CHAPTER-2

Fixed Points of Integral Type of Contraction in Metric Spaces
Introduction

In the last two decades, multitudes of papers have defined various types of compatible maps including weakly compatible maps in metric space. In 1993, Jungck et al. [61] defined the concept of compatible maps of type (A) which is equivalent to the concept of compatible maps under some conditions. In 1995, Pathak et al. [88] introduced the notion of compatible maps of type (P) and compared with compatible maps (respectively, compatible maps of type (A)), and improved some results of Jungck [58] and Jungck et al. [61]. In the meantime, Pathak and Khan [90] introduced another new concept of compatible maps of type (B) as a generalization of compatible mappings of type (A) and proved some fixed point theorems of Greguš type. One year later, Jungck [59] weakened the notion of compatible maps to weakly compatible maps and asserted that there exists a pair of weakly compatible maps, but not compatible maps (respectively, compatible maps of type (A), (P)) (also refer to [64]). In 1998, Pathak et al. [89] extended compatible maps of type (A) to a new notion of compatible maps of type (C) and compared with the compatible maps (respectively, compatible maps of type (A), (B)) and proved common fixed point theorems of Greguš type. In 2002, Aamri and Moutawakil [1] introduced a property (E.A) generalizing the notion of noncompatible maps in metric space. Recently, M. R. Singh and Mahendra [110] introduced the concept of compatible maps of type (E) and compared with compatible maps (respectively, compatible maps of type (A), (P)), and showed that this very notion is equivalent to them under certain conditions.

On the other hand, in 2002, Branciari [20] obtained a fixed point theorem for a single map satisfying an analogue of contraction mapping principle of integral type inequality. One year later, Rhoades [100] proved two fixed point theorems involving more general contractive conditions of integral type. In 2005, Vijayaraju et al. [118] obtained fixed point theorems for a pair of maps satisfying a contractive inequality of integral type. As an extension of Branciari’s theorem, in 2007,
Boikanyo [18] proved some common fixed point theorems for a self-map satisfying a general contractive condition of integral type. In the meantime, Altun et al. [9] extended a result of Vijayaraju et al. [118] to two pairs of weakly compatible maps with the assumption on a sequence of iterates. In the same year, Zhang [121] introduced a generalized contractive condition in which the integral operator is replaced by a monotone non decreasing function and proved common fixed point theorems that extended results of [20], [100], [118].

One year later, Bari and Vetro [15] extended the result of Altun et al. [9] by introducing a generalized contractive condition which is versatile of deducing contractive condition of integral type.

This chapter consists of four sections. The first is the introductory in which relevant definitions and results needed in the following sections are furnished.

In the second, we extend some early results of Boikanyo [18] from one self-map to a pair of self-maps. The result of this section has been published in Kathmandu University Journal of Science, Engineering and Technology (KUSET), vol. 6, no. II, November, 2010, pp. 20-27.

The third section deals with a comparative study of various types of compatible maps, such as type (A), (B), (C), (E) and (P), including weakly compatible maps in metric space with examples. The direct implication, implication via continuity, and relationships among themselves are precisely and concisely sketched in diagrams and in tabular form. The result of this section has been presented in National Conference on History of Mathematics, organized by Indian Society for History of Mathematics and Manipur University, Imphal NCHM, 2008.

In the fourth, we improve the main result of Bari and Vetro [15] by employing property (E.A) and closeness of space in lieu of the assumption on a sequence of iterates and completeness of space respectively. The result of this
We retrieve some relevant definitions and results in the sequel.

**Definition 2.1.1** ([61]). Self-maps $S$ and $T$ of a metric space $(X, d)$ are called *compatible of type $(A)$* if
\[
\lim_{n \to \infty} d(STx_n, TTx_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(TSx_n, SSx_n) = 0
\]
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

By [28], it follows that the notions of compatible maps and compatible maps of type $(A)$ are independent.

**Definition 2.1.2** ([88]). Self-maps $S$ and $T$ of a metric space $(X, d)$ are called *compatible of type $(P)$* if
\[
\lim_{n \to \infty} d(SSx_n, TTx_n) = 0
\]
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

**Definition 2.1.3** ([90]). Self-maps $S$ and $T$ of a metric space $(X, d)$ are called *compatible of type $(B)$* if
\[
\lim_{n \to \infty} d(STx_n, TTx_n) \leq \frac{1}{2} \left\{ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n) \right\}
\]
\[
\lim_{n \to \infty} d(TSx_n, SSx_n) \leq \frac{1}{2} \left\{ \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, TTx_n) \right\}
\]
whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

From [90, Proposition 2.4], it is clear to see that compatible maps of type $(A)$ are compatible maps of type $(B)$. From [90, Example 2.4], it follows that implication is not reversible.

**Definition 2.1.4** ([89]). Self maps $S$ and $T$ of a metric space $(X, d)$ are called *compatible of type $(C)$* if
\[
\lim_{n \to \infty} d(STx_n, TTx_n) \leq \frac{1}{3} \left\{ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n) + \lim_{n \to \infty} d(St, TTx_n) \right\}
\]
and \( \lim_{n \to \infty} d(TSx_n, SSx_n) \leq \frac{1}{3} \{ \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, TTx_n) + \lim_{n \to \infty} d(Tt, SSx_n) \} \)

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

In [19], it is clear to see that compatible maps of type (A) are compatible maps of type (C), but the converse is not true.

**Definition 2.1.5 ([110]).** Self-maps \( S \) and \( T \) of a metric space \( (X, d) \) are called compatible maps of type \((E)\) if \( \lim_{n \to \infty} TTx_n = \lim_{n \to \infty} TSx_n = St \) and \( \lim_{n \to \infty} SSx_n = \lim_{n \to \infty} STx_n = Tt \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

**Remark 2.1.6.** It is noted that at the coincidence point, compatible maps of type \((E)\) implies intuitively compatible maps (respectively, compatible maps of type (A), (P), (B), (C)) however, the converse may not be true (refer to remark 2.16 of [110]).

**2.2. Extension to a pair of self-maps satisfying a general contractive condition of integral type in complete metric space**

After appearing contraction mapping principle, Kannan [68], Chatterjea [26], Hardy and Rogers [47], and Sen Gupta [108] established many theorems of maps satisfying various types of contractive inequalities. Besides, Rhoades [98] compared various definitions of contractive maps. Branciari [20] established a fixed point theorem of a self-map satisfying an analogue contraction mapping principle of integral type inequality in metric space. As an extension of Branciari’s theorem, Boikanyo [18] proved some fixed point theorems of a self-map satisfying a general contractive condition of integral type in metric space.

We extend the results of Boikanyo [18, Theorems 1, 2, 3, 4 and 5] to a pair of self-maps and proved the existence and uniqueness of a common fixed point
Theorem of a pair of self-maps satisfying a general contractive condition of integral type in metric space.

Now, we state our theorem as follows.

**Theorem 2.2.1.** Let \((X, d)\) be a complete metric space. Let \(a_i (i = 1, 2, 3, 4, 5)\) be non-negative real numbers such that \(\sum_{i=1}^{5} a_i < 1\), \(T_1\) and \(T_2\) be a pair of self-maps of the metric space \(X\) satisfying for each \(x, y \in X\),

\[
\int_{0}^{d(T_1x, T_2y)} \varphi(t) \, dt \leq a_1 \int_{0}^{d(x, y)} \varphi(t) \, dt + a_2 \int_{0}^{d(x, T_1x)} \varphi(t) \, dt + a_3 \int_{0}^{d(y, T_2y)} \varphi(t) \, dt + a_4 \int_{0}^{d(y, T_1y)} \varphi(t) \, dt + a_5 \int_{0}^{d(y, T_2y)} \varphi(t) \, dt \quad \ldots \quad (2.2.1)
\]

where \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) is a Lebesgue-integrable map which is summable, non-negative and such that \(\int_{0}^{\varepsilon} \varphi(t) \, dt > 0\) for each \(\varepsilon > 0\). Then \(T_1\) and \(T_2\) have a unique common fixed point \(z \in X\).

**Proof.** Let \(x_0\) be any point in \(X\). Let us consider a sequence \(\{x_n\}\) defined by \(x_{2n-1} = T_1x_{2n-2}\) and \(x_{2n} = T_2x_{2n-1}\), where \(n \in \mathbb{N}\).

We claim that \(\lim_{n \to \infty} d(x_n, x_{n+1}) = 0\). It is enough to show that

\[
\int_{0}^{d(x_n, x_{n+1})} \varphi(t) \, dt \leq r^2 \int_{0}^{d(x_n, x_1)} \varphi(t) \, dt \quad \text{where} \quad r = \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5} < 1.
\]

For this, interchanging \(x\) by \(y\) and \(T_1\) by \(T_2\) in (2.2.1), we obtain

\[
\int_{0}^{d(T_1y, T_2x)} \varphi(t) \, dt \leq a_1 \int_{0}^{d(y, x)} \varphi(t) \, dt + a_2 \int_{0}^{d(y, T_1y)} \varphi(t) \, dt + a_3 \int_{0}^{d(x, T_2x)} \varphi(t) \, dt + a_4 \int_{0}^{d(x, T_1x)} \varphi(t) \, dt + a_5 \int_{0}^{d(x, T_2x)} \varphi(t) \, dt \quad \ldots \quad (2.2.2)
\]

Now, from (2.2.1), (2.2.2) and using symmetric property, we obtain
\[
\int_0^{d(T_2, T_x)} \varphi(t) \, dt \leq a_1 \int_0^{d(x, T_2)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x, T_2)} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(y, T_2)} \varphi(t) \, dt
\]

\[
= a_1 \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n-1}, y_{n-1})} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(y_{n-1}, T_{x_n})} \varphi(t) \, dt
\]

Using (2.2.3) for odd \(n\), we obtain

\[
\int_0^{d(x_n, x_{n+1})} \varphi(t) \, dt = \int_0^{d(T_{x_{n-1}}, T_{x_n})} \varphi(t) \, dt
\]

\[
\leq a_1 \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n-1}, y_{n-1})} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(y_{n-1}, T_{x_n})} \varphi(t) \, dt
\]

\[
\int_0^{d(x_{n+1}, x_{n+2})} \varphi(t) \, dt = \int_0^{d(T_{x_{n+1}}, T_{x_{n+2}})} \varphi(t) \, dt
\]

\[
\leq a_1 \int_0^{d(x_{n+1}, x_{n+2})} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n+1}, x_{n+2})} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n+1}, y_{n+1})} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(y_{n+1}, T_{x_{n+2}})} \varphi(t) \, dt
\]

Again, using (2.2.3) for even \(n\), we obtain

\[
\int_0^{d(x_n, x_{n+1})} \varphi(t) \, dt = \int_0^{d(T_{x_{n-1}}, T_{x_n})} \varphi(t) \, dt
\]

\[
\leq a_1 \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n-1}, y_{n-1})} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(y_{n-1}, T_{x_n})} \varphi(t) \, dt
\]
\[+ \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_n, x_{n+1})} \varphi(t) \, dt \]
\[= a_1 \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n-2}, x_n)} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt \]
\[+ \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n-2}, x_n)} \varphi(t) \, dt \]

From the above two cases, one can see that
\[\int_0^{d(x_n, x_{n+1})} \varphi(t) \, dt \leq a_1 \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt + \left( \frac{a_2 + a_3}{2} \right) \int_0^{d(x_{n-2}, x_n)} \varphi(t) \, dt + \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt \]
\[+ \left( \frac{a_4 + a_5}{2} \right) \int_0^{d(x_{n-2}, x_n)} \varphi(t) \, dt \]

It follows that
\[\int_0^{d(x_n, x_{n+1})} \varphi(t) \, dt \leq \left( 2a_1 + a_2 + a_3 + a_4 + a_5 \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt \]
\[= r \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt \text{, where } r < 1 \]
\[\leq r^n \int_0^{d(x_1, x_n)} \varphi(t) \, dt \to 0 \text{ as } n \to \infty \text{ since } r < 1 \text{, owing to the assumption } \sum_{i=1}^{5} a_i < 1 \text{. Therefore, } \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \text{.} \]
Now, we show that \( \{x_n\} \) is a Cauchy sequence in \( X \). Let \( m > n \) where \( m, n \in \mathbb{N} \). Without any loss of generality, two cases arise:

(i) \( m \) is even when \( n \) is odd, and

(ii) \( m \) is odd when \( n \) is even.

Case I. We choose \( n \) and \( m \) to be odd and even respectively. Then we have

\[
\int_0^{d(x_n,x_m)} \varphi(t) \, dt = \int_0^{d(T_{x_{n-1}}T_{x_{n-1}})} \varphi(t) \, dt \\
\leq a_1 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_2 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt + a_3 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt \\
+ a_4 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt + a_5 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt \\
= a_1 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_2 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_3 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt \\
+ a_4 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_5 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt .
\]

Case II. We choose \( n \) and \( m \) to be even and odd respectively. Then we have

\[
\int_0^{d(x_n,x_m)} \varphi(t) \, dt = \int_0^{d(T_{x_{n-1}}T_{x_{n-1}})} \varphi(t) \, dt \\
\leq a_1 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_2 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt + a_3 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt \\
+ a_4 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt + a_5 \int_0^{d(x_{n-1},T_{x_{n-1}})} \varphi(t) \, dt \\
= a_1 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_2 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_3 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt \\
+ a_4 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_5 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt .
\]

From both the cases, we have

\[
\int_0^{d(x_n,x_m)} \varphi(t) \, dt \leq a_1 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_2 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_3 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt \\
+ a_4 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_5 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt \\
\leq a_1 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_2 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt + a_3 \int_0^{d(x_{n-1},x_{n-1})} \varphi(t) \, dt
\]
It follows that
\[
\int_0^{d(x_n, x_m)} \varphi(t) \, dt \leq \left( \frac{a_1 + a_2 + a_4}{1 - a_1 - a_4 - a_5} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt
\]
\[
+ \left( \frac{a_1 + a_2 + a_5}{1 - a_1 - a_4 - a_5} \right) \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt
\]
\[
\leq \left( \frac{a_1 + a_2 + a_4}{1 - a_1 - a_4 - a_5} \right) r^{n-1} \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt
\]
\[
+ \left( \frac{a_1 + a_2 + a_5}{1 - a_1 - a_4 - a_5} \right) r^{m-1} \int_0^{d(x_{n-1}, x_n)} \varphi(t) \, dt \to 0
\]
as \(n, m \to \infty\), since \(r < 1\). Hence, \(\{x_n\}\) is a Cauchy sequence in the complete metric space \(X\), so it is convergent in \(X\).

Let its limit be \(z\), i.e. \(\lim_{n \to \infty} x_n = z\). We show that \(T_1z = T_2z = z\). Now, we have
\[
\int_0^{d(x_n, Tz)} \varphi(t) \, dt = \int_0^{d(T_2x_n, Tz)} \varphi(t) \, dt
\]
\[
\leq a_1 \int_0^{d(z_n, z)} \varphi(t) \, dt + a_2 \int_0^{d(z_n, Tz_n, Tz_n)} \varphi(t) \, dt + a_3 \int_0^{d(z, Tz)} \varphi(t) \, dt
\]
\[
+ a_4 \int_0^{d(z_n, Tz_n, Tz_n)} \varphi(t) \, dt + a_5 \int_0^{d(z_n, Tz_n)} \varphi(t) \, dt
\]
\[
= a_1 \int_0^{d(z_n, z)} \varphi(t) \, dt + a_2 \int_0^{d(z_n, Tz_n, Tz_n)} \varphi(t) \, dt + a_3 \int_0^{d(z, Tz)} \varphi(t) \, dt
\]
\[
+ a_4 \int_0^{d(z_n, Tz_n, Tz_n)} \varphi(t) \, dt + a_5 \int_0^{d(z_n, Tz_n)} \varphi(t) \, dt
\]
Taking the limits as \(n \to \infty\), we get
\[
\int_0^{d(z, Tz)} \varphi(t) \, dt \leq a_1 \int_0^{d(z, Tz)} \varphi(t) \, dt + a_4 \int_0^{d(z, Tz)} \varphi(t) \, dt
\].
It follows that \( \int_0^{d(z, Tz)} \varphi(t) \, dt = 0 \) which yields that \( T_1 z = z \). Similarly, one can show that \( T_2 z = z \). Thus, \( T_1 \) and \( T_2 \) have a common fixed point.

For uniqueness, if possible, let \( w \) be another common fixed point of \( T_1 \) and \( T_2 \) such that \( w \neq z \). Now, we have
\[
\int_0^{d(z, w)} \varphi(t) \, dt = \int_0^{d(Tz, Tw)} \varphi(t) \, dt \\
\leq a_1 \int_0^{d(z, w)} \varphi(t) \, dt + a_2 \int_0^{d(z, Tz)} \varphi(t) \, dt + a_3 \int_0^{d(w, Tw)} \varphi(t) \, dt \\
+ a_4 \int_0^{d(z, Tw)} \varphi(t) \, dt + a_5 \int_0^{d(w, Tz)} \varphi(t) \, dt \\
= a_1 \int_0^{d(z, w)} \varphi(t) \, dt + a_4 \int_0^{d(z, w)} \varphi(t) \, dt + a_5 \int_0^{d(z, w)} \varphi(t) \, dt \\
= 0 , \text{ a contradiction.}
\]
So, \( z = w \). Hence, \( T_1 \) and \( T_2 \) have a unique common fixed point. This completes the proof.

**Corollary 2.2.2.** Let \((X, d)\) be a complete metric space. Let \( a, b, c \) be non-negative real numbers satisfying \( a + b + c < 1 \), \( T_1 \) and \( T_2 \) be a pair of self-maps of the metric space \( X \) such that for each \( x, y \in X \),
\[
\int_0^{d(Tx, Ty)} \varphi(t) \, dt \leq a_1 \int_0^{d(x, Tx)} \varphi(t) \, dt + a_2 \int_0^{d(y, Ty)} \varphi(t) \, dt + a_3 \int_0^{d(x, y)} \varphi(t) \, dt + a_4 \int_0^{d(y, x)} \varphi(t) \, dt + a_5 \int_0^{d(x, y)} \varphi(t) \, dt \ldots \quad (2.2.4)
\]
where \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) is a Lebesgue-integrable mapping which is summable, non-negative and such that \( \int_0^{\varepsilon} \varphi(t) \, dt > 0 \) for each \( \varepsilon > 0 \). Then \( T_1 \) and \( T_2 \) have a unique common fixed point \( z \in X \).

**Proof.** Since the contractive condition (2.2.4) is obviously a special case of (2.2.1), the result follows from Theorem 2.2.1 by setting \( a_1 = c \), \( a_2 = a \), \( a_3 = b \) and \( a_4 = a_5 = 0 \).
Corollary 2.2.3. Let $(X,d)$ be a complete metric space. Let $a, b, c$ be non-negative real numbers satisfying $a+b+c < 1$, $T_1$ and $T_2$ be a pair of self-maps of the metric space $X$ such that for each $x, y \in X$,
\[
\int_0^{d(T_1x,T_2y)} \phi(t) dt \leq a \int_0^{d(x,T_2y)} \phi(t) dt + b \int_0^{d(y,T_1x)} \phi(t) dt + c \int_0^{d(x,y)} \phi(t) dt \quad \ldots \quad (2.2.5)
\]
where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that $\int_0^\varepsilon \phi(t) dt > 0$ for each $\varepsilon > 0$. Then $T_1$ and $T_2$ have a unique common fixed point $z \in X$.

Proof. Since the contractive condition (2.2.5) is also a special case of (2.2.1), the result follows from Theorem 2.2.1 by putting $a_1 = c$, $a_4 = a$, $a_5 = b$ and $a_2 = a_3 = 0$.

Remark 2.2.4. Theorem 1 and Theorem 2 (cf. [18]) are special cases of Corollary 2.2.2 and Corollary 2.2.3 respectively if we set $T_1 = T_2$, $a = b$ and $c = 0$.

Remark 2.2.5. By taking $T_1 = T_2$, Corollary 2.2.2 and Corollary 2.2.3 reduce to Theorem 3 and Theorem 4 (cf. [18]) respectively.

Remark 2.2.6. Theorem 5 (cf. [18]) is a consequence of Theorem 2.2.1 if we take $T_1 = T_2$.

2.3. Comparability of various types of compatible maps

In this section, we attempt to delineate a comparative study of various types of compatible maps, such as type $(A)$, $(B)$, $(C)$, $(E)$ and $(P)$, including weakly compatible maps in metric space by furnishing examples. To discern direct implication, implication via continuity and relationships among themselves precisely, we exhibit the implications in diagrams and their inter-relationships in a tabular form.
In the following, we use “$a \Rightarrow b$” to mean that any two maps satisfying condition “$a$” also satisfy condition “$b$”.

**Theorem 2.3.1.** Considering the maps and space defined in sections 1.5 and 2.1, the following relationships among the definitions hold.

(i) $1.5.9 \Rightarrow 2.1.1$; $2.1.1 \Rightarrow 1.5.9$; (ii) $1.5.9 \Rightarrow 2.1.2$; $2.1.2 \Rightarrow 1.5.9$; (iii) $1.5.9 \Rightarrow 2.1.3$; $2.1.3 \Rightarrow 1.5.9$; (iv) $1.5.9 \Rightarrow 2.1.4$; $2.1.4 \Rightarrow 1.5.9$; (v) $1.5.9 \Rightarrow 2.1.5$; $2.1.5 \Rightarrow 1.5.9$; (vi) $2.1.1 \Rightarrow 2.1.2$; $2.1.2 \Rightarrow 2.1.1$; (vii) $2.1.1 \Rightarrow 2.1.3$; $2.1.3 \Rightarrow 2.1.1$; (viii) $2.1.1 \Rightarrow 2.1.4$; $2.1.4 \Rightarrow 2.1.1$; (ix) $2.1.1 \Rightarrow 2.1.5$; $2.1.5 \Rightarrow 2.1.1$; (x) $2.1.2 \Rightarrow 2.1.3$; $2.1.3 \Rightarrow 2.1.2$; (xi) $2.1.2 \Rightarrow 2.1.4$; $2.1.4 \Rightarrow 2.1.2$; (xii) $2.1.2 \Rightarrow 2.1.5$; $2.1.5 \Rightarrow 2.1.2$; (xiii) $2.1.3 \Rightarrow 2.1.4$; $2.1.4 \Rightarrow 2.1.3$; (xiv) $2.1.4 \Rightarrow 2.1.5$; $2.1.5 \Rightarrow 2.1.4$; (xv) $2.1.5 \Rightarrow 2.1.3$; $2.1.3 \Rightarrow 2.1.5$.

**Proof.** We prove most of the above statements by giving counter examples.

(i) From [28, Example 2.1] and [28, Example 2.2], it follows respectively.

(ii) To show that $1.5.9 \Rightarrow 2.1.2$, let $X = \mathbb{R}$, with the usual metric $d(x,y) = |x-y|$. Define $S, T : X \to X$ as follows:

$$Sx = \begin{cases} 1/x & , x \neq 0 \\ 1 & , x = 0 \end{cases} \text{ and } Tx = \begin{cases} 1/x^2 & , x \neq 0 \\ 2 & , x = 0 \end{cases}.$$

Then $S$ and $T$ are discontinuous at $x = 0$. Consider a sequence $\{x_n\}$ in $X$ defined by $x_n = n^2$, $n \in \mathbb{N}$. We see that $Sx_n = 1/x_n = 1/n^2 \to 0 = t$, $Tx_n = 1/x_n^2 = 1/n^4 \to 0 = 0 = t$, $STx_n = S(1/n^4) = n^4$, $TSx_n = T(1/n^2) = n^4$, $SSx_n = S(1/n^2) = n^2$ and $TTx_n = T(1/n^4) = n^8$.

Now, we obtain $\lim_{n \to \infty} d(STx_n, TSx_n) = 0$ which shows that $S$ and $T$ are compatible. Also, we have $\lim_{n \to \infty} d(SSx_n, TTx_n) = \lim_{n \to \infty} d(n^2, n^8) \to \infty$ and hence $S$ and $T$ are not compatible of type $(P)$. 
To show that $2.1.2 \Rightarrow 1.5.9$, let $X = [2, 20]$ with the usual metric $d(x, y) = |x - y|$. Define $S, T : X \rightarrow X$ by

$$
Sx = \begin{cases} 
2, & x = 2 \\
1, & 2 < x \leq 4 \\
x - 2, & 4 < x \leq 20 
\end{cases}
$$
and

$$
Tx = \begin{cases} 
2, & x = 2 \text{ or } 4 < x \leq 20 \\
8, & 2 < x \leq 4 
\end{cases}
$$

Then $S$ and $T$ are discontinuous at $x = 4$. Let $\{x_n\}$ be a decreasing sequence defined by $x_n = 4 + 1/n$, $n \in \mathbb{N}$. We observe that $Sx_n = x_n - 2 \rightarrow 2 = t$, $Tx_n \rightarrow 2 = t$, $STx_n = S(2) = 2$, $TSx_n = T(x_n - 2) = 8$, $SSx_n = S(x_n - 2) = x_n - 2 \rightarrow 2$ and $TTx_n = T(2) = 2$.

For compatible of type (P), we show that $\lim_{n \to \infty} d(SSx_n, TTx_n) = 0$. But, $S$ and $T$ are not compatible as $\lim_{n \to \infty} d(STx_n, TSx_n) = 6 \neq 0$.

(iii) Follow from [90, Example 2.3] and [90, Example 2.4] respectively.

(iv) To show that $1.5.9 \Rightarrow 2.1.4$, choose $S$ and $T$ in (ii) for which $1.5.9 \Rightarrow 2.1.2$. We observe that $S$ and $T$ are compatible. We have $St = S(0) = 1$ and $Tt = T(0) = 2$.

Now we see that

$$
\lim_{n \to \infty} d(TSx_n, SSx_n) \leq \{\lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, TTx_n) + \lim_{n \to \infty} d(Tt, SSx_n)\} / 3
$$

as

$$
\{\lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, TTx_n) + \lim_{n \to \infty} d(Tt, SSx_n)\} / 3 \to \infty \quad \text{and} \quad \lim_{n \to \infty} d(TSx_n, SSx_n) \to \infty.
$$

So, $S$ and $T$ are not compatible of type (C).

Example 2.1 of [19] shows that $2.1.4 \Rightarrow 1.5.9$.

(v) To show that $1.5.9 \Rightarrow 2.1.5$, choose $S$ and $T$ in (ii) for which $1.5.9 \Rightarrow 2.1.2$. We observe that $S$ and $T$ are compatible. Now, $St = S(0) = 1$ and $Tt = T(0) = 2$ and further, $\lim_{n \to \infty} TTx_n = \lim_{n \to \infty} n^8 \to \infty \neq 1 = St$. So, $S$ and $T$ are not compatible of type (E).

Example 3.6 of [110] shows that $2.1.5 \Rightarrow 1.5.9$.

(vi) To show that $2.1.1 \Rightarrow 2.1.2$, let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define $S, T : X \rightarrow X$ by
\[ S(x) = \begin{cases} x, & 0 \leq x < 1/2 \\ 1, & 1/2 \leq x \leq 1 \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 1-x, & 0 \leq x < 1/2 \\ 1, & 1/2 \leq x \leq 1 \end{cases}. \]

Then \( S \) and \( T \) are discontinuous at \( t = 1/2 \). Suppose that \( \{x_n\} \subseteq [0,1] \) and that \( T x_n, S x_n \to t \). By definition of \( S \) and \( T \), \( t \in \{1/2,1\} \). Since \( S \) and \( T \) agree on \( \{1/2,1\} \), we need to consider only \( t = 1/2 \). So, we suppose that \( x_n \to 1/2 \) and \( x_n < 1/2 \) for all \( n \in \mathbb{N} \). Then \( T x_n = 1 - x_n \to 1/2 \) from the right, and \( S x_n = x_n \to 1/2 \) from the left. Since \( 1 - x_n > 1/2 \) for all \( n \in \mathbb{N} \), we have \( S T x_n \to 1, \quad T S x_n \to 1/2, \quad S S x_n \to 1/2 \) and \( T T x_n \to 1 \).

For compatible of type \((A)\), we show that \( \lim \limits_{n \to \infty} d(ST x_n, T T x_n) = 0 \)
\( = \lim \limits_{n \to \infty} d(T S x_n, S S x_n) \). But, \( \lim \limits_{n \to \infty} d(S S x_n, T T x_n) = 1/2 \) which tells that \( S \) and \( T \) are not compatible of type \((P)\).

To show that \( 2.1.2 \Rightarrow 2.1.1 \), choose \( S \) and \( T \) in (ii) for which \( 2.1.2 \Rightarrow 1.5.9 \). We get \( \lim \limits_{n \to \infty} d(S S x_n, T T x_n) = 0 \) which shows that \( S \) and \( T \) are compatible of type \((P)\).

But, \( \lim \limits_{n \to \infty} d(T S x_n, S S x_n) = 6 \) which tells that \( S \) and \( T \) are not compatible of type \((A)\).

(vii) Implication is obvious from the definitions (refer to [90, Proposition 2.4]).

From [90, Example 2.4], it follows that \( 2.1.3 \Rightarrow 2.1.1 \).

(viii) Implication is obvious from the definitions (refer to [19]).

From [19, Example 2.1], it follows that \( 2.1.4 \Rightarrow 2.1.1 \).

(ix) To show that \( 2.1.1 \Rightarrow 2.1.5 \), choose \( S \) and \( T \) in (vi) for which \( 2.1.1 \Rightarrow 2.1.2 \).

We have \( S t = S(1/2) = 1 \) and \( T t = T(1/2) = 1 \). Now, we see that \( \lim \limits_{n \to \infty} d(ST x_n, T T x_n) = 0 = \lim \limits_{n \to \infty} d(T S x_n, S S x_n) \) which shows that \( S \) and \( T \) are compatible of type \((A)\). Further, we see that \( \lim \limits_{n \to \infty} T S x_n = 1/2 \neq 1 = S t \) which shows that \( S \) and \( T \) are not compatible of type \((E)\).

Example 3.6 of [110] shows that \( 2.1.5 \Rightarrow 2.1.1 \).
(x) To show that $2.1.2 \Rightarrow 2.1.3$, choose $S$ and $T$ in (ii) for which $2.1.2 \Rightarrow 1.5.9$. We obtain $St = S(2) = 2$ and $Tt = T(2) = 2$. For compatible of type $(P)$, we show that 
\[ \lim\limits_{n \to \infty} d(SSx_n, TTx_n) = 0. \]
Further, we have
\[ 6 = \lim\limits_{n \to \infty} d(TSx_n, SSx_n) \leq \{ \lim\limits_{n \to \infty} d(TSx_n, Tt) + \lim\limits_{n \to \infty} d(Tt, TTx_n) \} / 2 = 3 \] which shows that $S$ and $T$ are not compatible of type $(B)$.

To show that $2.1.3 \Rightarrow 2.1.2$, let $X = [0, 2]$ with usual metric $d(x, y) = |x - y|$. Define $S, T : X \to X$ by
\[
Sx = \begin{cases} 
1/2 + x, & 0 \leq x < 1/2 \\
2, & x = 1/2 \\
1, & 1/2 < x \leq 1
\end{cases}
\]
and
\[
Tx = \begin{cases} 
1/2 - x, & 0 \leq x < 1/2 \\
1, & x = 1/2 \\
0, & 1/2 < x \leq 2
\end{cases}
\]
Then $S$ and $T$ are discontinuous at $t = 1/2$. Suppose that \( \{ x_n \} \subseteq [0, 2] \) and $Sx_n, Tx_n \to t = 1/2$. By definition of $S$ and $T$, $t = 1/2$. So, we suppose that $x_n \to 0$.

Then $Sx_n = 1/2 + x_n \to 1/2$ from the right and $Tx_n = 1/2 - x_n \to 1/2$ from the left.

We have $STx_n = S(1/2 - x_n) = 1 - x_n$, $TSx_n = T(1/2 + x_n) = 0$, $SSx_n = S(1/2 + x_n) = 1$ and $TTx_n = T(1/2 - x_n) = x_n$. Further, we obtain $St = S(1/2) = 2$ and $Tt = T(1/2) = 1$.

For compatibility of type $(B)$, we show that
\[
1 = \lim\limits_{n \to \infty} d(STx_n, TTx_n) \leq \{ \lim\limits_{n \to \infty} d(STx_n, St) + d(St, SSx_n) \} / 2 = 1
\] and
\[
1 = \lim\limits_{n \to \infty} d(TSx_n, SSx_n) \leq \{ \lim\limits_{n \to \infty} d(TSx_n, Tt) + d(Tt, TTx_n) \} / 2 = 1.
\]
But, \( \lim\limits_{n \to \infty} d(SSx_n, TTx_n) = 1 \neq 0 \) which shows that $S$ and $T$ are not compatible of type $(P)$.

(xi) To show that $2.1.2 \Rightarrow 2.1.4$, choose $S$ and $T$ in (ii) for which $2.1.2 \Rightarrow 1.5.9$. We observe that $S$ and $T$ are compatible mappings of type $(P)$. Further, we see that
\[
6 = \lim\limits_{n \to \infty} d(TSx_n, SSx_n) \leq \{ \lim\limits_{n \to \infty} d(TSx_n, Tt) + \lim\limits_{n \to \infty} d(Tt, TTx_n) + \lim\limits_{n \to \infty} d(Tt, SSx_n) \} / 3 = 2
\] which shows that $S$ and $T$ are not compatible of type $(C)$. 

To show that \(2.1.4 \Rightarrow 2.1.2\), let \(X = [1, 20]\) with the usual metric \(d(x, y) = |x - y|\). Define \(S, T : X \to X\) by

\[
S_x = \begin{cases} 
1, & x = 1 \\
3, & 1 < x \leq 7 \\
x - 6, & 7 < x \leq 20
\end{cases} \quad \text{and} \quad T_x = \begin{cases} 
1, & x = 1 \text{ or } 7 < x \leq 20 \\
2, & 1 < x \leq 7
\end{cases}.
\]

Let \(\{x_n\}\) be a sequence defined by \(x_n = 7 + 1/n\) for all \(n \in \mathbb{N}\). Then \(S\) and \(T\) are discontinuous at \(x = 7\). Now we have \(Sx_n = x_n - 6 \to 1 = t\), \(Tx_n \to 1 = t\), \(STx_n = S(1) = 1\), \(TSx_n = T(x_n - 6) = 2\), \(SSx_n = S(x_n - 6) = 3\), \(TTx_n = T(1) = 1\), \(St = S(1) = 1\) and \(Tt = T(1) = 1\).

For compatible of type \((C)\), we show that

\[
1 = \lim_{n \to \infty} d(TSx_n, SSx_n) \leq \left\{ \lim_{n \to \infty} d(TSx_n, St) + \lim_{n \to \infty} d(Tt, TTx_n) + \lim_{n \to \infty} d(Tt, SSx_n) \right\} / 3 = 1
\]

and

\[
0 = \lim_{n \to \infty} d(STx_n, TTx_n) \leq \left\{ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n) + \lim_{n \to \infty} d(St, TTx_n) \right\} / 3 = 2/3.
\]

Also, we have \(\lim_{n \to \infty} d(SSx_n, TTx_n) = 2 \neq 0\) which shows that \(S\) and \(T\) are not compatible of type \((P)\).

(xii) To show that \(2.1.2 \Rightarrow 2.1.5\), choose \(S\) and \(T\) in (ii) for which \(2.1.2 \Rightarrow 1.5.9\). We observe that \(S\) and \(T\) are compatible of type \((P)\). We see that \(\lim_{n \to \infty} TSx_n = 8 \neq 2 = St\) which shows that \(S\) and \(T\) are not compatible of type \((E)\).

Example 3.6 of [110] shows that \(2.1.5 \Rightarrow 2.1.2\).

(xiii) To show that \(2.1.3 \Rightarrow 2.1.4\), choose \(S\) and \(T\) in (x) for which \(2.1.3 \Rightarrow 2.1.2\). One can show that

\[
\lim_{n \to \infty} d(STx_n, TTx_n) \leq \left\{ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n) \right\} / 2 \quad \text{and}
\]

\[
\lim_{n \to \infty} d(TSx_n, SSx_n) \leq \left\{ \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, TTx_n) \right\} / 2 \quad \text{so that} \quad S \text{ and } T \text{ are compatible of type \((B)\).}
\]

Further, one can check that
\lim d(TS_{x_n}, SS_{x_n}) \leq \{ \lim d(TS_{x_n}, Tt) + \lim d(Tt, TT_{x_n}) + \lim d(Tt, SS_{x_n}) \} / 3 \quad \text{so that } S \text{ and } T \text{ are not compatible of type (C)}.

From [19, Example 2.1], it follows that 2.1.4 \Rightarrow 2.1.3.

(xiv) To show that 2.1.4 \Rightarrow 2.1.5, choose \( S \) and \( T \) in (xi) for which 2.1.4 \Rightarrow 2.1.2. We observe that \( S \) and \( T \) are compatible of type \((C)\). One can show that \( \lim TS_{x_n} = 2 \neq 1 = S t \) so that \( S \) and \( T \) are not compatible of type \((E)\).

To show that 2.1.5 \Rightarrow 2.1.4, let \( X = [0,1] \) with usual metric \( d(x,y) = |x-y| \).

Define \( S, T : X \to X \) by

\[
Sx = \begin{cases} 
1, & 0 \leq x < 1/4 \text{ or } 1/4 < x \leq 1/2 \\
0, & x = 1/4 \\
(1-x)/2, & 1/2 < x \leq 1 
\end{cases}
\]

and

\[
Tx = \begin{cases} 
0, & 0 \leq x < 1/4 \text{ or } 1/4 < x \leq 1/2 \\
1, & x = 1/4 \\
x/2, & 1/2 < x \leq 1 
\end{cases}
\]

Then \( S \) and \( T \) are discontinuous at \( x = 1/4, 1/2 \). Suppose that \( x_n \to 1/2 \) and \( x_n > 1/2 \) for all \( n \in \mathbb{N} \). We get \( Sx_n \to 1/4 = t, Tx_n \to 1/4 = t, STx_n \to 1, TSx_n \to 0 \), \( SSx_n \to 1 \) and \( TTx_n \to 0 \). Further, we obtain \( St = S(1/4) = 0 \) and \( Tt = T(1/4) = 1 \).

For compatible of type \((E)\), we show that \( \lim TTx_n = \lim TSx_n = 0 = St \) and \( \lim SSx_n = \lim STx_n = 1 = Tt \).

One can see that

\[
\lim d \left( STx_n, TTx_n \right) \leq \{ \lim d \left( STx_n, St \right) + \lim d \left( St, SSx_n \right) + \lim d \left( St, TTx_n \right) \} / 3
\]

so that \( S \) and \( T \) are not compatible of type \((C)\).

(xv) Implication is obvious from the definition.

For, \( \lim d \left( STx_n, TTx_n \right) = \{ \lim d \left( STx_n, Tt \right) + \lim d \left( STx_n, TTx_n \right) \} / 2 \)

\[
\leq \{ \lim d \left( STx_n, St \right) + \lim d \left( St, TTx_n \right) + \lim d \left( STx_n, SSx_n \right) \}
\]
\[+ \lim_{n \to \infty} d(SSx_n, St) + \lim_{n \to \infty} d(St, TTx_n) / 2.\]

Therefore, \(\lim_{n \to \infty} d(STx_n, TTx_n) = \left\{ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n) \right\} / 2\) and similarly, we deduce the other condition.

To show that 2.1.3 \(\Rightarrow\) 2.1.5, choose \(S\) and \(T\) in (x) for which 2.1.3 \(\Rightarrow\) 2.1.2. We observe that \(S\) and \(T\) are compatible of type (B). We see that \(\lim_{n \to \infty} TTx_n = \lim_{n \to \infty} x_n = 0 \neq 2 = St\) which shows that \(S\) and \(T\) are not compatible of type (E).

**Remark** 2.3.2. In view of the above statements, we conclude the following: Compatible maps, compatible maps of type (B), (C), (P) do not imply one another; Compatible maps, compatible maps of type (A), (P) do not imply one another; Compatible maps, compatible maps of type (P), (C), (E) do not imply one another.

**Remark** 2.3.3. If \(S\) and \(T\) are compatible maps of type (A) and one of \(S\) and \(T\) is continuous, then \(S\) and \(T\) are compatible (respectively, compatible of type (P), (B) (C)). Further, if \(S\) and \(T\) are compatible maps of type (E) and one of \(S\) and \(T\) is continuous, then \(S\) and \(T\) are compatible (respectively, compatible of type (A), (P), (B), (C)) (refer to [19], [61], [88], [90], [110]).

**Remark** 2.3.4. If both maps are continuous, the compatible and its types (defined in 1.5.9, 2.1.1 to 2.1.5) are equivalent.

**Theorem** 2.3.5. Considering the maps and space defined in section 1.5 and 2.1, the following relationships among the definitions hold.

(i) 1.5.12 \(\Rightarrow\) 1.5.9; 1.5.9 \(\Rightarrow\) 1.5.12; (ii) 1.5.12 \(\Rightarrow\) 2.1.1; 2.1.1 \(\Rightarrow\) 1.5.12;

(iii) 1.5.12 \(\Rightarrow\) 2.1.2; 2.1.2 \(\Rightarrow\) 1.5.12; (iv) 1.5.12 \(\Rightarrow\) 2.1.3; 2.1.3 \(\Rightarrow\) 1.5.12;

(v) 1.5.12 \(\Rightarrow\) 2.1.4; 2.1.4 \(\Rightarrow\) 1.5.12; (vi) 1.5.12 \(\Rightarrow\) 2.1.5; 2.1.5 \(\Rightarrow\) 1.5.12.

**Proof.** We prove the above statements by giving examples.

(i) The example of [94, page no. 34] shows that 1.5.12 \(\Rightarrow\) 1.5.9.

From [110, Example 2.11], it follows that 1.5.9 \(\Rightarrow\) 1.5.12.

(ii) The example of [94, page no. 34] shows that 1.5.12 \(\Rightarrow\) 2.1.1.
From [110, Example 2.13], it follows that $2.1.1 \nRightarrow 1.5.12$.

(iii) The example of [94, page no. 34] shows that $1.5.12 \nRightarrow 2.1.2$.

From [110, Example 2.15], it follows that $2.1.2 \nRightarrow 1.5.12$.

(iv) The example of [94, page no. 34] shows that $1.5.12 \nRightarrow 2.1.3$.

To show that $2.1.3 \nRightarrow 1.5.12$, let $X = [0,1]$ with usual metric $d(x,y) = |x-y|$. Define $S, T : X \rightarrow X$ by

$$\begin{align*}
Sx &= \begin{cases} 
1, & x=0 \\
0, & 0 < x \leq 1/2 \text{ and } 3/4, & x=0 \\
(1-x)/2, & 1/2 < x \leq 1
\end{cases} \\
Tx &= \begin{cases} 
0, & 0 < x \leq 1/2. \\
3/4, & x=0 \\
x/2, & 1/2 < x \leq 1
\end{cases}
\end{align*}$$

Then $S$ and $T$ are discontinuous at $x=0,1/2$. Suppose that $\{x_n\}$ is a sequence defined by $x_n = 1/2 + 1/n$ for all $n \in \mathbb{N}$ $(n > 2)$. Now we see that $Sx_n = (1-x_n)/2 \rightarrow 1/4 = t$, $Tx_n = x_n/2 \rightarrow 1/4 = t$, $STx_n \rightarrow 0$, $TSx_n \rightarrow 0$, $SSx_n \rightarrow 0$ and $TTx_n \rightarrow 0$. Further, we obtain $St = S(1/4) = 0$ and $Tt = T(1/4) = 0$.

For compatible of type $(B)$, we show that

$$\begin{align*}
0 &= \lim_{n \rightarrow \infty} d(STx_n, TTx_n) \leq \lim_{n \rightarrow \infty} d(STx_n, St) + \lim_{n \rightarrow \infty} d(Sx_n, SSx_n) / 2 = 0 \text{ and} \\
0 &= \lim_{n \rightarrow \infty} d(TSx_n, Sx_n) \leq \lim_{n \rightarrow \infty} d(TSx_n, Tt) + \lim_{n \rightarrow \infty} d(Tx_n, TTx_n) / 2 = 0.
\end{align*}$$

At coincidence point $x = 1/4$, $ST(1/4) = S(0) = 1$ and $TS(1/4) = T(0) = 3/4$. So, $ST(1/4) \neq TS(1/4)$ when $S(1/4) = T(1/4)$. Therefore, $S$ and $T$ are not weakly compatible maps.

(v) To show that $1.5.12 \nRightarrow 2.1.4$, let $X = [2,20]$ with the usual metric $d(x,y) = |x-y|$. Define $S, T : X \rightarrow X$ by

$$\begin{align*}
Sx &= \begin{cases} 
2, & x = 2 \text{ or } 5 < x \leq 20 \\
6, & 2 < x \leq 5
\end{cases} \quad \text{and} \quad Tx &= \begin{cases} 
2, & x = 2 \\
12 + x, & 2 < x \leq 5 \\
x - 3, & 5 < x \leq 20
\end{cases}
\end{align*}$$

Then $S$ and $T$ are discontinuous at $x = 2,5$. The only coincidence point of the maps $S$ and $T$ is $x = 2$. We see that $S(2) = T(2)$ implies $ST(2) = TS(2)$ which shows that $S$ and $T$ are weakly compatible.
Let \( \{x_n\} \) be a decreasing sequence defined by \( x_n = 5 + 1/n \) for all \( n \in \mathbb{N} \).

Then \( Sx_n \to 2=t \), \( Tx_n \to 2=t \), \( STx_n \to 6 \), \( TStx_n \to 2 \), \( SSx_n \to 2 \) and \( TTx_n \to 14 \).

Moreover, \( S(2) = 2 \) and \( T(2) = 2 \).

Now, we see that \( \lim_{n \to \infty} d\left(STx_n, TTx_n\right) = 8 \) and

\[
\frac{1}{3}\left\{ \lim_{n \to \infty} d\left(STx_n, St\right) + \lim_{n \to \infty} d\left(St, SSx_n\right) + \lim_{n \to \infty} d\left(St, TTx_n\right) \right\} = \frac{16}{3}.
\]

One can conclude that \( S \) and \( T \) are not compatible of type (C).

To show that \( 2.1.4 \not\Rightarrow 1.5.12 \), choose \( S \) and \( T \) in (iv) for which \( 2.1.3 \not\Rightarrow 1.5.12 \). We obtain

\[
\lim_{n \to \infty} d\left(TSx_n, SSx_n\right) = 0,
\]

\[
\frac{1}{3}\left\{ \lim_{n \to \infty} d\left(TSx_n, Tt\right) + \lim_{n \to \infty} d\left(Tt, TTx_n\right) + \lim_{n \to \infty} d\left(Tt, SSx_n\right) \right\} = 0,
\]

\[
\lim_{n \to \infty} d\left(STx_n, TTx_n\right) = 0 \quad \text{and}
\]

\[
\frac{1}{3}\left\{ \lim_{n \to \infty} d\left(STx_n, St\right) + \lim_{n \to \infty} d\left(St, SSx_n\right) + \lim_{n \to \infty} d\left(St, TTx_n\right) \right\} = 0.
\]

One can conclude that \( S \) and \( T \) are compatible of type (C). But, \( S \) and \( T \) are not weakly compatible maps.

(vi) To show that \( 1.5.12 \not\Rightarrow 2.1.5 \), choose \( S \) and \( T \) in (v) for which \( 1.5.12 \not\Rightarrow 2.1.4 \). So, \( S \) and \( T \) are weakly compatible maps. Besides, we have \( \lim_{n \to \infty} TTx_n = 14 \neq 2 = St \) which tells that \( S \) and \( T \) are not compatible of type (E).

To show that \( 2.1.5 \not\Rightarrow 1.5.12 \), choose \( S \) and \( T \) in (iv) for which \( 2.1.3 \not\Rightarrow 1.5.12 \). We obtain \( \lim_{n \to \infty} TTx_n = \lim_{n \to \infty} TSx_n = 0 = St \) and \( \lim_{n \to \infty} SSx_n = \lim_{n \to \infty} STx_n = 0 = Tt \).

Therefore, \( S \) and \( T \) are compatible of type (E). But, \( S \) and \( T \) are not weakly compatible maps.

**Remark** 2.3.6. In view of above statements in Theorem 2.3.5, it is overt that compatible (respectively, compatible of type (A), (P), (E), (B), (C)) maps and weakly compatible maps do not imply each other on the contrary to Popa’s remark [94, Remark 1] and [93, Page no. 467] (also refer to [110, Remark 2.16]).
2.3.7. Diagrammatic representation of implication among various type of compatible maps

(A) Direct implication

(B) Implication via continuity of one map (or directly)

(C) Implication via continuity of both maps (or directly)
### Inter-relationship table of various types of compatible maps

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Type</th>
<th>Year</th>
<th>Relationship with other</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td>Directly</td>
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<td></td>
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<td>via continuity of $S$ and $T$</td>
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<td>when both mappings are continuous</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$S_t = T_t$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Compatible maps [57]</td>
<td>1986</td>
<td>Compatible of type $(A), (P), (B), (C), (E)$</td>
<td>At coincidence point, pair is not commuting</td>
</tr>
<tr>
<td>2</td>
<td>Compatible maps of type $(A)$ [61]</td>
<td>1993</td>
<td>Compatible, Compatible of type $(P), (B), (C)$</td>
<td>$S_{T_t} \neq T_{S_t}$</td>
</tr>
<tr>
<td>3</td>
<td>Compatible maps of type $(P)$ [88]</td>
<td>1995</td>
<td>Compatible, Compatible of type $(A), (B), (C), (E)$</td>
<td>$S_{T_t} \neq T_{S_t}$</td>
</tr>
<tr>
<td>4</td>
<td>Compatible maps of type $(B)$ [90]</td>
<td>1995</td>
<td>Compatible, Compatible of type $(A), (P), (C), (E)$</td>
<td>At coincidence point, pair is not commuting</td>
</tr>
<tr>
<td>5</td>
<td>Compatible maps of type $(C)$ [89]</td>
<td>1998</td>
<td>Compatible, Compatible of type $(A), (P), (B), (E)$</td>
<td>$S_{T_t} \neq T_{S_t}$</td>
</tr>
<tr>
<td>6</td>
<td>Compatible maps of type $(E)$ [110]</td>
<td>2007</td>
<td>Compatible, Compatible of type $(A), (P), (B), (C)$</td>
<td>At coincidence point, compatible of type $(E)$ reduces to compatible, compatible of type $(A), (B), (C), (P)$</td>
</tr>
<tr>
<td>7</td>
<td>Weakly compatible maps [59]</td>
<td>1996</td>
<td>Compatible (resp. of type $(A), (P), (E), (B), (C)$) and weakly compatible do not imply each other.</td>
<td></td>
</tr>
</tbody>
</table>
2.4. Fixed points of weakly compatible maps in metric spaces

Altun et al. [9] generalized the main theorem of Branciari [20], Rhoades [100, Theorem 2], and Vijayaraju et al. [118, Theorem 2] by employing a generalized contractive condition of integral type for two pairs of maps in metric space. Meanwhile, Zhang [121] unified and extended many results of Branciari [20], Rhoades [100] and Vijayaraju et al. [118] by introducing a more generalized contractive condition for a pair of maps in metric space. Bari and Vetro [15] extended the results of Zhang [121, Theorem 1] and Altun et al. [9, Theorem 2.1] by employing a new generalized contractive condition in metric space.

We improve the result of Bari and Vetro [15] by using property (E.A) and closeness of space in lieu of the assumption on a sequence of iterates and completeness of space respectively. Subsequently, we show that many fixed point theorems under contractive conditions of integral type can be deduced as particular cases.

Following Bari and Vetro [15], let $F$ be a family of all functions $F : [0, \infty) \to [0, \infty)$ such that $F$ is non decreasing, continuous and $F(0) = 0 < F(t)$ for every $t > 0$.

By $\Psi$, we denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi$ is non decreasing, right continuous and $\psi(t) < t$ for every $t > 0$.

If $A$, $B$, $S$ and $T$ are self-maps of a metric space $(X,d)$, in the sequel, we set

$$M(x,y) = \max \{d(Sx,Ty),d(Ax,Sx),d(By,Ty),[d(Sx,By)+d(Ax,Ty)]/2\} \ldots (2.4.1)$$

Providing some considerable improvements to the result of Bari and Vetro [15], we state our theorem as follows.

**Theorem 2.4.1.** Let $A$, $B$, $S$ and $T$ be self-maps of a metric space $(X,d)$ such that

(i) $AX \subset TX$ and $BX \subset SX$;

(ii) one of the pairs $\{A,S\}$ and $\{B,T\}$ satisfies the property (E.A);
(iii) the pairs \{A, S\} and \{B, T\} are weakly compatible;

(iv) for any \(x, y \in X\),
\[
F(d(Ax, By)) \leq \psi(F(M(x, y)))
\]
where \(\psi \in \Psi\) and \(F \in F\).

If one range of the maps \(A, B, S\) and \(T\) is a closed subspace of \(X\), then \(A, B, S\) and \(T\) have a unique common fixed point.

**Proof.** Suppose that the pair \(\{B, T\}\) satisfies the property (E.A). Then there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = t\) for some \(t \in X\).

Since \(BX \subseteq SX\), there exists a sequence \(\{y_n\}\) in \(X\) such that \(Bx_n = Sy_n\) for all \(n \in \mathbb{N}\), and hence \(\lim Sy_n = t\). Now, we claim that \(\lim A y_n = t\). Since \(d(Ay_n, t) \leq d(Ay_n, Bx_n) + d(Bx_n, t)\), it is enough to show that \(\lim_{n \to \infty} d(Ay_n, Bx_n) = 0\).

If not, then there exists a real number \(\varepsilon > 0\) such that \(\lim_{n \to \infty} d(Ay_n, Bx_n) = \varepsilon\). This assures that there exists a subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) in \(X\) such that for each positive integer \(k \geq n\), \(\lim_{k \to \infty} d(Ay_{n_k}, Bx_{n_k}) = \varepsilon\). Using (2.4.1), we obtain
\[
M(y_{n_k}, x_{n_k}) = \max \{d(Sy_{n_k}, Tx_{n_k}), d(Ay_{n_k}, Sy_{n_k}), d(Bx_{n_k}, Tx_{n_k})\},
\]
and subsequently, from (iv) it follows that \(F(d(Ay_{n_k}, Bx_{n_k}) \leq \psi(F(M(y_{n_k}, x_{n_k})))\).

Letting \(k \to \infty\) and as \(F\) being continuous and \(\psi\) right continuous, we obtain \(F(\varepsilon) \leq \psi(F(\varepsilon))\)
\[
< F(\varepsilon), \text{ a contradiction.}
\]
Therefore, \(\lim_{k \to \infty} d(Ay_{n_k}, Bx_{n_k}) = 0\) which yields that \(\lim_{n \to \infty} d(Ay_n, Bx_n) = 0\). So, \(\lim_{n \to \infty} d(Ay_n, t) = 0\).
Suppose that SX is a closed subspace of X. Then there exists a point \( u \in X \) such that \( t = Su \). So, we obtain \( \lim_{n \to \infty} Ay = \lim_{n \to \infty} Bx = \lim_{n \to \infty} Sy = \lim_{n \to \infty} Tx = t = Su \).

We claim that \( Au = Su \). For this, suppose that \( Au \neq Su \). Using (2.4.1), we have

\[
M(u, x_n) = \max \{d(Su, Tx_n), d(Au, Su), d(Bx_n, Tx_n), [d(Su, Bx_n) + d(Au, Tx_n)]/2 \}
\]

and therefore, from (iv) it follows that \( F(d(Au, Bx_n)) \leq \psi(F(M(u, x_n))) \).

As \( n \to \infty \), by definitions of \( F \) and \( \psi \), we have

\[
F(d(Au, Su)) \leq \psi(F(d(Au, Su))) < F(d(Au, Su)), \text{ a contradiction.}
\]

Therefore, \( Au = Su \). Now, we need to show that \( Au = Bv \). Since \( AX \subset TX \), there exists a point \( v \in X \) such that \( Au = Tv \). We claim that \( Bv = Tv \). Suppose that \( Bv \neq Tv \). Using (2.4.1), we have

\[
M(u, v) = \max \{d(Su, Tv), d(Au, Su), d(Bv, Tv), [d(Su, Bv) + d(Au, Tv)]/2 \}
\]

and hence from (iv) it follows that

\[
F(d(Tv, Bv)) = F(d(Au, Bv)) \leq \psi(F(M(u, v))) = \psi(F(d(Tv, Bv))) < F(d(Tv, Bv)), \text{ a contradiction.}
\]

So, \( Bv = Tv \) which yields that \( Au = Su = t = Bv = Tv \). The same result holds if we suppose that one of \( AX \), \( BX \) and \( TX \) is a closed subspace of \( X \). The weak compatibility of \( A \) and \( S \) implies that \( ASu = SAu \), i.e. \( At = St \).

We claim that \( d(At, t) = 0 \). If \( d(At, t) \neq 0 \), then the condition (2.4.1) gives

\[
M(t, v) = \max \{d(St, Tv), d(At, St), d(Bv, Tv), [d(St, Bv) + d(At, Tv)]/2 \}
\]

\[
= d(At, t).
\]

From (iv), it follows that
\[ F(d(At,t)) = F(d(At,Bv)) \]
\[ \leq \psi(F(M(t,v))) \]
\[ = \psi(F(d(At,t))) \]
\[ < F(d(At,t)), \text{a contradiction.} \]

Therefore, \( d(At,t) = 0 \) which implies that \( At = St = t \). Similarly, one can prove that \( Bt = Tt = t \). Thus, \( t \) is a common fixed point of \( A, B, S \) and \( T \).

If \( z \in X \) is also a common fixed point of \( A, B, S \) and \( T \) with \( t \neq z \), then using (2.4.1), we have

\[
M(t,z) = \max \{ d(St,Tz), d(At,St), d(Bz,Tz), d(St,Bz) + d(At,Tz) \}/2
\]

\[ = d(t,z). \]

Therefore, from (iv) it follows that

\[
F(d(t,z)) = F(d(At,Bz))
\]
\[ \leq \psi(F(M(t,z))) \]
\[ = \psi(F(d(t,z))) \]
\[ < F(d(t,z)) \]

which is a contradiction and hence \( t = z \). This completes the proof.

If we assume that one range of the maps is complete instead of being closed in Theorem 2.4.1, we have the following corollary.

**Corollary 2.4.2.** Let \( A, B, S \) and \( T \) be self-maps satisfying (i)-(iv) as in Theorem 2.4.1.

If one range of the maps \( A, B, S \) and \( T \) is a complete subspace of \( X \), then \( A, B, S \) and \( T \) have a unique common fixed point.

The proof of the Corollary 2.4.2 parallels to that of Theorem 2.4.1.

We need to demonstrate how the generalized contractive condition is being versatile of deducing contractive condition of integral type.
By $A$, we denote the set of all non-negative, Lebesgue-integrable function $\lambda : [0, \infty) \to [0, \infty)$ such that

$$
\int_0^\varepsilon \lambda(t) \, dt > 0, \text{ for every } \varepsilon > 0.
$$

Let $\lambda \in A$. If in Theorem 2.4.1, $F$ is defined by $F(s) = \int_0^s \lambda(t) \, dt$ for any $s \geq 0$, then the following theorem is deduced.

**Theorem 2.4.3.** Let $\psi \in \Psi$, $\lambda \in \Lambda$ and let $A, B, S$ and $T$ be self-maps of a metric space $(X, d)$ such that

(i) $AX \subset TX$ and $BX \subset SX$;

(ii) one of the pairs $\{A, S\}$ and $\{B, T\}$ satisfies the property (E.A);

(iii) the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible;

(iv) $\int_0^{d(Ax, By)} \lambda(t) \, dt \leq \psi \left( \int_0^{M(x, y)} \lambda(t) \, dt \right)$ for any $x, y \in X$.

If one range of the maps $A, B, S$ and $T$ is a closed subspace of $X$, then $A, B, S$ and $T$ have a unique common fixed point.

**Proof.** One can follow on the lines of the proof of Theorem 2.4.1.

**Corollary 2.4.4.** Let $\psi \in \Psi$ and $\lambda \in \Lambda$ and let $A, B, S$ and $T$ be self-maps satisfying (i)-(iv) as in Theorem 2.4.3.

If one range of the maps $A, B, S$ and $T$ is a complete subspace of $X$, then $A, B, S$ and $T$ have a unique common fixed point.

**Remark 2.4.5.** An analogue of Corollary 2.4.4 appears in Altun et al. [9, Theorem 2.1] by assuming a sequence of $\{S, T\}$-iteration of $x_0$ under $A$ and $B$, in lieu of the assumption of the property (E.A).

**Remark 2.4.6.** By setting $S = T = I_X$ in Corollary 2.4.4, one can obtain the generalization of common fixed point theorem of Vijayaraju et al. [118, Theorem 2].
Remark 2.4.7. If $S = T = I_x$, $A = B$ and $\psi(t) = kt$ where $k \in [0,1)$, in Corollary 2.4.4, we obtain a common fixed point theorem generalizing the result of Rhoades [100, Theorem 2], and further, one can find the result of Branciari [20, Theorem 2.1] as a special case.

Conclusion. The application of property (E.A) enables us to curtail the proof for the existence of Cauchy sequence of iterates and use of closeness of the space is more general than the use of completeness.