CHAPTER 4
FUZZY P-SPACES

4.1 Introduction

The class of ‘p- spaces’ generalizes both metrizable spaces and compact spaces. Various theorems which hold for both metrizable spaces and compact spaces can often be generalized and hence unified by showing that they hold for p- spaces. The concept of ‘p- spaces’ due to Arhangelskii is in terms of a sequence of open covers in some compactification of the space rather than the space itself (for details, cf[GG]). In this chapter we define fuzzy p-spaces, strict fuzzy p-spaces and prove some characterizations of both of these spaces. We refer [MH1] for fuzzy Stone-Čech compactification. We also define fuzzy k-spaces and establish some relation between fuzzy p-spaces and fuzzy k-spaces. We say that a fuzzy topological space (X, F) is completely regular if the topological space (X, L(F)) is completely regular, where L(F) is the weakest topology on X which makes every member of F lower semicontinuous function from (X, L(F)) → [0,1]. All the fuzzy topological spaces considered in this chapter are assumed to be completely regular.

4.2 Fuzzy P-Spaces

Definition 4.2.1 (Fuzzy compactification) [MH1]

Let (X, F) be a fuzzy topological space. Let (βX, T) be any compact topological space which contains (X, L(F)) as a dense subspace. Then F_T, the set of
all lower semicontinuous mappings $g : (\beta X, T) \to [0, 1]$ such that $g \chi_X \in F$, is a fuzzy topology on $\beta X$ and $(\beta X, F_T)$ is fuzzy compact.

If $(\beta X, T)$ is the Stone-Čech compactification of $(X, \tau(F))$ then $(\beta X, F_T)$ is the fuzzy Stone-Čech compactification of $(X, F)$.

**Definition 4.2.2**

A completely regular fuzzy topological space $(X, F)$ is called a fuzzy $p$-space if there exists a sequence $(\mathcal{B}_n)$ of families of fuzzy open sets on $(\beta X, F_T)$ such that

(i) for each $n$, $(\bigvee_{B \in \mathcal{B}_n} B) \geq \chi_X$.

(ii) for each $x \in X$, $\alpha \in (0, 1]$, $\bigvee_{n} \operatorname{st}(x_\alpha, \mathcal{B}_n) \leq \chi_X$. If we also have

(iii) for $x \in X$, $\alpha \in (0, 1]$, $\bigvee_{n} \operatorname{st}(x_\alpha, \mathcal{B}_n) = \bigwedge_{n} \operatorname{st}(x_\alpha, \overline{\mathcal{B}_n})$, then $(X, F)$ is said to be a strict fuzzy $p$-space.

**Theorem 4.2.3**

A fuzzy topological space $(X, F)$ is a strict fuzzy $p$-space if and only if there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of $(X, F)$ such that for each $x \in X$, $\alpha \in (0, 1]$

(a) $C_{x_\alpha} = \bigwedge_{n} \operatorname{st}(x_\alpha, \mathcal{A}_n)$ is fuzzy compact.
(b) \{st(x_\alpha, \mathcal{A}_n) : n \in \mathbb{N}\} forms a base for Cx_\alpha. (That is if Cx_\alpha \leq G \in F, there exists some n_0 such that st(x_\alpha, \mathcal{A}_{n_0}) \leq G).

Proof

(Necessity)

First suppose that (X, F) is a strict fuzzy p-space. Then there exists a sequence (\mathcal{B}_n) of families of fuzzy open sets on (\beta X, F_T) which satisfies (i), (ii) and (iii) in the definition of a strict fuzzy p-space. We can assume that \mathcal{B}_{n+1} refines \mathcal{B}_n. For if (\mathcal{B}_n)'s are not so, then (\mathcal{B}_n') where \mathcal{B}_n' = \{ \bigwedge_{i \leq n} B_i | B_i \in \mathcal{B}_i \} will do so.

For each \mathcal{B}_n \subseteq \mathcal{B}_n choose A_n \in F such that A_n = B_n \chi_X and denote such collection of A_n by \mathcal{A}_n. By (i) (\mathcal{A}_n) forms a sequence of fuzzy open covers of (X, F). By (iii) \bigwedge_n st\left(x_\alpha, \mathcal{B}_n\right) = \bigwedge_n st\left(x_\alpha, \mathcal{B}_n\right) is closed in (\beta X, F_T) and hence fuzzy compact. Therefore Cx_\alpha = \bigwedge_n st\left(x_\alpha, \mathcal{A}_n\right) = \bigwedge_n st\left(x_\alpha, \mathcal{A}_n\right) is fuzzy compact in (X, F), since (\beta X, F_T) is the fuzzy Stone-Čech compactification of (X, F). This proves (a). Also as \mathcal{B}_{n+1} refines \mathcal{B}_n, we have \mathcal{A}_{n+1} refines \mathcal{A}_n.

Take U_n = st\left(x_\alpha, \mathcal{A}_n\right). Now \bigwedge_n U_n = \bigwedge_n U_n and as Cx_\alpha = \bigwedge_n U_n is fuzzy compact, \{(x_n)_{\alpha} : n \in \mathbb{N}\} has a cluster point, where \(x_n)_{\alpha} are the fuzzy points with
support $x_n$, value $\alpha$ and $(x_n)_{\alpha} \leq U_n$. Therefore by lemma 2.3.5, \( \{U_n\} \) is a base for the fuzzy set $\bigwedge_n U_n$. That is \( \{ \text{st}(x_n, A_n) : n \in \mathbb{N} \} \) is a base for $C_{x_{\alpha}}$. This proves (b).

**(Sufficiency)**

Conversely assume that $(A_n)$ is a sequence of open covers of $(X, F)$ satisfying (a) and (b). For each $A_n \in A_n$, we can find a $B_{A_n} \in F_T$ and can form $(B_n)$ in the following way.

For each $\alpha \in (0, 1]$, take $V_{\alpha} = A_n^{-1} (\alpha, 1] \in \ell(F)$. Let $V_{\alpha} = \beta X - cl_{\beta x} (X - V_{\alpha})$.

Then $V_{\alpha} \in T$ and $V_{\alpha} = V_{\alpha} \cap X$. Define $B_{A_n} : \beta X \rightarrow [0, 1]$ by $B_{A_n}(x) = \sup \{ \alpha : x \in V_{\alpha} \}$. Then for $x \in X$, $B_{A_n}(x) = \sup \{ \alpha : x \in V_{\alpha} \} = A_n(x)$. Therefore $B_{A_n} \chi_x = A_n$. Now we show that $B_{A_n}$'s are lower semicontinuous. Take $b \in [0, 1]$. Let $G = B_{A_n}^{-1}(b, 1]$.

Then $x \in G \Rightarrow B_{A_n}(x) > b$

$\Rightarrow B_{A_n}(x) > \alpha > b$ for some $\alpha \in (0, 1)$

$\Rightarrow x \in V_{\alpha} \subset B_{A_n}^{-1}(b, 1] = G$. 

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Therefore \( G \in T \). Hence \( B_{A_n} \) is lower semicontinuous. That is \( B_{A_n} \in F_T \). Take \( \mathcal{B}_n = \{ B_{A_n} \mid A_n \in \mathcal{A}_n \} \). Then \( (\mathcal{B}_n) \) forms a sequence of families of open fuzzy sets on \((\beta X, F_T)\) which satisfies condition (i) of a fuzzy p-space.

For each \( A_n \in \mathcal{A}_n \), define \( A_n \) as \( A_n^*(x) = \begin{cases} A_n(x) & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases} \)

Denote \( \bigwedge_n\text{st} \left( x, \mathcal{A}_n \right) \) as \( C_{x, A} \) where \( \mathcal{A}_n = \{ A_n^* \mid A_n \in \mathcal{A}_n \} \).

We show that \( C_{x, A} = \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) = \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) \). This gives conditions (ii) and (iii) for strict fuzzy p-space.

Now \( C_{x, A} = \bigwedge_n\text{st} \left( x, \mathcal{A}_n \right) \leq \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) \leq \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) \rightarrow (i) \)

since each \( B \in \mathcal{B}_n \) is such that \( B \chi_x = A \in \mathcal{A}_n \). Suppose that \( \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) \notin C_{x, A} \).

Then there exists some \( y \in \beta X \) with \( \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) (y) > C_{x, A} (y) \). Let \( \gamma = \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) (y) \). Therefore \( y_\gamma \leq \bigwedge_n\text{st} \left( x, \mathcal{B}_n \right) \) but \( y_\gamma \notin C_{x, A} \). Since \( C_{x, A} \) is fuzzy compact \( C_{x, A} \) is also fuzzy compact. Therefore there exists some open fuzzy set \( W \) on \((\beta X, F_T)\) such that \( y_\gamma \leq W \) and \( W \bigwedge C_{x, A} = 0 \). Also as the \( \{ \text{st}(x, \mathcal{A}_n) : n \in \mathbb{N} \} \) forms a base for \( C_{x, A} \), \( \{ \text{st}(x, \mathcal{A}_n) : n \in \mathbb{N} \} \) forms a base.
for \( Cx_\alpha \). Therefore there exists some \( m \in \mathbb{N} \) such that \( \text{st}\left(x_\alpha, A^*_m\right) \cap W = \emptyset \) Hence

\[
\text{st}\left(x_\alpha, A^*_m\right) \cap W = \emptyset . \quad \text{In particular} \quad \text{st}\left(x_\alpha, B_m\right) \cap W = \emptyset . \quad \text{This is a contradiction as}
\]

\[
y_\gamma \leq \wedge \text{st}\left(x_\alpha, B_n\right) . \quad \text{Thus} \quad Cx_\alpha \succeq \wedge \text{st}\left(x_\alpha, B_n\right) \rightarrow (2) .
\]

From (1) and (2) \( Cx_\alpha = \wedge \text{st}\left(x_\alpha, A_n\right) = \wedge \text{st}\left(x_\alpha, B_n\right) \). Therefore \((X, F)\) is a strict fuzzy p-space.

**Corollary 4.2.4**

Every fuzzy Moore space is a strict fuzzy p-space.

**Proof**

Let \((X, F)\) be a fuzzy Moore space. Let \((A_n)\) be a fuzzy development for \((X, F)\). Therefore for \( \alpha \in (0, 1]\), a fuzzy point \( x_\alpha \), \( \{\text{st}\{x_\alpha, A_n\}; n \in \mathbb{N}\} \) forms a base for \( x_\alpha \). Let \( \{B_i | i \in I\} \), where \( I \) is an index set, be an open cover for \( Cx_\alpha = \wedge \text{st}\left(x_\alpha, A_n\right) \). Then there exists some \( B_{i_0} \) such that \( x_\alpha \leq B_{i_0} \). Since \( \{\text{st}\{x_\alpha, A_n\}; n \in \mathbb{N}\} \) is a base at \( x_\alpha \), there exists \( m \in \mathbb{N} \) such that \( x_\alpha \leq \text{st}\{x_\alpha, A_m\} \leq B_{i_0} \). Hence \( Cx_\alpha \leq B_{i_0} \), so that \( Cx_\alpha \) is fuzzy compact.

For any fuzzy open set \( G \) with \( Cx_\alpha \leq G \), we can also have \( x_\alpha \leq G \). Therefore there exists \( n_0 \) such that \( x_\alpha \leq \text{st}\{x_\alpha, A_{n_0}\} \leq G \). Hence
\{\text{st}(x_\alpha, \mathcal{A}_n) : n \in \mathbb{N}\} is a base for \text{C}_{x_\alpha}. Thus \((X, F)\) satisfies conditions (a) and (b) of theorem 4.2.3. Thus \((X, F)\) is a strict fuzzy p-space.

**Corollary 4.2.5**

Every strict fuzzy p-space is a fuzzy w\(\Delta\)-space.

**Proof**

Assume that \((X, F)\) is a strict fuzzy p-space. Then by theorem 4.2.3, there exists a \((\mathcal{A}_n)\) a fuzzy open covers of \(X\) such that \(\{\text{st}(x_\alpha, \mathcal{A}_n) : n \in \mathbb{N}\}\) forms a base for \(\text{C}_{x_\alpha} = \landslash \text{st}(x_\alpha, \mathcal{A}_n)\) and \(\text{C}_{x_\alpha}\) is fuzzy compact. Therefore if we choose fuzzy points \((x_n)_\alpha\) with \((x_n)_\alpha \leq \text{st}(x_\alpha, \mathcal{A}_n)\), \(\{(x_n)_\alpha : n \in \mathbb{N}\}\) has a cluster point in \(\text{C}_{x_\alpha}\). Thus \((X, F)\) is a fuzzy w\(\Delta\)-space.

**Theorem 4.2.6**

Every regular fuzzy submetacompact, fuzzy p-space is a strict fuzzy p-space.

**Proof**

Let \((X, F)\) be a regular fuzzy submetacompact space which is also a fuzzy p-space. Let \((\mathcal{B}_n)\) be a sequence of families of fuzzy open subsets on \((\beta X, F_\tau)\) which satisfies conditions (i) and (ii) for the fuzzy p-space. For each \(B_n \in \mathcal{B}_n\), choose \(A_n \in F\) such that \(A_n = B_n \chi_{x_\alpha}\) and form \(\mathcal{A}_n\). Then \((\mathcal{A}_n)\) forms a
sequence of fuzzy open covers of $X$. Consider $\mathcal{A}_1$. By the fuzzy submetacompactness of $X$, $\mathcal{A}_1$ has a sequence of open refinements, say $(U_{1n})_{n\in\mathbb{N}}$, such that for a fuzzy point $x_\alpha$, there exists $n\in\mathbb{N}$ such that $x_\alpha \preceq U_{1n} \in U_{1n}$ holds only for finitely many elements of $U_{1n}$. Let $U_{11}$ be one such open refinement corresponding to the fuzzy point $x_\alpha$. By regularity of $X$, for each $U_{1n}$, we can find an open refinement, say $\mathcal{A}_{1n}$, such that, for $x_\alpha \preceq U_{1n} \in U_{1n}$ there exists $A_{1n} \in \mathcal{A}_{1n}$ with $x_\alpha \preceq A_{1n} \subseteq \overline{A_{1n}} \subseteq U_{1n}$. Similarly for $\mathcal{A}_2$, by fuzzy submetacompactness, there exists $(U_{2n})_{n\in\mathbb{N}}$ and by regularly each $U_{2n}$ has a refinement $\mathcal{B}_{2n}$. Then take $(\mathcal{A}_{2n})_{n\in\mathbb{N}}$ as follows

$$\mathcal{A}_{2n} = \mathcal{B}_{2n} \wedge \mathcal{A}_{11} = \{ B \wedge U / B \in \mathcal{B}_{2n}, U \in \mathcal{A}_{11} \}.$$ 

For $\mathcal{A}_3$, by fuzzy submetacompactness, there exists a sequence of open refinements $(U_{3n})_{n\in\mathbb{N}}$ and by regularly each $U_{3n}$ has an open refinement $\mathcal{B}_{3n}$. Take $(\mathcal{A}_{3n})_{n\in\mathbb{N}}$ as follows

$$\mathcal{A}_{3n} = \mathcal{B}_{3n} \wedge \mathcal{A}_{11} \wedge \mathcal{A}_{12} \wedge \mathcal{A}_{21} \wedge \mathcal{A}_{22}.$$ 

Repeating this process for each $m$, we have a sequence $(\mathcal{A}_{mn})_{n\in\mathbb{N}}$ of fuzzy open covers of $X$ such that
(a) \((\mathcal{A}_{m,n})_{n \in \mathbb{N}}\) is a refinement of each \(\mathcal{A}_{i,j}\) such that \(i < m, j < m\) and for every fuzzy point \(x_\alpha\), there exists \(n \in \mathbb{N}\) such that \(x_\alpha\) is in only finitely many members of \(\mathcal{A}_{m,n}\).

(b) If \(V \in \mathcal{A}_{m,n}\) and \(i, j < m\), there exists \(w \in \mathcal{A}_{i,j}\) such that \(\overline{V} \leq W\) and for \(k \leq m\) there exists \(A \in \mathcal{A}_k\) such that \(\overline{V} \leq A\).

Let \(y_\alpha \leq \bigwedge_{i,j} \text{st}(x_\alpha, \mathcal{A}_{i,j})\). Fix \(i\) and \(j\) and let \(m > \max \{i, j\}\). Then there exists \(n \in \mathbb{N}\) such that \(x_\alpha\) is in only finitely many members of \(\mathcal{A}_{m,n}\).

That is \(y_\alpha \leq \text{st}(x_\alpha, \mathcal{A}_{m,n}) = \bigvee \{\overline{V} : x_\alpha \leq V \in \mathcal{A}_{m,n}\}\)

\[\leq \bigvee \{w : x_\alpha \leq \overline{V} < w \in \mathcal{A}_{i,j}\}, \text{ by (b)}\]

\[= \text{st}(x_\alpha, \mathcal{A}_{i,j})\]

Therefore for each \(i, j \in \mathbb{N}\),

\[\bigwedge_{i,j} \text{st}(x_\alpha, \mathcal{A}_{i,j}) = \bigwedge \text{st}(x_\alpha, \mathcal{A}_{i,j}) \leq \bigwedge_{n} \text{st}(x_\alpha, \mathcal{A}_{n}).\]

Corresponding to each \(A_{ij} \in \mathcal{A}_{i,j}\), we can find a \(B_{Ai,j} \in F_T\) such that \(A_{ij} = B_{Ai,j} \mathcal{X}_X\) and can form \(B_{i,j}\). That is \(B_{i,j} = \{B_{Ai,j} \mid A_{ij} \in \mathcal{A}_{i,j}\}\). Now

\[\bigwedge \text{st}(x_\alpha, B_n) = \bigwedge \text{st}(x_\alpha, B_{ij}) = \bigwedge \text{st}(x_\alpha, \mathcal{A}_{ij}).\]
\[ \wedge \text{st} \left( x_\alpha, \mathcal{A}_n \right) \leq \wedge \text{st} \left( x_\alpha, \mathcal{B}_n \right) \] where \( \mathcal{A}_n(x) = \begin{cases} A_n(x) & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases} \)

and \( \mathcal{A}_n = \{ A_n | A_n \in \mathcal{A}_n \} \). Therefore \( \wedge \text{st} \left( x_\alpha, \mathcal{B}_n \right) = \wedge \text{st} \left( x_\alpha, \mathcal{B}_n \right) \).

is \((X, F)\) is a strict fuzzy p- space.

**Theorem 4.2.7**

Every regular fuzzy paracompact, fuzzy p-space is a fuzzy M-space.

**Proof**

Let \((X, F)\) be a regular fuzzy paracompact space, which is also a fuzzy p- space. Since every fuzzy paracompact space is a fuzzy subparacompact space and hence submetacompact space, it follows from theorem 4.2.6 that \((X, F)\) is a strict fuzzy p- space. Hence by corollary 4.2.5, \((X, F)\) is a fuzzy w\( \Delta \)- space. Also fuzzy paracompact, fuzzy w\( \Delta \)- spaces are fuzzy M-spaces, since using fuzzy paracompactness one can modify \( \mathcal{H}_n \) such that \( \mathcal{H}_{n+1} \) star refines \( \mathcal{H}_n \). Thus \((X, F)\) is a fuzzy M-space.

4.3 Fuzzy P-Spaces and Fuzzy K-Spaces.

In this section we prove some characterizations for fuzzy p-spaces and prove some relationship between fuzzy p-spaces and fuzzy k-spaces.
Theorem 4.3.1

A fuzzy topological space \((X, F)\) is a fuzzy p-space if and only if there exists a sequence \(\{\mathcal{A}_n\}\) of fuzzy open covers of \((X, F)\) satisfying the following conditions. If for each \(n\), fuzzy point \(x_\alpha\) with support \(x \in X\), value \(\alpha \in (0, 1]\) and \(x_\alpha \leq A_n \leq \mathcal{A}_n\)

\begin{enumerate}
  \item \(\bigwedge_n \mathcal{A}_n\) is fuzzy compact
  \item \(\{\bigwedge_{i \leq n} \mathcal{A}_i \mid n \in \mathbb{N}\}\) is an outer network for the fuzzy set \(\bigwedge_n \mathcal{A}_n\). That is for every fuzzy open set \(G\) with \(\bigwedge_n \mathcal{A}_n \leq G\), there exists some \(\bigwedge_{i \leq n} \mathcal{A}_i\) with \(\bigwedge_{i \leq n} \mathcal{A}_i \leq G\).
\end{enumerate}

Proof

(Necessity)

Assume that \((X, F)\) is a fuzzy p-space. Let \((\mathcal{B}_n)\) be a sequence of fuzzy open sets in \((\beta X, F_\beta)\) which satisfies (i) and (ii) in the definition of fuzzy p-space. Then we can choose a sequence \(\{\mathcal{A}_n\}\) of fuzzy open covers of \((X, F)\) such that \(\big\{ \mathcal{A}_n^* \mid A_n \in \mathcal{A}_n \big\}\) refines \(\mathcal{B}_n\) where \(A_n^*(x) = \begin{cases} A_n(x) & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases}\)
Let \( \{G_i\} \) be a fuzzy open cover of \( \bigwedge_n A_n \). For each \( G_i \), choose \( H_i \in F_T \) such that \( G_i = H_i \bigvee_X \). Now these \( H_i \)'s forms a fuzzy open cover of \( \bigwedge_n A_n^* \) and hence possesses a finite subfamily which cover \( \bigwedge_n A_n^* \). The corresponding \( G_i \)'s then form a subfamily of \( \{G_i\} \) which cover \( \bigwedge_n A_n \). Therefore \( \bigwedge_n A_n \) is fuzzy compact, which proves (a).

Let \( G \) be any fuzzy open set in \( (X, F) \) with \( \bigwedge_n A_n \leq G \). Suppose that for each \( n \), \( \bigwedge_{i \leq n} A_i \neq G \). Take \( K_n = \bigwedge_{i \leq n} A_i \). For each \( n \), choose \( x_n \in X \) such that \( K_n(x_n) > G(x_n) \). Let \( \alpha_n = K_n(x_n) \). Then the fuzzy points \( (x_n)_{\alpha_n} \), where \( (x_n)_{\alpha_n} \) is a fuzzy point with support \( x_n \) and value \( \alpha_n \), are such that \( (x_n)_{\alpha_n} \leq K_n \) and \( (x_n)_{\alpha_n} \neq G \). Since sequence \( (K_n) \) is decreasing, the set of fuzzy points \( \{(x_n)_{\alpha_n} : n \in \mathbb{N}\} \) has a cluster point, say \( x_\alpha \). Now \( \bigwedge_n K_n = \bigwedge_n A_n \) which is fuzzy compact, so that \( x_\alpha \leq \bigwedge_n K_n \). That is \( x_\alpha \leq \bigwedge_n A_n \leq G \), which is a contradiction to the choice of \( (x_n)_{\alpha_n} \). Thus for some \( n \), \( \bigwedge_{i \leq n} A_i \leq G \), which proves (b).

(Sufficiency)

Assume that there exists a sequence \( \{A_n\} \) of fuzzy open covers of \( (X, F) \) with satisfies (a) and (b). Let \( A_n \in \mathcal{A}_n \). Then for each \( \alpha \in [0, 1] \),
\(A_n^{-1}(\alpha, 1) \in \mathcal{L}(F)\). Take \(U_{(\alpha)} = A_n^{-1}(\alpha, 1)\). Then \(U_{(\alpha)} \subset X\) and let \(U^*_{(\alpha)} = \beta X - CL(X - U_{(\alpha)})\). Now \(U^*_{(\alpha)} \in T\), \(\beta X\) is the Stone-Čech compactification of \((X, \mathcal{L}(F))\) and \(U_{(\alpha)} = U^*_{(\alpha)} \cap X\). Define \(B_{A_n} : \beta X \to [0, 1]\) by \(B_{A_n}(x) = \sup \{\alpha : x \in U^*_{(\alpha)}\}\). Then for \(x \in X\), \(B_{A_n}(x) = \sup \{\alpha : x \in U_{(\alpha)}\} = A_n(x)\).

Therefore \(B_{A_n} \chi_X = A_n\). Now \(B_{A_n}\)'s are lower semicontinuous [see sufficiency part of theorem 4.2.3]. That is \(B_{A_n} \in F_T\). Take \(\mathcal{B}_n = \{ B_{A_n} \mid A_n \in \mathcal{A}_n\}\). Then \(\mathcal{B}_n\) forms a sequence of families of fuzzy open sets on \((\beta X, F_T)\) which satisfies condition (i) for a fuzzy p-space.

Now we show that \(\bigwedge_n st(x_\alpha, \mathcal{B}_n) \leq \chi_X\) for every fuzzy point \(x_\alpha\) with support \(x \in X\) and value \(\alpha \in (0, 1]\). For \(A_n \in \mathcal{A}_n\) we have \(A_n^* \leq B_{A_n}\) where

\[
A_n^*(x) = \begin{cases} \overline{A_n(x)} & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases}
\]

Let \(x_\alpha \leq A_n \leq \mathcal{A}_n\). Then \(\overline{A_n}^* \leq \bigwedge_n st(x_\alpha, \mathcal{B}_n) \rightarrow (1)\)

Now suppose that \(\bigwedge_n st(x_\alpha, \mathcal{B}_n) \neq \overline{A_n}^*\). Then there exists some \(y \in \beta X\) such that \(\bigwedge_n st(x_\alpha, \mathcal{B}_n)(y) > \overline{A_n}^*(y)\). Let \(\bigwedge_n st(x_\alpha, \mathcal{B}_n) = \gamma\). Therefore \(y \leq \bigwedge_n st(x_\alpha, \mathcal{B}_n)\), but \(y \neq \overline{A_n}^*\). Since \(\overline{A_n}^*\) is fuzzy compact, there exists a fuzzy...
open set $W$ in $\beta X$ with $y_1 \leq W$ and $\overline{W} \wedge (\bigwedge_{i=1}^{n} \overline{A}_i^*) = 0$. Therefore $\overline{W} \wedge (\bigwedge_{i=1}^{n} \overline{A}_i^*) = 0 \Rightarrow (2)$ by (b). We claim that $\overline{W} \wedge (\bigwedge_{i=1}^{n} \text{st}(x_1, \overline{A}_i^*)) = 0$ where $\overline{A}_i^* = \{ \overline{A}_i^* | A_i \in \mathcal{A}_i \}$. For if $\{ \overline{W} \wedge (\bigwedge_{i=1}^{n} \text{st}(x_1, \overline{A}_i^*)) \} (z) > 0$, then $\overline{W} (z) > 0$ and $\bigwedge_{i=1}^{n} \text{st}(x_1, \overline{A}_i^*) (z) > 0$. Hence $\text{st}(x_1, \overline{A}_i^*) (z) > 0$ for all $i = 1, 2, \ldots, n$.

Therefore we can choose $\overline{A}_i^* \in \mathcal{A}_i^*$ with $\overline{A}_i^* (z) > 0$ for all $i = 1, 2, \ldots, n$, which is a contradiction to equation (2). Thus $\overline{W} \wedge (\bigwedge_{i=1}^{n} \text{st}(x_1, \overline{A}_i^*)) = 0$. In particular $\overline{W} \wedge (\bigwedge_{i=1}^{n} \text{st}(x_1, \overline{A}_n^*)) = 0$. Therefore $\overline{W} \wedge (\bigwedge_{i=1}^{n} \text{st}(x_1, \mathcal{B}_n)) = 0$. Therefore $y_1 \neq \bigwedge_{n} \text{st}(x_1, \mathcal{B}_n)$, which is a contradiction.

Hence $\bigwedge_{n} \overline{A}_n^* > \bigwedge_{n} \text{st}(x_1, \mathcal{B}_n) \Rightarrow (3)$. Thus from (1) and (3) it follows that $\bigwedge_{n} \overline{A}_n^* = \bigwedge_{n} \text{st}(x_1, \mathcal{B}_n)$. For $x \in X$, $A_1^* (x) = A_1 (x)$ and $\bigwedge_{n} \overline{A}_n^*$ is fuzzy compact. Therefore it follows that $\bigwedge_{n} \text{st}(x_1, \mathcal{B}_n) \leq \mathcal{K}_X$, which is condition (ii) for a fuzzy $p$-space.

Hence $(X, F)$ is a fuzzy $p$-space.

**Definition 4.3.2**

A fuzzy topological space $(X, F)$ is called a fuzzy $k$-space if the fuzzy set $A$ is closed in $(X, F)$ whenever $(A \wedge K) / Y : Y \rightarrow [0,1]$ is a closed fuzzy set in $(Y, F_Y)$, where $Y = \text{supp } K$ and $F_Y = \{ A / Y | A \in F \}$, for each compact fuzzy set $K$. 67
Theorem 3.3

Every regular fuzzy p-space is a fuzzy k-space.

Proof

Let \((X, F)\) be a fuzzy p-space. Let \(A\) be not a closed fuzzy set on \((X, F)\). We find a compact fuzzy set \(K\) in \(X\) such that \((A \land K)/Y\) is not a closed fuzzy set on \((Y, F_Y)\). By theorem 4.3.1 there exists a \((\mathcal{A}_n)\) of fuzzy open covers of \((X, F)\) such that for each \(n\), fuzzy point \(x_{a_n}\) with support \(x \in X\) value \(a \in (0,1]\) and \(x_{a_n} \leq A_n \leq \mathcal{A}_n\),

\[
\bigwedge_n \tilde{A}_n
\]

is fuzzy compact. (b) \(\{\bigwedge_i \tilde{A}_i \mid n \in \mathbb{N}\}\) is an outer network for the fuzzy set \(\bigwedge_n \tilde{A}_n\).

Let \(y \in \text{support of } \tilde{A}\) \(\text{support of } A\) and \(a \in (0,1]\). For each \(n\), choose a fuzzy point \(y_{a_n}\) and a fuzzy open set \(G_n\) with \(y_{a_n} \subseteq G_n\) and \(A_n \in \mathcal{A}_n\) such that \(y_{a_n} \subseteq G_{n+1} \subseteq G_n \subseteq \bigwedge_i \tilde{A}_i \subseteq \bigwedge_n \tilde{A}_n\). Then \(\bigwedge_n G_n = \bigwedge_n \tilde{G}_n \leq \bigwedge_n \tilde{A}_n\). By (a) \(\bigwedge_n \tilde{G}_n\) is fuzzy compact and hence \(\bigwedge_n G_n\) is fuzzy compact. If \((x_n)_{a_n}\) denote a fuzzy point with support \(x_n\) and value \(\alpha\) and \((x_n)_{a_n} \leq G_n\), then by (b), \(\{(x_n)_{a_n} : n \in \mathbb{N}\}\) has a cluster point. This is because for every fuzzy open set \(W\) with \(\bigwedge_n \tilde{A}_n \leq W\) there exists some \(n \in \mathbb{N}\) with \(\bigwedge_i \tilde{A}_i \leq W\). Therefore all but finitely many fuzzy points of \((x_n)_{a_n}\) are such that \((x_n)_{a_n} \leq W\). Hence some point of \(\bigwedge_n \tilde{A}_n\) must be a cluster point.
Hence by lemma 2.3.5, \{G_n \mid n \in \mathbb{N}\} is a base for the fuzzy compact set \(K = \bigwedge_n G_n\). If \((A \land K)/Y\) is not a closed fuzzy set on \((Y, F_Y)\), then the proof is over.

Suppose that \((A \land K)/Y\) is a closed fuzzy set on \((Y, F_Y)\). Then there exists a closed fuzzy set \(B\) on \((X, F)\) such that \((A \land K)/Y = B/Y\). Denote \(B\) as \((A \land K)\).

Let \((H_n)\) be a sequence of fuzzy open sets such that \(y_n \leq H_n\) for all \(n\) and \(\overline{H}_{n+1} \leq H_n\) and \(H_0 \land (A \land K) = 0\). Then as above we can show that \(\{G_n \land H_n \mid n \in \mathbb{N}\}\) is a base for the fuzzy compact set \(K' = \bigwedge_n (G_n \land H_n)\). Also \(K' \land A = 0\). Choose fuzzy point \((x_n)_\alpha\) with \((x_n)_\alpha \leq G_n \land H_n \land A\). Let \(K'' = K' \lor \{(x_n)_\alpha \mid n \in \mathbb{N}\}\). Then we show that \(K''\) is fuzzy compact and \(K'' \land A\) is not a closed fuzzy set on \((X, F)\), so that \((K'' \land A)/Y\) is not a closed fuzzy set on \((Y, F_Y)\), which will complete the proof.

Now \(G_n \land H_n \land A\) is decreasing and since \(K' = \bigwedge_n (G_n \land H_n)\) is fuzzy compact, \(\{(x_n)_\alpha \mid n \in \mathbb{N}\}\) has a cluster point \(x_\alpha\) such that \(x_\alpha \leq K'\). Therefore any fuzzy open cover of \(K''\) contains a finite sub-cover of the compact fuzzy set \(K'\) and at least one member in the sub-cover, say \(G\), is such that \(x_\alpha \leq G\). Therefore all but finitely many \((x_n)_\alpha\) are such that \((x_n)_\alpha \leq G\). Hence the finite sub-cover of \(K'\) together with fuzzy open set corresponding to those finitely many \((x_n)_\alpha \neq G\), forms a finite sub-cover of the original cover of \(K''\). Hence \(K''\) is fuzzy compact. But \(K'' \land A\) is not a closed fuzzy set on \((X, F)\), since the cluster point \(x_\alpha\) of \(\{(x_n)_\alpha \mid n \in \mathbb{N}\}\) is such that \(x_\alpha \leq K'\) and \(K' \land A = 0\). Hence the theorem.