CHAPTER 2
FUZZY WΔ-SPACES AND FUZZY MOORE SPACES

2.1 Introduction

The notion of generalized metric spaces is closely related to metrization theory. The concepts like wΔ-spaces, developable spaces and Moore spaces were extensively studied by various authors, as a part of the study of generalized metric spaces. In this chapter we introduce fuzzy submetacompact spaces, fuzzy subparacompact spaces, fuzzy wΔ-spaces, fuzzy Moore spaces and investigate some of their properties.

2.2 Fuzzy sub metacompact spaces.

A topological space $X$ is submetacompact if for each open cover $\mathcal{U}$ of $X$ there is a sequence $(\psi_n)$ of open refinements of $\mathcal{U}$ such that for each $x \in X$, there exists $n \in \mathbb{N}$ such that $x$ is in only finitely many elements of $\psi_n$. In this section we define fuzzy submetacompact spaces and study some of its properties.

We have included some results of this chapter in the paper titled 'On Fuzzy wΔ-Spaces and fuzzy Moore Spaces' Journal of Thirupura Mathematical Society Vol.4 (2002) 47-52.
Definition 2.2.1

A sequence $(\mathcal{A}_n)$ of fuzzy open covers of a fuzzy topological space $(X, F)$ is called a $G_\delta^*$-diagonal sequence if for each $x \in X$, $\alpha \in (0, 1]$ $\bigwedge_n \text{st}(x_\alpha, \mathcal{A}_n) = \bigwedge_n \text{st}(x_\alpha, \mathcal{A}_n) = x_\alpha$.

Definition 2.2.2

A fuzzy topological space $(X, F)$ is said to be fuzzy submetacompact, if for each fuzzy open cover $\mathcal{A}$ of $X$, there exists a sequence $(\psi_n)$ of fuzzy open refinements of $\mathcal{A}$ such that, for each fuzzy point $x_\alpha$, $x \in X$, $\alpha \in (0, 1]$, there exists $n \in \mathbb{N}$ such that $x_\alpha \leq v_n \in \psi_n$ holds for finitely many elements of $\psi_n$.

Theorem 2.2.3

A regular fuzzy submetacompact space with a $G_\delta$-diagonal has a $G_\delta^*$-diagonal.

Proof

Let $(X, F)$ be a regular fuzzy submetacompact space with a $G_\delta$-diagonal. Then there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of $(X, F)$ such that for a fuzzy point $x_\alpha$, $\bigwedge_n \text{st}(x_\alpha, \mathcal{A}_n)$ $= x_\alpha$. Consider $\mathcal{A}_1$. Then by submetacompactness of $X$, $\mathcal{A}_1$ has a sequence of open refinements say $(U_{1n})_{n \in \mathbb{N}}$. 

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such that for each fuzzy point $x_a$ there exists $n \in \mathbb{N}$ such that $x_a$ is only in finitely many elements of $U_{1n}$. Let $U_{11}$ be one such open refinement corresponding to the fuzzy point $x_a$. By regularity of $X$, for each $U_{1n}$, we can find an open refinement, say $\psi_{1,n}$ such that, for $x_a \leq U_{1n} \in U_{1n}$, there exists $V_{1n} \in \psi_{1,n}$ with $x_a \leq V_{1n} \leq \overline{V_{1n}} \leq U_{1n}$.

Similarly for $A_2$, by submetacompactness, there exist $(B_{2n})_{n \in \mathbb{N}}$ and by regularly each $B_{2n}$ has an open refinement $U_{2n}$. Then take $(\psi_{2,n})_{n \in \mathbb{N}}$ as follows.

$$\psi_{2,n} = U_{2n} \cap \psi_{1,1} = \{ U \cap V | U \in U_{2n}, V \in \psi_{1,1} \}.$$ For $A_3$, by submetacompactness, there exists a sequence of open refinements $(B_{3n})_{n \in \mathbb{N}}$ and by regularity each $B_{3n}$ has an open refinement $U_{3n}$. Take $(\psi_{3,n})_{n \in \mathbb{N}}$ as follows.

$$\psi_{3,n} = U_{3n} \cap \psi_{1,1} \cap \psi_{1,2} \cap \psi_{2,1} \cap \psi_{2,2}$$

Repeating this process for each $m$, we have a sequence $(\psi_{mn})_{n \in \mathbb{N}}$ of open covers of $(X, \mathcal{F})$ such that

(i) $(\psi_{mn})_{n \in \mathbb{N}}$ is a refinement of each $\psi_{ij}$ such that $i < m$, $j < m$ and for each fuzzy point $x_a$ there exists $n \in \mathbb{N}$ such that $x_a$ is in only finitely many members of $\psi_{mn}$.
(ii) If $V \in \psi_{mn}$ and $i, j < m$ there exists $w \in \psi_{ij}$ such that, $\overline{V} \leq W$ and for $k \leq m$ there exists $A \in \mathcal{H}_k$ such that $\overline{V} \leq A$.

Let $y_a \leq \bigwedge_{ij} \text{st}(x_a, \psi_{ij})$, for the fuzzy points $x_a, y_a$. Fix $i$ and $j$ and let $m > \max \{i, j\}$. Now the fuzzy point $x_a$ is in only finitely many members of $\psi_{mn}$ for some $n \in \mathbb{N}$.

Therefore $y_a \leq \text{st}(x_a, \psi_{mn}) = \bigvee \{ \overline{V} : x_a \leq V \in \psi_{mn} \}$

$\leq \bigvee \{ w : x_a \leq V \leq \overline{V} < w \in \psi_{ij} \}$, by (ii)

$= \text{st}(x_a, \psi_{ij})$

Therefore, for each $i, j \in \mathbb{N}$, $y_a \leq \bigwedge_{ij} \text{st}(x_a, \psi_{ij}) \Rightarrow y_a \leq \text{st}(x_a, \psi_{ij})$.

Therefore by using (i) $\bigwedge_{ij} \text{st}(x_a, \psi_{ij}) = \bigwedge_{ij} \text{st}(x_a, \psi_{ij}) \leq \bigwedge_{n} \text{st}(x_a, \mathcal{A}_n) = x_a$.

Therefore $(\psi_{ij})_{i,j \in \mathbb{N}}$ is a $G_{\delta}^*$-diagonal sequence for $(X, F)$. Hence $(X, F)$ has a $G_{\delta}^*$-diagonal.

### 2.3 Fuzzy w\(\Delta\)-Spaces and Fuzzy Developable Spaces.

In this section we define fuzzy developable spaces, fuzzy w\(\Delta\)-spaces and find some relationship between them.
Definition 2.3.1

Let $(X, F)$ be a fuzzy topological space. A fuzzy point $x_\alpha, \alpha \in (0,1]$ is said to be a cluster point of the set $\{(x_n)_\alpha : n \in \mathbb{N}\}$, where $(x_n)_\alpha$ is a fuzzy set with support $x_n$ and value $\alpha$, if for each fuzzy set $G \in F$ such that $x_\alpha \not\subseteq G$, there exists $n_0 \in \mathbb{N}$ with $x_{n_0} \neq x$ and $(x_{n_0})_\alpha \subseteq G$.

Definition 2.3.2

A fuzzy topological space $(X, F)$ is called a fuzzy w$\Delta$-space if there exists sequence $(\mathcal{A}_n)$ of fuzzy open covers of $X$ such that for each $n \in \mathbb{N}$, fuzzy points $(x_n)_\alpha$ with support $x_n \in X$ and value $\alpha$ and $(x_n)_\alpha \subseteq st(x_\alpha, \mathcal{A}_n)$, set $\{(x_n)_\alpha : n \in \mathbb{N}\}$ has a cluster point.

Definition 2.3.3

A sequence $(\mathcal{A}_n)$ of fuzzy open covers of $(X, F)$ is a fuzzy development for $X$, if for $\alpha \in (0,1]$, a fuzzy point $x_\alpha$, the set $\{st(x_\alpha, \mathcal{A}_n) : n \in \mathbb{N}\}$ is a base at $x_\alpha$.

A fuzzy topological space $(X, F)$ is fuzzy developable if it has a fuzzy development.

Example 2.3.4

Consider $X = \mathbb{R}$ with usual topology $T$. Let $F$ be the topology generated by the set $\{\chi_U | U \text{ open in the usual topology on } \mathbb{R}\}$. For each $x \in X$, consider $\chi_{\left(\frac{x-\frac{1}{n}}{n}, \frac{x+\frac{1}{n}}{n}\right)}$ and form...
\( \mathcal{A}_n = \{ \chi_{(\frac{x-1}{n}, \frac{x+1}{n})} \mid x \in X \} \). Then \( (\mathcal{A}_n) \) forms a sequence of fuzzy open covers of 

\((X, F)\). For \( \alpha \in (0, 1] \), a fuzzy point \( x_\alpha \), \( \text{st} (x_\alpha, \mathcal{A}_n) = \vee \{ A_n \in \mathcal{A}_n \mid A_n(x) \geq \alpha \} \)

\[ = \vee \{ \chi_{(\frac{x-1}{n}, \frac{x+1}{n})} \mid \chi_{(\frac{x-1}{n}, \frac{x+1}{n})}(x) \geq \alpha \} \]

\[ = \chi_{(\frac{x-\frac{2}{n}}{n}, \frac{x+\frac{2}{n}}{n})}. \]

If \( G \) is an open fuzzy set in \((X, F)\) and \( x_\alpha \leq G \), then there exists \( U \in \mathcal{T} \) such that \( x_\alpha \leq \chi_U \leq G \). Now \( x \in U \) so that there exists \( n \in \mathbb{N} \) such that \( \left( x - \frac{2}{n}, x + \frac{2}{n} \right) \subset U \).

Therefore \( x_\alpha \leq \chi_{(\frac{x-\frac{2}{n}}{n}, \frac{x+\frac{2}{n}}{n})} \leq \chi_U \leq G \). Hence the set \( \{ \text{st} (x_\alpha, \mathcal{A}_n) : n \in \mathbb{N} \} \) is a base at \( x_\alpha \). Therefore \((X, F)\) is a fuzzy developable space.

For \( \alpha \in (0, 1] \), if we choose fuzzy points \((x_n)_\alpha \) with \( (x_n)_\alpha \leq \text{st}(x_\alpha, \mathcal{A}_n) \), then \( x_\alpha \) is a cluster point of the set \( \{(x_n)_\alpha : n \in \mathbb{N} \} \). Therefore \((X, F)\) is a fuzzy \( w \Delta \)-space.

Lemma 2.3.5

Let \((X, F)\) be a fuzzy topological space. Suppose \( \{ U_n \} \) is a decreasing sequence of fuzzy open sets such that \( \bigwedge_n U_n = \overline{\bigcap_n U_n} \) and for \( \alpha \in (0,1) \), fuzzy points \((x_n)_\alpha \) with \( (x_n)_\alpha \leq U_n \) implies the set \( \{(x_n)_\alpha : n \in \mathbb{N} \} \) has a cluster point.
Then \( \{U_n\} \) is a base for the fuzzy set \( \bigwedge_n U_n \). (That is for every open fuzzy set \( V \) with \( \bigwedge_n U_n \leq V \), there exists some \( U_n \) such that \( U_n \leq V \)).

**Proof**

Suppose that \( \{U_n\} \) satisfies the hypothesis of the lemma, but not the conclusion. Then we can find a fuzzy open set \( V \) such that \( \bigwedge_n U_n \leq V \) and for each \( n \), there exists fuzzy points \( (x_n)_\alpha \) with \( (x_n)_\alpha \leq U_n \), but \( (x_n)_\alpha \not\leq V \). Since \( \{U_n\} \) is decreasing, any cluster point of \( \{(x_n)_\alpha : n \in \mathbb{N}\} \) must be in \( \bigwedge_n U_n \) and \( \bigwedge_n U_n = \bigwedge_n \overline{U_n} \). But as \( \bigwedge_n U_n \leq V \), this implies that the set \( \{(x_n)_\alpha : n \in \mathbb{N}\} \) has no cluster point, which is a contradiction. Thus for every open fuzzy set \( V \) with \( \bigwedge_n U_n \leq V \), there exists \( U_n \) such that \( U_n \leq V \).

**Theorem 2.3.6**

A regular fuzzy topological space \((X, F)\) is Fuzzy developable if and only if it is a fuzzy \( w\Delta \) -space with a \( G_\delta^* \) - diagonal.

**Proof**

First suppose that \((X, F)\) is a fuzzy developable space. Let \((\mathcal{A}_n)\) be a fuzzy development. Then for \( \alpha \in (0,1] \), a fuzzy point \( x_\alpha \) with \( (x_n)_\alpha \leq \text{st} (x_\alpha, \mathcal{A}_n) \), the set \( \{(x_n)_\alpha : n \in \mathbb{N}\} \) has a cluster point \( 'x_\alpha' \). This is because the set
\{\text{st}(x_\alpha, \mathcal{A}_n) : n \in \mathbb{N}\} \text{ forma a base at } x_\alpha. \text{ Therefore if } x_\alpha \leq G \text{ with } G \in \mathcal{F} \text{ there exists } n \in \mathbb{N} \text{ such that } x_\alpha \leq \text{st}(x_\alpha, \mathcal{A}_n) \leq G. \text{ Hence } (x_n)_\alpha \leq G. \text{ Therefore } (X, \mathcal{F}) \text{ is a fuzzy w}_\Delta \text{- space. Also for } x \neq y, \alpha \in (0,1], \{y_\alpha\}' \text{ is an open fuzzy set and } x_\alpha \leq \{y_\alpha\}'. \text{ Therefore } x_\alpha \leq \text{st}(x_\alpha, \mathcal{A}_n) \leq \{y_\alpha\}' \text{ for some } n. \text{ Hence } y_\alpha \neq \text{st}(x_\alpha, \mathcal{A}_n) \text{ for some } n. \text{ Since } (X, \mathcal{F}) \text{ is regular it follows that } \bigwedge_\alpha \text{st}(x_\alpha, \mathcal{A}_n) = x_\alpha. \text{ Therefore } (X, \mathcal{F}) \text{ has a G}_5^* \text{- diagonal.}

Conversely assume that \((X, \mathcal{F})\) is a fuzzy w\(\Delta\)-space with a G\(_5^\ast\) - diagonal. Let \((\mathcal{A}_n)\) be a sequence of fuzzy open covers of \(X\) such that, for \(\alpha \in (0,1], \) a fuzzy point \(x_\alpha\) with \((x_n)_\alpha \leq \text{st}(x_\alpha, \mathcal{A}_n), \) the set \(\{(x_n)_\alpha : n \in \mathbb{N}\}\) has a cluster point and \(\bigwedge_\alpha \text{st}(x_\alpha, \mathcal{A}_n) = x_\alpha. \) In lemma 2.3.5, take \(U_n = \text{st}(x_\alpha, \mathcal{A}_n). \) By passing onto refinement, one can make \(\{U_n\}\) decreasing. Therefore by lemma 2.3.5 \(\{\text{st}(x_\alpha, \mathcal{A}_n) : n \in \mathbb{N}\}\) is a base at \(x_\alpha. \) Thus \((\mathcal{A}_n)\) is a fuzzy development for \((X, \mathcal{F}).\) Hence \((X, \mathcal{F})\) is fuzzy developable.

### 2.4 Fuzzy Moore Spaces

A topological space \(X\) is said to be subparacompact if for each open cover \(\mathcal{U}\) of \(X\) there exists a sequence \((\psi_n)\) of open covers such that, for each \(x \in X, \) there exists \(n \in \mathbb{N}\) such that \(\text{st}(x, \psi_n)\) is contained in some members of \(\mathcal{U}.\)
In this section we define fuzzy subparacompact spaces, fuzzy Moore spaces and study their properties.

**Definition 2.4.1**

A fuzzy topological space $(X,F)$ is said to be fuzzy subparacompact if for every fuzzy open cover $U$ of $X$, there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of $X$ such that for $\alpha \in (0,1]$, a fuzzy point of $x_\alpha$, there exists $n \in \mathbb{N}$ such that $st(x_\alpha, \mathcal{A}_n) \leq U_n$ for some $U_n \in U$.

**Remark 2.4.2**

Every fuzzy subparacompact space is a fuzzy submetacompact space.

**Proof**

Let $U$ be any fuzzy open cover of the fuzzy subparacompact space $(X,F)$. Then by the definition there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of $X$ such that for $\alpha \in (0,1]$, a fuzzy point of $x_\alpha$, there exists $n \in \mathbb{N}$ such that $st(x_\alpha, \mathcal{A}_n) \leq U_n$ for some $U_n \in U$. Take $\psi_n = \{ st(x_\alpha, \mathcal{A}_n) : x \in X, \alpha \in (0,1] \}$. Then $(\psi_n)$ forms a sequence of open refinements of $U$ such that for each fuzzy point of $x_\alpha$, there exists only one $st(x_\alpha, \mathcal{A}_n)$ such that $x_\alpha \leq st(x_\alpha, \mathcal{A}_n) \in \psi_n$. Therefore $(X,F)$ is a fuzzy submetacompact space.
Definition 2.4.3

A fuzzy topological space \((X, F)\) is said to be a fuzzy Moore space, if it is regular and fuzzy developable.

Remark

By Theorem 2.3.6 it follows that \((X, F)\) is a Moore space if and only if it is a fuzzy \(w\Delta\) -space with a \(G_\delta\) * - diagonal.

Example 2.4.4

In Example 2.3.4 \(\{\text{st}(x_a, \mathcal{A}_n) : n \in \mathbb{N}\}\) forms a base at \(x_a\). Let \(U\) be any open cover of \(X\). Then for fuzzy point \(x_a\) there exists \(U \in U\) such that \(x_a \leq U\). Then there exists \(n \in \mathbb{N}\) such that \(\text{st}(x_a, \mathcal{A}_n) \leq U\). Therefore \((X, F)\) is fuzzy subparacompact. Also \((X, F)\) is regular and fuzzy developable. Therefore it follows that \((X, F)\) is a fuzzy Moore space.

Remark 2.4.5

Every fuzzy Moore space is a fuzzy subparacompact space.

Proof

Let \((X, F)\) be a fuzzy Moore space. Let \((\mathcal{A}_n)\) be a development for \(X\) and let \(U\) be any open cover of \((X,F)\). For \(\alpha \in (0,1]\), a fuzzy point of \(x_a\), \(\{\text{st}(x_a, \mathcal{A}_n) : n \in \mathbb{N}\}\) forms a base at \(x_a\). Therefore if \(x_a \leq U\) with \(U \in U\)
there exists \( n \in \mathbb{N} \) with \( \text{st} (x, \mathcal{A}_n) \subseteq U \). Hence \((X, F)\) is a fuzzy subparacompact space.

**Theorem 2.4.6**

A regular fuzzy topological space \((X, F)\) is a fuzzy Moore space if and only if it is a fuzzy submetacompact, fuzzy \( \omega \)-space with a \( G_{\delta} \)-diagonal.

**Proof**

First assume that \((X, F)\) is a fuzzy Moore space. Then \((X, F)\) is fuzzy subparacompact [by remark 2.4.5]. Therefore \((X, F)\) is fuzzy submetacompact [by Remark 2.4.2]. Since \((X, F)\) is fuzzy developable, by Theorem 2.3.6, it is a fuzzy \( \omega \)-space with a \( G_{\delta} \)-diagonal and hence a fuzzy \( \omega \)-space with a \( G_{\delta} \)-diagonal.

Conversely assume that \((X, F)\) is a fuzzy submetacompact, fuzzy \( \omega \)-space with a \( G_{\delta} \)-diagonal. By Theorem 2.3.3, \((X, F)\) has a \( G_{\delta} \)-diagonal. Therefore it follows that \((X, F)\) is a fuzzy \( \omega \)-space with a \( G_{\delta} \)-diagonal and hence fuzzy developable, by Theorem 2.3.6. Hence as \((X, F)\) is regular, it is a fuzzy Moore space.