = x' \) and \( g(x, y) = y' \)

From equation (4.3.1) and equation (4.3.2), the autonomous system of equations have five critical points namely A, B, C, D and E:

- Point A \((1, 0)\)
- Point B \((-1, 0)\)
- Point C \(\left( \frac{2 \beta k'}{3}, 0 \right)\)
- Point D \(\left( \frac{\lambda \lambda}{3}, 1 - \left( \frac{\lambda \lambda}{3} \right)^2 \right)^{1/2} \)
- Point E \(\left( -\frac{3}{2(\beta k' - \lambda')}, \frac{\lambda' \beta \beta}{2(\beta k' - \lambda')} \frac{1}{2(\beta k' - \lambda')} \right) \)

where \( k' = \sqrt{\frac{\sqrt{2(n+1)}}{n+2}} \) and \( \lambda' = \frac{2 \sqrt{2n+1}}{\sqrt{2(n+2)}} \) i.e. \( k' = \sqrt{\frac{2}{3 \lambda}} \). The cosmological parameters at critical points are given in Table 4.1.

where \( \lambda' \sqrt{2(n+1)} = 16 \beta^2 \lambda^3 k'^2 \) \( l'^2 + 45 \beta^2 k'^2 + 8 \beta \lambda k' + 27 \beta \lambda k' + \frac{5 \lambda^2 k'^2}{4} + 81 \).

Here it is observed that, from Table 4.1, the critical points A and B are always exist for all values of \( \lambda \) and \( \gamma \). The critical points C, D and E exist when the restrictions are
4.3. Coupling Model (I): $Q = \sqrt{\frac{2}{3}} k \beta \rho_x \dot{\phi}$

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_*$</th>
<th>$y_*$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>3 - 2$\beta k'$</td>
<td>3 - $\lambda l'$</td>
</tr>
<tr>
<td>B</td>
<td>−1</td>
<td>0</td>
<td>3 + 2$\beta k'$</td>
<td>3 + $\lambda l'$</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{2\beta k'}{3}$</td>
<td>0</td>
<td>$-\frac{3}{2} + \frac{2\beta^2 k'^2}{3}$</td>
<td>$\frac{3}{2} - \frac{2\beta k'}{3}(\lambda l' - \beta k')$</td>
</tr>
<tr>
<td>D</td>
<td>$\frac{\nu \lambda}{3}$</td>
<td>$\left(1 - \frac{\nu^2 \beta^2}{9}\right)^{1/2}$</td>
<td>$-3 + \frac{\lambda k'^2}{3}$</td>
<td>$-3 - \frac{2\lambda'}{\beta k' - \lambda l'}$</td>
</tr>
<tr>
<td>E</td>
<td>$-\frac{3}{2(\beta k' - \lambda l')}$</td>
<td>$\frac{3(\beta k' - 4\beta l' k' + 9)^{1/2}}{2(\beta k' - \lambda l')}$</td>
<td>$\frac{3(\lambda k' - 2(\lambda k')^{1/2})}{4(\beta k' - \lambda l')} - \frac{3}{2}$</td>
<td>$-\frac{3}{2}$</td>
</tr>
</tbody>
</table>

Table 4.2: Critical points and eigenvalues for model I.

$\beta \leq \frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}, \lambda \leq \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ and $\lambda \geq \frac{\sqrt{3(n+2)}}{\sqrt{2n+1}} + \frac{2}{\sqrt{3\beta}}$ respectively. The nature of critical points and eigenvalues of the Jacobian matrix at critical points are presented in Table 4.2.

4.3.2 Local and classical stability analysis of model I

In this subsection, we will discuss the local and classical stability of model I.

Nature of point A, unstable point if $\beta < \frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ whereas this point becomes stable if $\beta > \frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$. This critical point is saddle (unstable) if either $\beta > \frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ or $\beta < \frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$. From equation (4.3.4), we can obtain $\Omega_\phi = 1$ at point A, this means the kinetic dominated solution exists near this point. From equation (4.3.7), one can obtain $q = 2$ (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as $a \propto t^{1/3}$ at this point A. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{n}{(n+2)t}$ and $\frac{1}{(n+2)t}$ respectively. In view of equation (4.3.6) with $x = 1$ & $y = 0$, the effective equation of state parameter $\gamma_\phi$ is equal to one.

Nature of point B, unstable point if $\beta > -\frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda > -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ whereas this point becomes stable if $\beta < -\frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda < -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$. This critical point is saddle (unstable) if either $\beta < -\frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda > -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ or $\beta > -\frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}}$ and $\lambda < -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$. From equation (4.3.4), we can obtain $\Omega_\phi = 1$ at point B, this means the kinetic dominated solution exists near this point. From equation (4.3.7), one can obtain $q = 2$ (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as $a \propto t^{1/3}$ at this point B. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{n}{(n+2)t}$ and $\frac{1}{(n+2)t}$ respectively. In view of equation (4.3.6) with $x = -1$ & $y = 0$, the effective equation of state parameter $\gamma_\phi$ is equal to one.

Nature of point C, unstable point if $\beta^2 > \frac{3(n+2)^2}{2(2n+1)}$ and $\lambda < \frac{\sqrt{3(n+2)^2}}{2\sqrt{2(2n+1)}\beta} + \sqrt{\frac{2}{\sqrt{3}}} \beta$ whereas this point becomes stable if $\beta^2 < \frac{3(n+2)^2}{4(2n+1)}$ and $\lambda > \frac{\sqrt{3(n+2)^2}}{2\sqrt{2(2n+1)}\beta} + \sqrt{\frac{2}{\sqrt{3}}} \beta$. This critical point is
saddle (unstable) point if either \( \beta^2 > \frac{3(n+2)^2}{2(2n+1)} \) and \( \lambda > \frac{\sqrt{3(n+2)^2}}{2\sqrt{2(2n+1)}} + \frac{\sqrt{7}}{\sqrt{3}} \beta \) or \( \beta^2 < \frac{3(n+2)^2}{2(2n+1)} \) and \( \lambda < \frac{\sqrt{3(n+2)^2}}{2\sqrt{2(2n+1)}} + \frac{\sqrt{7}}{\sqrt{3}} \beta \). From equation (4.3.7), one can obtain there is no accelerating phase of the Universe near point C for all values of \( \beta \) and \( n \) and the average scale factor of the model is evolves as \( a \propto \frac{3}{2} \left( 1 + \frac{4\beta^2 k^2}{9} \right) t^3 \sqrt{1 + \frac{4\beta^2 k^2}{9}} \) at this point C. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{2n}{(n+2)(1 + \frac{4\beta^2 k^2}{9}) t} \) and \( \frac{2}{n+2} \left( 1 + \frac{4\beta^2 k^2}{9} \right) t \), respectively. In view of equation (4.3.6), the effective equation of state parameter \( \gamma_\phi \) is equal to one at point C.

Nature of the point D, unstable point if \( \lambda^2 > \frac{2(2n+2)^2}{(2n+1)} \) and \( \beta < -\frac{\sqrt{3}}{\sqrt{2\lambda}} \left( \frac{n^2+3}{2n+1} \right) \) whereas this point becomes stable if \( \lambda^2 < \frac{2(2n+2)^2}{(2n+1)} \) and \( \beta > -\frac{\sqrt{3}}{\sqrt{2\lambda}} \left( \frac{n^2+3}{2n+1} \right) \). This critical point is saddle (unstable) if either \( \lambda^2 < \frac{2(2n+2)^2}{(2n+1)} \) and \( \beta < -\frac{\sqrt{3}}{\sqrt{2\lambda}} \left( \frac{n^2+3}{2n+1} \right) \) or \( \lambda^2 > \frac{2(2n+2)^2}{(2n+1)} \) and \( \beta > -\frac{\sqrt{3}}{\sqrt{2\lambda}} \left( \frac{n^2+3}{2n+1} \right) \).

From equation (4.3.7), one can obtain there is no accelerating phase of the Universe near point D for \( \lambda^2 > \frac{2(2n+2)^2}{(2n+1)} \) while goes to accelerating phase when \( \lambda^2 < \frac{2(2n+2)^2}{(2n+1)} \). Nature of the point E, unstable point if \( (X_E)^{1/2} > -\frac{3}{2} (\lambda' - 2\beta k') \) and \( (X_E)^{1/2} < \frac{3}{2} (\lambda' - 2\beta k') \) where as this point becomes stable if \( (X_E)^{1/2} < -\frac{3}{2} (\lambda' - 2\beta k') \) and \( (X_E)^{1/2} > \frac{3}{2} (\lambda' - 2\beta k') \). This critical point is saddle (unstable) if either \( (X_E)^{1/2} > -\frac{3}{2} (\lambda' - 2\beta k') \) and \( (X_E)^{1/2} < \frac{3}{2} (\lambda' - 2\beta k') \) or \( (X_E)^{1/2} < -\frac{3}{2} (\lambda' - 2\beta k') \) and \( (X_E)^{1/2} > \frac{3}{2} (\lambda' - 2\beta k') \).

From equation (4.3.7), one can obtain there is accelerating phase of the Universe near point E for \( \lambda < \frac{\sqrt{3} \lambda}{\sqrt{3} + \frac{2n}{(n+2)(\lambda'^2 + 2\beta k'^2)}} \) and the average scale factor of the model is evolves as \( a \propto \frac{3}{2} \left( 1 + \frac{4\beta^2 k^2}{9} \right) t^3 \sqrt{1 + \frac{4\beta^2 k^2}{9}} \) at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{2n}{(n+2)(\lambda'^2 + 2\beta k'^2)} \) and \( \frac{2}{n+2} \left( \lambda'^2 + 2\beta k'^2 \right) \), respectively. In view of equation (4.3.6), the effective equation of state parameter is given by \( \gamma_\phi = \frac{(2n+1)}{(n+2)} \lambda^2 - 1 \) at point D.

In view of classical stability, the model is stable when speed of sound \( (C_s)^2 \) holds the condition \( 0 \leq (C_s)^2 \leq 1 \). In the present cosmological scenario we have

\[
(C_s)^2 = \frac{2x^2 - 2 \sqrt{\frac{2n+1}{n+2}} \beta x(1 - x^2 - y^2) - 2 \sqrt{\frac{2n+1}{n+2}} \lambda x y^2}{1 + x^2 - y^2}
\]  

(4.3.10)

We have given both local and classical stability for model I in Table 4.3. It is observed that points A and B are not locally stable when \( \beta < \frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}} \) and \( \lambda < \frac{\sqrt{3(n+2)}}{2\sqrt{2n+1}} \) respectively while these points are classically stable for all \( \lambda \) and \( \beta \). It is also observed that point C is locally unstable when \( \beta^2 > \frac{3(n+2)^2}{2(2n+1)} \) and \( \lambda < \frac{\sqrt{3(n+2)^2}}{2\sqrt{2(n+1)}} + \frac{\sqrt{7}}{\sqrt{3}} \beta \) but classically stable when \( \beta^2 < \frac{3(n+2)^2}{4(2n+1)} \). From Table 4.3, it is noted
### 4.3. Coupling Model (I): $Q = \sqrt{2} \kappa \beta \rho \phi$

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_*$</th>
<th>$y_*$</th>
<th>Local stability</th>
<th>Classical stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>Stable point if $\beta &gt; \sqrt{3(n+2)} \frac{2}{\sqrt{2n+1}}$ and $\lambda &gt; \frac{\sqrt{2n+1}}{\sqrt{2(n+2)}}$ \newline Unstable otherwise</td>
<td>Stable</td>
</tr>
<tr>
<td>B</td>
<td>-1</td>
<td>0</td>
<td>Stable point if $\beta &lt; -\sqrt{3(n+2)} \frac{2}{\sqrt{2n+1}}$ and $\lambda &lt; -\sqrt{2(n+2)} \frac{\sqrt{2}}{\sqrt{2n+1}}$ \newline Unstable otherwise</td>
<td>Stable</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{2\beta k'}{3}$</td>
<td>0</td>
<td>Stable point if $\beta^2 &lt; \frac{3(n+2)^2}{4(2n+1)}$ and $\lambda &gt; \frac{\sqrt{3(n+2)^2}}{2\sqrt{2(2n+1)}} + \frac{\sqrt{2}}{\sqrt{3}}$ \newline Unstable otherwise</td>
<td>Stable if $\beta^2 &lt; \frac{3(n+2)^2}{4(2n+1)}$</td>
</tr>
<tr>
<td>D</td>
<td>$\frac{\nu\lambda}{3}$</td>
<td>$(1 - \frac{\nu^2\lambda^2}{9})^{1/2}$</td>
<td>Stable point if $\lambda^2 &lt; \frac{2(n+2)^2}{(2n+1)}$ and $\beta &gt; -\frac{\sqrt{3}}{\sqrt{2\lambda}} \frac{(n^2+3)}{2n+1}$ \newline Stable if $\lambda &lt; \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$</td>
<td>Stable if $\lambda &lt; \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$</td>
</tr>
<tr>
<td>E</td>
<td>$-\frac{3}{2(\beta k' - \lambda l')}$</td>
<td>$(\frac{4\beta^2 k'^2 - 4\beta \lambda k' + 9}{2(\beta k' - \lambda l')})^{1/3}$</td>
<td>Stable point if $(X_E)^{1/2} &lt; -\frac{3}{2}(\lambda - 2\beta k')$ and $(X_E)^{1/2} &gt; \frac{3}{2}(\lambda - 2\beta k')$ \newline Stable if $-4\beta^2 + 3\sqrt{6}\lambda \beta - 3\lambda^2 \leq 0$</td>
<td>Stable if $-4\beta^2 + 3\sqrt{6}\lambda \beta - 3\lambda^2 \leq 0$</td>
</tr>
</tbody>
</table>

Table 4.3: Local and classical stability criteria at fixed points for the model-I.

That point D is stable when $\lambda^2 < \frac{2(n+2)^2}{(2n+1)}$ and $\beta > -\frac{\sqrt{3}}{\sqrt{2\lambda}} \frac{(n^2+3)}{2n+1}$ while this point corresponds to classical stability when $\lambda < \frac{\sqrt{2n+1}}{\sqrt{2(n+2)}}$. It is interesting that the point E is locally and classically stable under some restrictions on $\lambda$ and $\beta$. 
4.3. Coupling Model (I): \( Q = \sqrt{\frac{2}{3}} \kappa \beta \rho \bar{\phi} \)

Figure 4.1: Phase plot of the system (model-I) when \( n = 0.05, \lambda = -2 \& \beta = -2 \),

Figure 4.2: Phase plot of the system (model-I) when \( n = 0.05, \lambda = 2 \& \beta = 2 \),

Figure 4.3: Phase plot of the system (model-I) when \( n = 1, \lambda = -2 \& \beta = -2 \),
4.3. Coupling Model (I): \[ Q = \sqrt{2\kappa \beta \rho} \dot{\phi} \]

Figure 4.4: Phase plot of the system (model-I) when \( n = 1, \lambda = 2 \) & \( \beta = 2 \),

Figure 4.5: Phase plot of the system (model-I) when \( n = 100, \lambda = -2 \) & \( \beta = -2 \),

Figure 4.6: Phase plot of the system (model-I) when \( n = 100, \lambda = 2 \) & \( \beta = 2 \),
4.4 Coupling Model (II): $Q = \alpha H \rho_\gamma$

Here $\alpha$ is dimensionless constant whose sign determines the direction of energy transfer. When $\alpha > 0$, the energy transfer from dark matter to dark energy whereas when $\alpha < 0$, the energy transfer from dark energy to dark matter.

In this section, we are considering the LRS BI cosmological model with dark sector interaction as $Q = \alpha H \rho_\gamma$.

The autonomous system of equations for model II are

\[
\frac{dx}{dN} \equiv x' = -3x + \frac{3\sqrt{(2n+1)}}{\sqrt{2(n+2)}} \lambda x^2 + \frac{3}{2}(1 + x^2 - y^2) + \frac{\alpha(1 - x^2 - y^2)}{2x} \quad (4.4.1)
\]

\[
\frac{dy}{dN} \equiv y' = -\lambda \frac{3\sqrt{(2n+1)}}{\sqrt{2(n+2)}} xy + \frac{3}{2} y(1 + x^2 - y^2) \quad (4.4.2)
\]

4.4.1 Phase-space analysis of model II

From equations (4.4.1) and (4.4.2), the autonomous system of equations have six critical points namely A, B, C, D, E and F:

- Point A $(1, 0)$
- Point B $(-1, 0)$
- Point C $\left(\frac{\sqrt{\alpha}}{\sqrt{3}}, 0\right)$
- Point D $\left(-\frac{\sqrt{\alpha}}{\sqrt{3}}, 0\right)$
- Point E $\left(\frac{\nu}{\sqrt{3}}, (1 - (\nu/3)^{1/2})\right)$
- Point F $\left(\frac{\alpha + 3}{2\lambda'}, \left(\frac{(\alpha + 3)^2 - 4\alpha \lambda' t^2}{4\lambda'}\right)^{1/2}\right)$

where $k' = \frac{\sqrt{2}}{\sqrt{3}} \rho' = \frac{\sqrt{2n+1}}{n+2}$.

where $X_F = -3\alpha^5 - \frac{15\lambda^4 t^2}{4} + 45\alpha^4 + \frac{2\alpha^3 \lambda^4 t^4}{3} - 51\alpha^3 \lambda^2 t^2 + 270\alpha^3 + \frac{4\alpha^2 \lambda^4 t^6}{9} + 12\alpha^2 \lambda^4 t^4 - \frac{441\alpha^2 \lambda^2 t^6}{2} + 810\alpha^2 + 30\alpha \lambda^4 t^2 - 351\alpha \lambda^2 t^2 + 1215\alpha - \frac{567\lambda^2 t^2}{4} + 729$.

Here it is noted, that from Table 4.4, the critical points A and B are always exist for all values of $\lambda$ and $\gamma$, while the critical points C and D exist for $0 \leq \alpha \leq 3$. The other critical points E & F exist when $\lambda \leq \frac{\sqrt{\gamma(3n+2)}}{\sqrt{3n+1}}$ and $\lambda \geq \frac{(n+2)(\alpha+3)}{3\sqrt{2n+1}}$ respectively. The nature of critical points and eigenvalues of the Jacobian matrix at critical points are given in Table 4.5.
4.4. Coupling Model (II): $Q = \alpha H \rho$ 

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_*$</th>
<th>$y_*$</th>
<th>$\Omega_\phi$</th>
<th>$\gamma_\phi$</th>
<th>$q$</th>
<th>Existence</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\forall \lambda, \gamma$</td>
</tr>
<tr>
<td>B</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\forall \lambda, \gamma$</td>
</tr>
<tr>
<td>C</td>
<td>$\sqrt[3]{\alpha}$</td>
<td>0</td>
<td>$\frac{\alpha}{3}$</td>
<td>1</td>
<td>$\frac{1+\alpha}{2}$</td>
<td>$0 \leq \alpha \leq 3$</td>
</tr>
<tr>
<td>D</td>
<td>$-\sqrt[3]{\alpha}$</td>
<td>0</td>
<td>$\frac{\alpha}{3}$</td>
<td>1</td>
<td>$\frac{1+\alpha}{2}$</td>
<td>$0 \leq \alpha \leq 3$</td>
</tr>
<tr>
<td>E</td>
<td>$\frac{\lambda l'}{3}$</td>
<td>$(1 - (\frac{\lambda l'}{3})^2)^{1/2}$</td>
<td>1</td>
<td>$-1 + \frac{2\lambda^2 l'^2}{9}$</td>
<td>$\lambda \leq \frac{\sqrt[2]{(n+2)}}{\sqrt[2]{2n+1}}$</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>$\frac{\alpha+3}{2\lambda l'}$</td>
<td>$ \frac{(\alpha+3)^2 - 4\alpha \lambda^2 l'^2}{2\lambda l'^2}^{1/2}$</td>
<td>$\frac{2(\alpha+3)^2 - 4\alpha \lambda^2 l'^2}{4\lambda^2 l'^2}$</td>
<td>$\frac{4\alpha \lambda^2 l'^2}{6(\alpha+3)^2 - 4\alpha \lambda^2 l'^2}$</td>
<td>$\lambda \geq \frac{(n+2)(\alpha+3)}{3\sqrt[2]{2n+1}}$, $\frac{(n+2)^2(\alpha+3)}{18(2n+1)}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: Physical parameters at fixed points for model II.

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_*$</th>
<th>$y_*$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>$3 - \alpha$</td>
<td>$3 - \lambda l'$</td>
</tr>
<tr>
<td>B</td>
<td>−1</td>
<td>0</td>
<td>$3 - 2\alpha$</td>
<td>$3 + \lambda l'$</td>
</tr>
<tr>
<td>C</td>
<td>$\sqrt[3]{\alpha}$</td>
<td>0</td>
<td>$-3 + \alpha$</td>
<td>$\frac{\alpha+3}{2} - \lambda l' \left(\frac{\alpha}{3}\right)^{1/2}$</td>
</tr>
<tr>
<td>D</td>
<td>$-\sqrt[3]{\alpha}$</td>
<td>0</td>
<td>$-3 + \alpha$</td>
<td>$\frac{\alpha+3}{2} + \lambda l' \left(\frac{\alpha}{3}\right)^{1/2}$</td>
</tr>
<tr>
<td>E</td>
<td>$\frac{\lambda l'}{3}$</td>
<td>$(1 - (\frac{\lambda l'}{3})^2)^{1/2}$</td>
<td>$-3 + \frac{\lambda^2 l'^2}{3}$</td>
<td>$-(\alpha + 3) + \frac{2\lambda^2 l'^2}{3}$</td>
</tr>
<tr>
<td>F</td>
<td>$\frac{\alpha+3}{2\lambda l'}$</td>
<td>$ \frac{(\alpha+3)^2 - 4\alpha \lambda^2 l'^2}{2\lambda l'^2}^{1/2}$</td>
<td>$\frac{(X_F)^{1/2}}{6\lambda l' + 2\alpha \lambda l'^2} + \frac{3\alpha^2 + 6\alpha - 9}{12 + 4\alpha}$</td>
<td>$\frac{3\alpha^2 + 6\alpha - 9}{12 + 4\alpha} - \frac{\alpha \lambda^2 l'^2}{9 \ast 3\alpha}$</td>
</tr>
</tbody>
</table>

Table 4.5: Critical points and eigenvalues for model II.
4.4.2 Local and classical stability analysis of model II

In this section we are discussing the stability of the critical points A, B, C, D, E and F of the autonomous system of equations (4.4.1) and (4.4.2).

Nature of point A, unstable if $\alpha < 3$ and $\lambda < \frac{2 \sqrt{3(n+2)}}{2n+1}$ whereas this point becomes stable if $\alpha > 3$ and $\lambda > \frac{2 \sqrt{3(n+2)}}{2n+1}$. This critical point A is saddle if either $\alpha < 3$ and $\lambda > \frac{2 \sqrt{3(n+2)}}{2n+1}$ or $\alpha > 3$ and $\lambda < \frac{2 \sqrt{3(n+2)}}{2n+1}$. From equation (4.3.4), we can obtain $\Omega_\phi = 1$ at point A, this means the kinetic dominated solution exists near this point. From equation (4.3.7), one can obtain $q = 2$ (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as $a \propto t^\frac{3}{2}$ at this point A. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{n}{(n+2)t}$ and $\frac{1}{(n+2)t}$ respectively. In view of equation (4.3.6) with $x = 1$ & $y = 0$, the effective equation of state parameter $\gamma_\phi$ is equal to one.

Nature of point B, unstable if $\alpha < 3$ and $\lambda > -\frac{2 \sqrt{3(n+2)}}{2n+1}$. This critical point is saddle if either $\alpha < 3$ and $\lambda < -\frac{2 \sqrt{3(n+2)}}{2n+1}$ or $\alpha > 3$ and $\lambda > -\frac{2 \sqrt{3(n+2)}}{2n+1}$. From equation (4.3.4), we can obtain $\Omega_\phi = 1$ at point B, this means the kinetic dominated solution exists near this point. From equation (4.3.7), one can obtain $q = 2$ (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as $a \propto t^\frac{3}{2}$ at this point B. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{n}{(n+2)t}$ and $\frac{1}{(n+2)t}$ respectively. In view of equation (4.3.6) with $x = -1$ & $y = 0$, the effective equation of state parameter $\gamma_\phi$ is equal to one.

Nature of point C, unstable if $\alpha > 3$ and $\lambda < \frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$ whereas this point becomes stable if $\alpha < 3$ and $\lambda > \frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$. This critical point is saddle if either $\alpha > 3$ and $\lambda > \frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$ or $\alpha < 3$ and $\lambda < \frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$. From equation (4.3.7), one can obtain there is accelerating phase of the Universe near point C for $\alpha < -1$ and the average scale factor of the model is evolves as $a \propto \frac{(3+\alpha)}{2} t^{\frac{2}{3+\alpha}}$ at this point C. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{6n}{(n+2)(3+\alpha)t}$ and $\frac{6}{(n+2)(3+\alpha)t}$ respectively. In view of equation (4.3.6), the effective equation of state parameter $\gamma_\phi$ is equal to one at point C.

Nature of point D, unstable if $\alpha > 3$ and $\lambda > -\frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$ whereas this point becomes stable if $\alpha < 3$ and $\lambda < -\frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$. This critical point is saddle if either $\alpha > 3$ and $\lambda < -\frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$ or $\alpha < 3$ and $\lambda > -\frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$. From equation (4.3.7), one can obtain there is accelerating phase of the Universe near point D for $\alpha < -1$ and the average scale factor of the model is evolves as $a \propto \frac{(3+\alpha)}{2} t^{\frac{2}{3+\alpha}}$ at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{6n}{(n+2)(3+\alpha)t}$ and $\frac{6}{(n+2)(3+\alpha)t}$ respectively.
It is interesting that the point F is locally and classically stable under some restrictions on \( \alpha > \lambda \) when \( \alpha > \gamma \). The effective equation of state parameter is given by equation (4.3.6), the effective equation of state parameter \( \gamma_{\phi} \) is equal to one at point D.

Nature of point E, unstable if \( \lambda^2 > \frac{2(\alpha + 2)^2}{(n + 2)^2} \) and \( \alpha < \frac{3(2n + 1)}{(n + 2)^2} \) whereas this point becomes stable if \( \lambda^2 < \frac{2(\alpha + 2)^2}{(n + 2)^2} \) and \( \alpha > \frac{3(2n + 1)}{(n + 2)^2} \). This critical point is saddle if either \( \lambda^2 > \frac{2(\alpha + 2)^2}{(n + 2)^2} \) and \( \alpha > \frac{3(2n + 1)}{(n + 2)^2} \) or \( \lambda^2 < \frac{2(\alpha + 2)^2}{(n + 2)^2} \) and \( \alpha < \frac{3(2n + 1)}{(n + 2)^2} \). From equation (4.3.7), one can obtain there is accelerating phase of the Universe near point E for \( \lambda^2 < \frac{2(\alpha + 2)^2}{3(2n + 1)} \) and the average scale factor of the model is evolves as \( a \propto \frac{l_{3(\alpha - 1)}}{\sqrt{3 \lambda^2 \alpha^2}} \) at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{9\lambda}{(n + 2)^2(\alpha^2 + \alpha^2)} \) and \( \frac{9}{(n + 2)^2(\alpha^2 + \alpha^2)} \) respectively. In view of equation (4.3.6), the effective equation of state parameter is given by \( \gamma_{\phi} = \frac{2\lambda^2}{9} - 1 \) at point E.

Nature of point F, unstable if \( \frac{(X_F)^{1/2}}{(n + 2)^2} + 3x^2 + \frac{6a - 9}{12 + 4a} - \frac{\alpha^2}{9 + 3a} > 0 \) whereas this point becomes stable if \( \frac{(X_F)^{1/2}}{(n + 2)^2} + 3x^2 + \frac{6a - 9}{12 + 4a} - \frac{\alpha^2}{9 + 3a} < 0 \). This critical point is saddle if either \( \frac{(X_F)^{1/2}}{(n + 2)^2} + 3x^2 + \frac{6a - 9}{12 + 4a} - \frac{\alpha^2}{9 + 3a} < 0 \) or \( \frac{(X_F)^{1/2}}{(n + 2)^2} + 3x^2 + \frac{6a - 9}{12 + 4a} - \frac{\alpha^2}{9 + 3a} > 0 \). From equation (4.3.7), one can obtain there is accelerating phase of the Universe near point F for \( \alpha < -1/3 \) and the average scale factor of the model is evolves as \( a \propto \frac{(3 + \alpha)^2}{l_{3(\alpha - 1)}} \) at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{6n}{(n + 2)^23 + \alpha^2} \) and \( \frac{6n}{(n + 2)^23 + \alpha^2} \) respectively. In view of equation (4.3.6), the effective equation of state parameter is given by \( \gamma_{\phi} = \frac{4a^2}{9 + 4a} \) at point F.

From the point of view of the classical stability for model II, the sound speed for this model is given by

\[
C_s^2 = \frac{6x^2 - 6\sqrt{2n + 1} \lambda xy^2 - \alpha(1 - x^2 - y^2)}{3 + 3x^2 - 3y^2}
\]

We have given both local and classical stability for model II in Table 4.6. It is observed that, when \( \alpha < 3 \), the points A and B are not locally stable if \( \lambda < \frac{\sqrt{2(n + 2)}}{\sqrt{2n + 1}} \) and \( \lambda > \frac{-\sqrt{2(n + 2)}}{\sqrt{2n + 1}} \) respectively while these points are classically stable for all \( \lambda \) and \( \alpha \). It is also observed that when \( \alpha > 3 \), point C and D are not unstable when \( \lambda < \frac{(3 + \alpha)(n + 2)}{\sqrt{6\alpha(2n + 1)}} \) and \( \lambda > \frac{(3 + \alpha)(n + 2)}{\sqrt{6\alpha(2n + 1)}} \) respectively while these critical points are classically stable under some restrictions on \( \alpha \) i.e. \( 0 < \alpha < 3 \). From Table 4.6, it is noted that the point E is stable when \( \lambda^2 < \frac{2(\alpha + 2)^2}{(n + 2)^2} \) and \( \alpha > \frac{3(2n + 1)}{(n + 2)^2} \). It is interesting that the point F is locally and classically stable under some restrictions on \( \lambda \) and \( \alpha \).
### 4.4. Coupling Model (II): $Q = \alpha H \rho_{\gamma}$

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_*$</th>
<th>$y_*$</th>
<th>Local stability</th>
<th>Classical stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>Stable point if $\alpha &gt; 3$ and $\lambda &gt; \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ otherwise unstable</td>
<td>Stable</td>
</tr>
<tr>
<td>B</td>
<td>−1</td>
<td>0</td>
<td>Stable point if $\alpha &gt; 3$ and $\lambda &lt; -\frac{\sqrt{2}(n+2)}{\sqrt{2n+1}}$ otherwise unstable</td>
<td>Stable</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{\sqrt{\alpha}}{\sqrt{3}}$</td>
<td>0</td>
<td>Stable if $\alpha &lt; 3$ and $\lambda &gt; \frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$ otherwise unstable</td>
<td>Stable if $0 &lt; \frac{\alpha}{3} &lt; 1$</td>
</tr>
<tr>
<td>D</td>
<td>$-\frac{\sqrt{\alpha}}{\sqrt{3}}$</td>
<td>0</td>
<td>Stable if $\alpha &lt; 3$ and $\lambda &lt; \frac{(3+\alpha)(n+2)}{6\alpha(2n+1)}$ otherwise unstable</td>
<td>Stable if $0 &lt; \frac{\alpha}{3} &lt; 1$</td>
</tr>
<tr>
<td>E</td>
<td>$\frac{\lambda^*}{3}$</td>
<td>$(1 - (\frac{\lambda^*}{3})^2)^{1/2}$</td>
<td>Stable if $\lambda^2 &lt; \frac{2(n+2)^2}{(2n+1)^2}$ and $\alpha &gt; \frac{3(n+1)}{n+2} \lambda^2 - 3$ otherwise unstable</td>
<td>Stable if $\lambda &lt; \frac{\sqrt{2}(n+2)}{\sqrt{2n+1}}$</td>
</tr>
<tr>
<td>F</td>
<td>$\frac{\alpha+3}{2\lambda^*}$</td>
<td>$\left(\frac{(\alpha+3)^2 - 4\alpha\lambda^<em>/3}{2\lambda^</em>}\right)^{1/2}$</td>
<td>Stable if $\frac{(X_F)^{1/2}}{6\lambda^2 + 2\alpha \lambda^<em>} + \frac{3\alpha^2 + 6\alpha - 9}{12+4\alpha} - \frac{\alpha\lambda^3}{9+3\alpha} &lt; 0$ and $-\frac{(X_F)^{1/2}}{6\lambda^2 + 2\alpha \lambda^</em>} + \frac{3\alpha^2 + 6\alpha - 9}{12+4\alpha} - \frac{\alpha\lambda^3}{9+3\alpha} &lt; 0$ otherwise unstable</td>
<td>Stable if $0 &lt; \frac{\alpha}{3} &lt; 1$</td>
</tr>
</tbody>
</table>

Table 4.6: Local and classical stability criteria at fixed points for the model-II.
4.4. Coupling Model (II): $Q = \alpha H \rho_\gamma$

Figure 4.7: Phase plot of the system (model-II) when $n = 0.001, \lambda = 4 \& \alpha = 4,$

Figure 4.8: Phase plot of the system (model-II) when $n = 1, \lambda = 4 \& \alpha = 3.1,$

Figure 4.9: Phase plot of the system (model-II) when $n = 100, \lambda = 4 \& \alpha = 3.5,$
4.5 **Coupling Model (III):** $Q = \Gamma \rho_\gamma$

Here $\Gamma$ is a constant. When $\Gamma > 0$, the energy transfer corresponds to decay of from dark matter to dark energy whereas when $\Gamma < 0$, a transfer of energy from dark energy to dark matter.

In this section, we are considering the LRS BI cosmological model with the dark sector interaction as $Q = \Gamma \rho_\gamma$. Here we introduce a new variable $z$, chosen so as to maintain compactness of the phase space.

$$z = \frac{H_0}{H + H_0}$$

(4.5.1)

where $0 \leq z \leq 1$. When $z \to 0$ corresponds to $H \to \infty$ then critical point is for the early Universe whereas for the late Universe $z \to 1$ corresponds $H \to 0$.

We also rescale to a dimensionless coupling constant

$$\gamma_* = \frac{\Gamma}{H_0}$$

(4.5.2)

The autonomous system of equations for model III are

$$\frac{dx}{dN} \equiv x' = -3x + \frac{3\sqrt{(2n + 1)}}{\sqrt{2(n + 2)}}(1 + x^2 - y^2) - \gamma_* \frac{(1 - x^2 - y^2)z}{2x(z - 1)}$$

(4.5.3)

$$\frac{dy}{dN} \equiv y' = -\lambda \frac{3\sqrt{(2n + 1)}}{\sqrt{2(n + 2)}}xy + \frac{3}{2}y(1 + x^2 - y^2)$$

(4.5.4)

$$\frac{dz}{dN} \equiv z' = \frac{3}{2}z(1 - z)(1 + x^2 - y^2)$$

(4.5.5)

### 4.5.1 Phase-space analysis of the model III

From equations (4.5.3), (4.5.4) and (4.5.5), the autonomous system of equations have seven critical points namely A, B, C, D, E, F and G:

- **Point A** $(1, 0, 0)$
- **Point B** $(-1, 0, 0)$
- **Point C** $\left(\frac{\rho_\Lambda}{3}, \left(1 - (\frac{\rho_\Lambda}{3})^2\right)^{1/2}, 0\right)$
- **Point D** $\left(\frac{3}{2\rho_\Lambda}, \frac{3}{2\rho_\Lambda}, 0\right)$
- **Point E** $(1, 0, 1)$
4.5. Coupling Model (III): $Q = \Gamma \rho_\gamma$

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_*$</th>
<th>$y_*$</th>
<th>$z_*$</th>
<th>$\Omega_\phi$</th>
<th>$\gamma_\phi$</th>
<th>$q$</th>
<th>Existence</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\forall \lambda, \gamma$</td>
</tr>
<tr>
<td>B</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\forall \lambda, \gamma$</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{\lambda'}{3}$ $(1 - \lambda^2 l^2)^{1/2}$</td>
<td>0</td>
<td>1</td>
<td>$-1 + \frac{2\lambda^2 l^2}{9}$</td>
<td>$-1 + \frac{\lambda^2 l^2}{3}$</td>
<td>$\lambda \leq \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>$\frac{3}{2\lambda'}$</td>
<td>$\frac{3}{2\lambda'}$</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\lambda \geq \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\forall \lambda, \gamma$</td>
</tr>
<tr>
<td>F</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\forall \lambda, \gamma$</td>
</tr>
<tr>
<td>G</td>
<td>$\frac{\lambda'}{3}$ $(1 - \lambda^2 l^2)^{1/2}$</td>
<td>1</td>
<td>1</td>
<td>$-1 + \frac{2\lambda^2 l^2}{9}$</td>
<td>$-1 + \frac{\lambda^2 l^2}{3}$</td>
<td>$\lambda \leq \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.7: Physical parameters at fixed points for model III.

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_*$</th>
<th>$y_*$</th>
<th>$z_*$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>$3 - \lambda'$</td>
</tr>
<tr>
<td>B</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>$3 + \lambda'$</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{\lambda'}{3}$ $(1 - \lambda^2 l^2)^{1/2}$</td>
<td>0</td>
<td>$\frac{\lambda^2 l^2}{3}$</td>
<td>$-3 + \frac{2\lambda^2 l^2}{3}$</td>
<td>$-3 + \frac{\lambda^2 l^2}{3}$</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>$\frac{3}{2\lambda'}$</td>
<td>$\frac{3}{2\lambda'}$</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>$-\frac{3}{4} + \left(\frac{\lambda - 6\lambda^2 l^2}{16}\lambda'\right)^{1/2}$</td>
<td>$-\frac{3}{4} - \left(\frac{\lambda - 6\lambda^2 l^2}{16}\lambda'\right)^{1/2}$</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-3$</td>
<td>$3 - \lambda'$</td>
<td>$-sgn(\gamma_*) \infty$</td>
</tr>
<tr>
<td>F</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>$-3$</td>
<td>$3 - \lambda'$</td>
<td>$-sgn(\gamma_*) \infty$</td>
</tr>
<tr>
<td>G</td>
<td>$\frac{\lambda'}{3}$ $(1 - \lambda^2 l^2)^{1/2}$</td>
<td>1</td>
<td>$-\frac{\lambda^2 l^2}{3}$</td>
<td>$-3 + \frac{\lambda^2 l^2}{3}$</td>
<td>$-sgn(\gamma_*) \infty$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8: Critical points and eigenvalues for model III.

- Point F $(-1, 0, 1)$
- Point G $\left(\frac{\lambda'}{3}, (1 - (\frac{\lambda'}{3})^2)^{1/2}, 1\right)$

Here it is noted, that from Table 4.7, the critical points A, B, E and F are always exist for all values of $\lambda$ and $\gamma$, while the critical points C and G exist when $\lambda \leq \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$. On the other hand, critical point D exists when $\lambda \geq \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$. The nature of critical points and eigenvalues of the Jacobian matrix at critical points are given in Table 4.8.

4.5.2 Local and classical stability analysis of model III

In this subsection, we are discussing the stability of the critical points A, B, C, D, E, F and G of the autonomous system of equations (4.5.3), (4.5.4) and (4.5.5).

Nature of point A, unstable if $\lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ whereas this point becomes saddle if $\lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$. From equation (4.3.4), we can obtain $\Omega_\phi = 1$ at point A, this means the kinetic
dominated solution exists near this point. From equation (4.3.7), one can obtain \( q = 2 \) (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as \( a \propto t^{\frac{1}{2}} \) at this point A. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{n}{(n+2)t} \) and \( \frac{1}{(n+2)t} \) respectively. In view of equation (4.3.6) with \( x = 1, y = 0 & z = 0 \), the effective equation of state parameter \( \gamma_\phi \) is equal to one.

Nature of point B, unstable if \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) whereas this point becomes saddle if \( \lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \). From equation (4.3.4), we can obtain \( \Omega_\phi = 1 \) at point B, this means the kinetic dominated solution exists near this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{n}{(n+2)t} \) and \( \frac{1}{(n+2)t} \) respectively. In view of equation (4.3.6) with \( x = -1, y = 0 \) & \( z = 0 \) the effective equation of state parameter \( \gamma_\phi \) is equal to one.

Nature of point C is saddle if \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \). From equation (4.3.7), one can obtain there is accelerating phase of the Universe near point C for \( \lambda^2 < \frac{2(2n+1)^2}{3(2n+1)} \) and the average scale factor of the model is evolves as \( a \propto \frac{\lambda^2 t}{\sqrt{2n+1}} \) at this point C. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{9n}{(n+2)\lambda^2 t^2} \) and \( \frac{9}{(n+2)\lambda^2 t^2} \) respectively. In view of equation (4.3.6), the effective equation of state parameter is \( \gamma_\phi = \frac{(2n+1)}{(n+2)} \lambda^2 - 1 \) at point C.

Nature of point D is saddle node for \( \frac{8(n+2)}{7(2n+1)} > \lambda^2 > \frac{n+2}{3(2n+1)} \) and saddle focus for \( \frac{36}{7} < \lambda^2 t^2 \). From equation (4.3.4), we can obtain \( \Omega_\phi = 1 \) at point D, this means the kinetic dominated solution exists near this point. From equation (4.3.7), one can obtain \( q = 1/2 \) (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as \( a \propto t^\frac{2}{3} \) at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{2n}{(n+2)t} \) and \( \frac{2}{(n+2)t} \) respectively. In view of equation (4.3.6), the effective equation of state parameter \( \gamma_\phi \) is zero at point D.

Nature of point E is stable node for \( \lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) and \( \gamma_\phi > 0 \) whereas this point becomes saddle node if either \( \lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) and \( \gamma_\phi < 0 \) or \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) and all \( \gamma_\phi \). From equation (4.3.4), we can obtain \( \Omega_\phi = 1 \) at point E, this means the kinetic dominated solution exists near this point. From equation (4.3.7), one can obtain \( q = 2 \) (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as \( a \propto t^\frac{1}{2} \) at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters \( H_1 \) and \( H_2 \) are given by \( \frac{n}{(n+2)t} \) and \( \frac{1}{(n+2)t} \) respectively. In view of equation (4.3.6), the effective equation of state parameter \( \gamma_\phi \) is equal to one at point E.

Nature of point F is stable node for \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) and \( \gamma_\phi > 0 \) whereas saddle node if either \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) and \( \gamma_\phi < 0 \) or \( \lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) and all \( \gamma_\phi \). From equation (4.3.4), we can
obtain $\Omega_\phi = 1$ at point F, this means the kinetic dominated solution exists near this point. From equation (4.3.7), one can obtain $q = 2$ (i.e. decelerating phase of the Universe) and the average scale factor of the model is evolves as $a \propto t^{2/3}$ at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{n}{(n+2)t}$ and $\frac{1}{(n+2)t}$ respectively. In view of equation (4.3.6), the effective equation of state parameter $\gamma$ is equal to one at point F.

Nature of point G is stable node for $\gamma_* > 0$ and saddle node $\gamma_* < 0$. From equation (4.3.7), one can obtain there is accelerating phase of the Universe near point G for $\lambda^2 < \frac{2(n+2)^2}{3(2n+1)}$ and the average scale factor of the model is evolves as $a \propto \lambda^2t^{3/2}$ at this point. Using above relation of scale factor with equations (4.2.2) and (4.2.3), the directional Hubble parameters $H_1$ and $H_2$ are given by $\frac{9n}{(n+2)\lambda^2t^2}$ and $\frac{9}{(n+2)\lambda^2t^2}$ respectively. In view of equation (4.3.6), the effective equation of state parameter is given by $\gamma_\phi = \frac{2\lambda^2t^2}{9} - 1$ at point G.

From the point of view of the classical stability of model III, the sound speed for this model is given by

$$C_s^2 = \frac{6x^2 - \frac{6\sqrt{2n+1}}{n+2}\lambda xy^2 - \gamma_* \frac{x z}{1-z} (1 - x^2 - y^2)}{3 + 3x^2 - 3y^2} \tag{4.5.6}$$

We have given both local and classical stability for model III in Table 4.9. It is observed that, points A and B are not locally stable when $\lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ and $\lambda > -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ respectively, while these critical points are classically stable for all $\lambda$. Points C and D are locally unstable while point C is classically stable for $\lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ and point D is classically stable for all $\lambda$. From Table 4.9, It is also observed that points E and F are stable node when $\lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ and $\lambda < -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}}$ respectively. It is interesting that the critical point G is locally stable when $\gamma_* > 0$ but classically undefined due to the limit $z_* \rightarrow 1$. 

4.5. Coupling Model (III): $Q = \Gamma \rho_\gamma$
4.5. Coupling Model (III): \( Q = \Gamma \rho \gamma \)  

\[
Q = \Gamma \rho \gamma
\]

Point \( x_\star \) \( y_\star \) \( z_\star \)  

<table>
<thead>
<tr>
<th>Point</th>
<th>( x_\star )</th>
<th>( y_\star )</th>
<th>( z_\star )</th>
<th>Local stability</th>
<th>Classical stability</th>
</tr>
</thead>
</table>
| A     | 1      | 0      | 0      | Unstable if \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \)  
Saddle if \( \lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) | Stable |
| B     | -1     | 0      | 0      | Unstable if \( \lambda > -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \)  
Saddle if \( \lambda < -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) | Stable |
| C     | \( \frac{\lambda^\prime}{3} \left( 1 - \frac{\lambda^2 l^2}{9} \right)^{1/2} \) | 0      | 0      | Saddle node if \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) | Stable if \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) |
| D     | \( \frac{3}{2X'^2} \) | \( \frac{3}{2X'^2} \) | 0      | Saddle node for \( \frac{36}{7} > \lambda^2 l^2 > \frac{3}{2} \)  
Saddle focus for \( \frac{36}{7} < \lambda^2 l^2 \) | Stable |
| E     | 1      | 0      | 1      | Stable node for \( \lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) & \( \gamma_\star > 0 \)  
Saddle node if either \( \lambda > \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) & \( \gamma_\star < 0 \)  
or \( \lambda < \frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) & for all \( \gamma_\star \) | Undefined |
| F     | -1     | 0      | 1      | Stable node for \( \lambda < -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) & \( \gamma_\star > 0 \)  
Saddle node if either \( \lambda > -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \); \( \gamma_\star < 0 \)  
or \( \lambda > -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) & for all \( \gamma_\star \) | Undefined |
| G     | \( \frac{\lambda'}{3} \left( 1 - \frac{\lambda^2 l^2}{9} \right)^{1/2} \) | 1      | 0      | Stable node for \( \gamma_\star > 0 \)  
Saddle node for \( \gamma_\star < 0 \) | Undefined |

Table 4.9: Local and classical stability criteria at fixed points for the model-III.
4.5. Coupling Model (III): \( Q = \Gamma \rho \gamma \)

Figure 4.10: Projection phase plot in YZ plane

Figure 4.11: Projection phase plot in XZ plane when \( \gamma_* = 0.001 \)

Figure 4.12: Projection phase plot in XZ plane when \( \gamma_* = 0.00001 \)
4.5. Coupling Model (III): $Q = \Gamma \rho \gamma$

Figure 4.13: Projection phase plot in XY plane when $n = 0.001$ & $\lambda = 0.5$

Figure 4.14: Projection phase plot in XY plane when $n = 0.001$ & $\lambda = -4$

Figure 4.15: Projection phase plot in XY plane when $n = 100$ & $\lambda = -4$
4.6 Conclusions

This chapter dealt the qualitative study of the LRS Bianchi I cosmological model under the background of the Universe dominated by dark energy (in the form of exponential quintessence) and cold dark matter, where the energy exchange in the dark sector is considered in three physically viable models as (i) \( Q = \sqrt{\frac{2}{3}} \kappa \beta \rho \dot{\phi} \) (ii) \( Q = \alpha H \rho \gamma \) and (iii) \( Q = \Gamma \rho \gamma \). We have calculated the critical points and their stability in terms of the parameter \( n \). It is interesting to note here that our model is reduced into Bohmer model [Bohmer et al. (2010)] when \( n = 1 \). We have examined all critical points and discussed their relevant physical parameters near each critical points in three different choices of dark sector coupling \( Q \).

In comparison to the model with last model there are two new fixed points. In model-I, it is observed that the critical points A, B and C are unstable in local stability analysis while stable in classical stability analysis, whereas other two critical points D and E are local and classical stable under some restrictions on \( \lambda \) and \( \beta \). On the other hand, the critical points A, B, C and D are unstable in local stability analysis while stable in classical stability analysis, whereas other two critical points E and F are local and classical stable under some restrictions on \( \lambda \) and \( \alpha \) in model-II. In the last model-III, it is observed that critical points A, B, C and D are locally unstable and classically stable whereas critical points E and F are stable node and critical point G is stable when \( \gamma_* > 0 \) but classically undefined due to the limit \( z_* \to 1 \).

When \( \lambda > -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \), the critical point B is unstable and point C is stable. As \( \lambda \) approaches to \( -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \), these two critical points B and C merge into single-fixed point. When \( \lambda < -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) (i.e. on the other side of bifurcation point) the behaviour of these critical points are interchanged. It is interesting to note here that there is a clear bifurcation in the system where \( \lambda \) is the bifurcation parameter and \( \lambda = -\frac{\sqrt{2(n+2)}}{\sqrt{2n+1}} \) is the bifurcation point. It is worth noting that the recently observed accelerated expansion of the Universe near the critical points E (in model-I), F (in model-II) and G (in model-III) can seen from the autonomous system in the quintessence scalar field with dark sector coupling.