CHAPTER II
CLASSICAL FOURIER THEORY

Fourier system forms a basis for the Hilbert space \( L^2[-\pi, \pi] \).

The classical Szegö's theorem [4,12] is based on Fourier System, \( \{ e_n : n \in \mathbb{Z} \} \), where \( e_n(x) = e^{inx} \). In this chapter we look into the effect of change in the ordering of the Fourier System on Szegö's classical observations of asymptotic distribution of eigenvalues of finite Toeplitz forms. This is done by checking proofs and Szegö's propositions in the new set up. Since the Fourier system is unconditional [19], any arbitrary ordering of the Fourier system forms a basis for the Hilbert space \( L^2[-\pi, \pi] \).

This chapter comprises of two sections of which the first one, deals with extremum (minimum) property of the Toeplitz forms and its limits in the changed system. Second one, deals with asymptotic distribution of eigenvalues of finite Toeplitz forms in the new system and the validity of Szegö's theorem.

2.1 Extremum properties of Toeplitz Forms (minimum)

In this section we define a system of orthogonal polynomials with respect to arbitrary ordered Fourier system and the associated Toeplitz forms and find its extremum properties as in the original work of Szegö [12]. Let the arbitrary ordered Fourier system be denoted by \( \{ e^{is_0 x}, n = 0, 1, \ldots, s_0 = 0 \} \) where 's' is the permutation on \( N \), the set of Natural numbers.

2.1.1 Definition

Let \( \alpha(x) \) be a distribution function of the infinite type, \(-\pi \leq x \leq \pi\) and
be its Fourier-Stieltjes coefficients in the new system. Using the orthogonalization procedure, we form a system of polynomials $\hat{\phi}_0(x), \hat{\phi}_1(x), \hat{\phi}_2(x), \ldots \hat{\phi}_n(x) \ldots$ of the complex variable $z$ which are orthogonal on the unit circle $|z|=1$ with the weight $\frac{d\alpha(x)}{2\pi}$.

The system \{\hat{\phi}_n(z)\} is uniquely determined by the conditions

(i) \[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\phi}_n(z) \hat{\phi}_m(z) d\alpha(x) = \delta_{nm} \]

(ii) $\hat{\phi}_n(z)$ is a polynomial in which coefficient of $z^r$ is real and positive.

Let

$$ f_n(x) = e^{iu_n x} \quad n = 0, 1, \ldots $$

$$ \langle f_n, f_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iu_n x} e^{-iu_m x} d\alpha(x) $$

$$ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(u_n - u_m)x} d\alpha(x) $$

$$ = c_{u_n - u_m} $$

Define

$$ \hat{D}_n = \det(c_{s_n - s_m})_{v,\mu=0}^{n} $$

$$ = \begin{vmatrix}
  c_0 & c_{-s_1} & \ldots & c_{-s_n-1} & c_{-s_n} \\
  c_{s_1} & c_0 & \ldots & c_{s_1-s_n-1} & c_{s_1-s_n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_{s_{n-1}} & c_{s_{n-1}-s_1} & \ldots & c_0 & c_{s_{n-1}-s_n} \\
  c_{s_n} & c_{s_n-s_1} & \ldots & c_{s_n-s_{n-1}} & c_0 \\
\end{vmatrix} $$
\[ \phi_n(x) = \left( \hat{D}_{n-1} \hat{D}_n \right)^{\frac{1}{2}} \begin{vmatrix} c_0 & c_{-s_1} & \cdots & c_{-s_{n-1}} & c_{-s_n} \\ c_{s_1} & c_0 & \cdots & c_{s_{n-1}} & c_{s_{n-1} - s_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{s_{n-1}} & c_{s_{n-1} - s_1} & \cdots & c_0 & c_{s_n - s_{n-1}} \\ 1 & z^{s_1} & \cdots & z^{s_{n-1}} & z^{s_n} \end{vmatrix} \]

where \( z = e^{ix} \).

The coefficient of \( z^{s_n} \) in \( \phi_n(z) \) is denoted by the special notation

\[ \hat{k}_n = \left( \frac{\hat{D}_{n-1}}{\hat{D}_n} \right)^{\frac{1}{2}}. \]

2.1.2 Definition

The Toeplitz forms [1, 25] with respect to the new system is defined as

\[ \hat{T}_n = \sum_{\mu, \nu=0}^n c_{s_\nu - s_\mu} \overline{u_\mu} u_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \mu_0 + u_1 z^{s_1} + \cdots + u_n z^{s_n} \right|^2 d\alpha(x), \quad \ldots \]  \hspace{1cm} (1)

Then \( \hat{D}_n = \det(c_{s_\nu - s_\mu}) \) is the determinant of the Toeplitz form. They are called Toeplitz determinants associated with \( \alpha(x) \) in the new system. Since (1) is positive definite, we have \( \hat{D}_n > 0 \quad \forall n \).

The next theorem gives the extremum property (minimum) of the Toeplitz forms, in the new system.

2.1.3 Theorem

The polynomial \( \hat{k}^{-1}_n \hat{\phi}_n(z) \) minimizes the integral

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 \, d\alpha(x), \quad z = e^{ix} \quad \text{where} \quad g(z) = z^{s_n} + a_1 z^{s_{n-1}} + \cdots + a_n \quad \text{is an arbitrary} \]
polynomial generated by $z^0, z^n, ..., z^s$ in which coefficient of $z^s = 1$. The minimum itself is $\hat{k}_{n}^{-2} = \frac{\hat{D}_n}{\hat{D}_{n-1}}$.

Proof:

This follows by representing $g(z)$ in the form

$$g(z) = \phi_0(z) + \phi_1(z) + ... + \phi_n(z)$$

where $\phi_0, \phi_1, ..., \phi_n$ are complex variables and $\phi_n$ is subjected to the condition

$$\phi_n \hat{k}_n = 1 \quad \therefore \phi_n = \hat{k}_n^{-1}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 d\alpha(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \phi_0(z) + \phi_1(z) + ... + \phi_n(z) \right|^2 d\alpha(x)$$

$$= |\phi_0|^2 + ... + |\phi_n|^2 \geq |\phi_n|^2$$

$$\geq \hat{k}_n^{-2} = \frac{\hat{D}_n}{\hat{D}_{n-1}} \quad \ldots \ (1)$$

When $g(z) = \hat{k}_n^{-1} \phi_n(z)$, then

coefficient of $z^s$ in $g(z) = \hat{k}_n^{-1} \phi_n = 1$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 d\alpha(x) = \hat{k}_n^{-2} \quad \ldots \ (2)$$

Hence from (1) and (2), we get

$$\min \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 d\alpha(x) = \hat{k}_n^{-2} = \frac{\hat{D}_n}{\hat{D}_{n-1}}$$

Therefore when $g(z) = \hat{k}_n^{-1} \phi_n(z)$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 d\alpha(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \hat{k}_n^{-1} \phi_n(z) \right|^2 d\alpha(x)$$

$$= \hat{k}_n^{-2} = \text{minimum value.}$$

Hence the polynomial $\hat{k}_n^{-1} \phi_n(z)$ minimizes the integral $\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2 d\alpha(x)$.
Now we find the limit of the minimum of the Toeplitz forms under the side condition \( u_0 = 1 \), which is given in the following limit theorem.

**2.1.4 Theorem**

Let \( \alpha(x) \) be a distribution function of the infinite type. We consider the Toeplitz forms

\[
\hat{T}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 z^{s_1} + \cdots + u_n z^{s_n} \right|^2 d\alpha(x)
\]

with the side condition \( u_0 = 1 \). Let \( \hat{\mu}_n \) denote the minimum. Then

\[
\lim_{n \to \infty} \hat{\mu}_n = \hat{\mu} = G(\omega) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\omega(x)) dx \right\}
\]

where \( \omega(x) \) is the almost everywhere existing derivative of \( \alpha(x) \).

**Proof:**

The minima \( \hat{\mu}_n \) are non-increasing as \( n \) increases. Hence \( \lim_{n \to \infty} \hat{\mu}_n = \hat{\mu} \) exists.

\[
\hat{\mu}_n = \min \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 z^{s_1} + \cdots + u_n z^{s_n} \right|^2 d\alpha(x), \quad u_0 = 1
\]

\[
\geq \min \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 z^1 + u_2 z^2 + \cdots + u_n z^{s_n} \right|^2 d\alpha(x), \quad u_0 = 1
\]

\[
= \min(T_{s_n})_{u_0=1} = \mu_{s_n}
\]

Taking limit we get,

\[
\hat{\mu} \geq \mu = G(\omega(x)) \quad [12, \text{Chapter 3}]
\]

In order to prove the reverse inequality, first we show that it is always possible to find a large enough \( m \) such that

\[
\{ 0, 1, 2, \ldots, n \} \subset \{ s_0, s_1, \ldots, s_m \}.
\]

There exist positive integers \( a_0, a_1, \ldots, a_n \) such that \( s_{a_0} = 0, s_{a_1} = 1, \ldots, s_{a_n} = n \).

Choose \( m \geq \max\{a_0, a_1, \ldots, a_n\} \).
Then
\[ \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \{0, 1, \ldots, m\} \]

Hence
\[ \{s_{\alpha_0}, s_{\alpha_1}, \ldots, s_{\alpha_n}\} \subset \{s_0, s_1, \ldots, s_m\} \]

ie.
\[ \{0, 1, \ldots, n\} \subset \{s_0, s_1, \ldots, s_m\} \]

Let
\[ T_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u_0 + u_1 z + \ldots + u_n z^n|^2 \, d\alpha(x) \]

Then
\[ \text{Min } T_n = \min \frac{1}{2\pi} \int_{-\pi}^{\pi} |u_0 + u_1 z + \ldots + u_n z^n|^2 \, d\alpha(x), \quad u_0 = 1 \]
\[ \geq \min \frac{1}{2\pi} \int_{-\pi}^{\pi} |u_0 + u_1 z^{n} + \ldots + u_m z^{n}|^2 \, d\alpha(x), \quad u_0 = 1 \]

Taking limit, the above inequality reduces to
\[ \lim_{n \to \infty} \mu_n \geq \lim_{m \to \infty} \hat{\mu}_m \]

ie.
\[ \mu = G(\omega(x)) \geq \hat{\mu} \]

From (1) and (2), we get
\[ \hat{\mu} = G(\omega) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\omega(x)) \, dx \right\} \]

2.1.5 Theorem
Consider the Toeplitz form
\[ \hat{T}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u_0 + u_1 z^n + \ldots + u_n z^{n}|^2 \, d\alpha(x), \quad z = e^{i\theta} . \]

Let \((\hat{\mu}_n)_{n=1}^{\infty}\) and \((\hat{\mu}_n)_{n=1}^{\infty}\) denote the minimum of \(\hat{T}_n\) under the side condition
\[ u_0 = 1 \text{ and } u_n = 1 \text{ respectively.} \]

Then
\[ \lim_{n \to \infty} (\hat{\mu}_n)_{n=1}^{\infty} = \lim_{n \to \infty} \frac{\hat{D}_n}{\hat{D}_{n-1}} = G(\omega) \]
\[ = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\omega(x)) \, dx \right\} \]
Proof

We show that \((\hat{\mu}_n)_{u_0=1} = (\hat{\mu}_n)_{u_1=1}\).

Then the theorem is evident from theorems 2.1.3 and 2.1.4.

Case 1: \(u_n\) is the leading coefficient

\[
\hat{T}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n} \right|^2 d\alpha(x), \quad u_0 = 1
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 z^{-s_n} + u_1 z^{-(s_n-s_1)} + \ldots + u_n \right|^2 d\alpha(x), \quad u_0 = 1
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| z^{s_n} + u_1 z^{s_n-s_1} + \ldots + u_n z^{s_n-s_{n-1}} + u_n \right|^2 d\alpha(x) \quad \ldots \quad (2)
\]

Therefore

\[
\min \hat{T}_n = \min \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| z^{s_n} + u_1 z^{s_n-s_1} + \ldots + u_n z^{s_n-s_{n-1}} + u_n \right|^2 d\alpha(x)
\]

Hence from theorem 2.1.3, we get

\[
(\hat{\mu}_n)_{u_0=1} = \frac{D_n}{D_{n-1}} \quad \ldots \quad (3)
\]

where \(\tilde{D}_n\) is the determinant of the Toeplitz form (2).

The Toeplitz forms (1) and (2) are same. Therefore their determinants are also same. This can be proved in the following way.

Evaluation of \(\tilde{D}_n\)

Let \(h(z) = z^{s_n} + u_1 z^{s_n-s_1} + \ldots + u_n\). Then \(h(z)\) is an arbitrary polynomial generated by \(z^0, z^{s_n-s_{n-1}}, z^{s_n-s_{n-2}}, \ldots, z^{s_n-s_1}, z^{s_n}\) such that coefficient of \(z^{s_n} = 1\).

The determinant of the Toeplitz form (1) is

\[
\tilde{D}_n = \begin{vmatrix}
    c_{s_0} & c_{s_n-s_0} & c_{s_n-s_1} & \ldots & c_{s_1-s_0} & c_{s_0-s_2} \\
    c_{s_n-s_{n-1}} & c_{s_{n-1}} & c_{s_{n-2}} & \ldots & c_{s_1-s_{n-1}} & c_{s_{n-2}-s_0} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_{s_n-s_1} & c_{s_{n-1}-s_1} & c_{s_{n-2}-s_2} & \ldots & c_0 & c_{s_1-s_0} \\
    c_{s_n} & c_{s_{n-1}} & c_{s_{n-2}} & \ldots & c_{s_1} & c_0
\end{vmatrix}
\]
Interchanging the rows $R_i$ and $R_{n-i}$, then the columns $C_i$ and $C_{n-i}$ for $i = 0, 1, 2, \ldots, n/2$ when $n$ is even and for $i = 0, 1, \ldots, (n-1)/2$ when $n$ is odd and then taking transpose we get,

$$
\tilde{D}_n = \begin{pmatrix}
    c_0 & c_{-s_1} & \ldots & c_{-s_{n-1}} & c_{-s_n} \\
    c_{s_1} & c_0 & \ldots & c_{-s_{n-1}} & c_{-s_n} \\
    \ldots & \ldots & \ldots & \ldots & \ldots \\
    c_{s_{n-1}-s_0} & c_{s_{n-1}-s_1} & \ldots & c_0 & c_{s_{n-1}-s_n} \\
    c_{s_n} & c_{s_n-s_1} & \ldots & c_{s_{n-1}-s_{n-1}} & c_0 \\
\end{pmatrix} = \hat{D}_n
$$

Hence equation (3) reduces to

$$
(\hat{\mu}_n)_{u_n=1} = \frac{\tilde{D}_n}{D_{n-1}} = \frac{\hat{D}_n}{\hat{D}_{n-1}} = \min \left( \tilde{t}_n \right)_{u_n=1} = (\hat{\mu}_n)_{u_n=1}
$$

Taking limit, then from theorem 2.1.4 we get,

$$
\lim_{n \to \infty} (\hat{\mu}_n)_{u_n=1} = \lim_{n \to \infty} \frac{\hat{D}_n}{\hat{D}_{n-1}} = G(\omega)
$$

$$
= \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\omega(x)) \, dx \right\}
$$

Case 2: when $u_n$ is not the leading coefficient.

Let $u_k$ be the leading coefficient. Then divide the polynomial by $z^k$. The rest of the proof can be carried out in the same way as in case 1.

Following are some observations obtained by comparing the results in the standard Fourier System and in the new System.
2.1.6 Remarks

It is observed that

(i) In the standard Fourier system, minimum \( T_n \) under the side condition \( u_0 = 1 \) is equal to the minimum of the same Toeplitz form \( T_n \) under the side condition \( u_n = 1 \). But in the new system, \( \min (\hat{T}_n)_{u_0=1} \) is equal to the minimum of another Toeplitz form under the side condition \( u_n = 1 \).

(ii) The trace of the matrix \( (c_{v,\mu})_{v,\mu=0}^n \) of the Toeplitz Form

\[
T_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_0 + u_1 z^1 + \ldots + u_n z^n \, d\alpha(x)
\]

in the standard Fourier system is same as the trace of the matrix \( (c_{s,\mu})_{s,\mu=0}^n \) of the Toeplitz form

\[
\hat{T}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_0 + u_1 z^1 + \ldots + u_n z^n \, d\alpha(x)
\]

in the arbitrary ordered Fourier System. That is, in any arbitrary ordering of the Fourier basis the trace of the Toeplitz matrix remains the same.

2.2 Asymptotic distribution of eigenvalues

In this section the validity of Szegö's Theorem is established. We do this by checking various stages of the proof of Szegö in the new set up. Toeplitz has studied the distribution of eigenvalues of an infinite matrix \( (c_{v,\mu}) \), where the indices \( v \) & \( \mu \) range from \(-\infty\) to \(\infty\) under the standard Fourier system. A value \( \lambda \) is called an eigenvalue of the matrix \( T \) if the matrix \( T - \lambda I \) has no bounded inverse, \( I \) denote the unit matrix.

Now we recall the definition of equal distribution of numbers.
2.2.1 Definition [1.2.2]

For each \( n \) we consider a set of \( n+1 \) real numbers \( a_1^{(n)}, a_2^{(n)}, \ldots, a_{n+1}^{(n)} \) and another set of the same kind \( b_1^{(n)}, b_2^{(n)}, \ldots, b_{n+1}^{(n)} \).

We assume that for each \( \nu \) and \( n \)

\[
|a_{\nu}^{(n)}| < K, \quad |b_{\nu}^{(n)}| < K,
\]

where \( K \) is independent of \( \nu \) and \( n \). We say that \( \{a_{\nu}^{(n)}\} \) and \( \{b_{\nu}^{(n)}\}, \ n \to \infty \), are equally distributed in the interval \([-K, K]\) if the following holds. Let \( F(t) \) be an arbitrary continuous function in the interval \([-K, K]\); we have then

\[
\lim_{n \to \infty} \frac{\sum_{\nu=1}^{n+1} [F(a_{\nu}^{(n)}) - F(b_{\nu}^{(n)})]}{n+1} = 0.
\]

Let \( f(x) \) be a real valued function of the class \( L \) and let

\[
c_{s_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iux} f(x) dx \quad n = 0, \pm 1, \pm 2 \ldots.
\]

We consider the finite Toeplitz forms

\[
\hat{T}_n(f) = \sum_{\mu,\nu=1}^{n} c_{s_{\mu}-s_{\nu}} u_{\mu} u_{\nu}^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 z^1 + \ldots + u_n z^n \right|^2 f(x) dx \quad \ldots \quad (1)
\]

The eigenvalues of \( \hat{T}_n(f) \) are defined as the root of the characteristic equation \( \det(\hat{T}_n(f - \lambda)) = 0 \). Hence the eigenvalues of \( \hat{T}_n(f) \) are the eigenvalues of the matrix

\[
\begin{bmatrix}
    c_{0} & c_{-s_{1}} & c_{-s_{2}} & \ldots & c_{-s_{n}} \\
    c_{s_{1}} & c_{0} & c_{s_{1}-s_{2}} & \ldots & c_{s_{1}-s_{n}} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_{s_{n-1}} & c_{s_{n-1}-s_{1}} & c_{s_{n-1}-s_{2}} & \ldots & c_{s_{n-1}-s_{n}} \\
    c_{s_{n}} & c_{s_{n}-s_{1}} & c_{s_{n}-s_{2}} & \ldots & c_{0}
\end{bmatrix}
\]
We denote them by $\beta_1, \beta_2, \ldots, \beta_{n+1}$. Also if $m \leq f(x) \leq M$ for all real $x$ then from (1) we have $m \leq \hat{T}_n(f) \leq M$. Also we have

$$m \leq \beta_{\nu} \leq M \quad \nu = 1, 2, \ldots, n+1.$$  

The main result of this chapter is the following theorem and it is the well known Szegö's Theorem in the new arbitrary ordered Fourier System.

2.2.2 Theorem

Let $f(x)$ be a real-valued function of the class $L$. We denote by $m$ and $M$ the ‘essential’ lower and upper bound of $f(x)$ respectively and assume that $m$ and $M$ are finite. If $F(\beta)$ is any continuous function defined in the finite interval $m \leq \beta \leq M$ we have

$$\lim_{n \to \infty} \frac{F(\beta_1) + F(\beta_2) + \ldots + F(\beta_{n+1})}{n + 1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(x))dx \quad \ldots \quad (2)$$

Proof

Using the definition of the equal distribution the above limit relation can be expressed as follows. The sets $\{\beta_{\nu}\}$ and $\left\{f\left(-\pi + \frac{2\nu\pi}{n+2}\right)\right\}$, $n \to \infty$ are equally distributed.

It is well known that the limit relation will be proved for all continuous functions $F(t)$ if it holds for certain special sets of continuous functions $F(t) = t^s \quad s = 0, 1, 2, \ldots$ and $F(t) = \log t$.

We show that the limit relation is true for $F(t) = \log t$. Then the result follows for $t^s$ also [12, Chapter V]. This will yield the required result (2).

Let $\hat{D}_n$ be the determinant of the Toeplitz form (1), then from theorem 2.1.5, we have
\[
\lim_{n \to \infty} \frac{\hat{D}_n}{\hat{D}_{n-1}} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx \right\}
\]

ie.
\[
\lim_{n \to \infty} [\hat{D}_n(f)]_{n+1}^{y} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx \right\}
\]

Therefore
\[
\lim_{n \to \infty} \log(\hat{D}_n)_{n+1}^{y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx
\]

Substituting \( \hat{D}_n = \beta \cdot \beta_2 \cdot \ldots \cdot \beta_{n+1} \), we get
\[
\lim_{n \to \infty} \log(\beta \cdot \beta_2 \cdot \ldots \cdot \beta_{n+1})_{n+1}^{y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx
\]

ie.
\[
\lim_{n \to \infty} \frac{\log \beta + \log \beta_2 + \ldots + \log \beta_{n+1}}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx
\]

Hence the result (2) is true for \( F(\beta) = \log \beta \), which completes the proof.