3.1 INTRODUCTION

A business either manufactures the products it sells or it purchases the products it sells from other businesses. In either case, an increase in inventory usually involves a corresponding increase in accounts payable. Raw materials used in the production process are purchased on credit, and many other manufacturing costs are not paid for immediately. Products from other businesses are bought on credit, instead of making immediate cash payment when inventory is increased, a business delays payment, perhaps by a month or so.

In today’s business tractions, it is more and more common to see that the customers are allowed some grace period before they settle the account with the supplier. This gives a advantage to the customers due to fact that they do not have to pay the supplier immediately after receiving the product, but instead, they can delay their payment until the end of the allowed period. The customer pays no interest during the fixed period they have to settle the account, but if the payment is delayed beyond that period, interest will be charged. The permissible delay in payments brings some advantages to the buyer, as he would try to earn some interest for the payment received during this period. When a supplier allows a fixed time for settling the account, he is actually giving a loan to the buyer without interest during this period.
Therefore, it is economical to delay in the settlement of accounts to the last moment of the permissible delay in payments.

The model incorporating the facility of permissible delay in payment was first discussed by Haley and Higgins (1973). Goyal (1985) explored a single item EOQ model under permissible delay in payments. In real life situations, there are products like volatile liquids, medicines, and materials, etc. in which the rate of deterioration is very large. Therefore, the loss due to deterioration should not be ignored. Aggarwal and Jaggi (1995) extended Goyal (1985) model to allow for deteriorating items. Shortages are of great importance especially in a model that considers a delay in payment due to the fact that shortages can affect the quantity ordered to benefit from the delay in payment. Jamal et al. (1997) generalized the model of Aggarwal and Jaggi (1995) to allow for shortages and make it more applicable in real the world.

In the above cited references, all the models were developed for a single warehouse and assumed that the available warehouse had unlimited capacity. However, this assumption is debatable in real life situations. A common practical situation is that of a limited storage space, whereas the extra storage capacity can be acquired in the form of rented warehouse if the retailer’s existing storage capacity is insufficient to store the ordered quantity. Motivated by the trade credit policy given by supplier, first time Shah and
Shah (1992) developed a deterministic inventory model under the permissible delay in payment with two storage facilities. In that model, the items were considered non-deteriorating without shortage and the time horizon was infinite. Chuang and Huang (2004) extended Goyal (1985) model by considering limited storage capacity of the retailer in which the items were considered non-deteriorating. In that model, shortage was not allowed and the time horizon was infinite with constant demand rate. Ouyang et al. (2006) developed an inventory model for deteriorating items with permissible delay in payments. The purpose of that study was to find an optimal replenishment policy for minimizing the total relevant inventory cost. Chuang and Huang (2007) presented a two-warehouse inventory model for deteriorating items under trade credit financing. In that model, shortages were not allowed and time horizon was infinite. The rate of deterioration in both warehouses was considered the same. Singh S.R. and Singh S. (2007) considered a lot size model for items having linear demand under the effect of permissible delay and inflation. Singh S.R. and Singh S. (2008) considered a production model for items under the effect of inflation and permissible delay in payments. Singh et al. (2008) presented a two-warehouse inventory model for deteriorating items with constant demand rate where shortages were allowed and partially backlogged. Singh and Jain
(2009) proposed a deterministic inventory model with time varying deterioration rate and a linear trend in demand over a finite planning horizon. They assumed that the supplier offers a credit limit to the retailer during which no interest is charged. However, the retailer has the reserve capital with him to make the payments at the beginning of the transaction, but he decides to take the benefit of the credit limit. Each cycle has shortages, which have been partially backlogged to suit present day competition in the market.

Numerical examples have been presented to explain the theory, while sensitivity of the optimal solution of the system has been studied with respect to various system parameters.

Geetha and Uthaya Kumar (2010) developed an Economic Order Quantity (EOQ) model for deteriorating items with permissible delay in payments and single storage facility. This model aids in minimizing the total inventory cost by finding an optimal replenishment policy. In that model, shortages were allowed and partially backlogged.

In many real-life situations, for certain types of consumer goods (e.g., fruits, vegetables, donuts, and others) the consumption rate is sometimes influenced by the stock-level. It is usually observed that a large pile of goods on shelf in a supermarket will lead the customer to buy more, this occurs because of its visibility, popularity or variety and then generate higher
demand. The consumption rate may go up or down with the on-hand stock level. These phenomena attract many marketing researchers to investigate inventory models related to stock-level. Conversely, low stocks of certain baked goods (e.g., donuts) might raise the perception that they are not fresh. Therefore, demand is often time and inventory-level dependent. In the last several years, a considerable body of literature has been written in the operational research area on how inventory-level-dependent demand should affect inventory control policies. The related analysis on such inventory system with stock-dependent consumption rate was studied by Levin et al. (1972) that “large piles of consumer goods displayed in a supermarket will lead customers to buy more. Yet, too many goods piled up in everyone’s way leaves a negative impression on buyers and employees alike.” Silver and Peterson (1985) also noted that sales at the retail level tend to be proportional to the amount of inventory displayed. To quantify this, Baker and Urban (1988) established an economic order quantity model (or EOQ) for a power-form inventory-level dependent demand pattern (i.e., the demand rate at time t is $D(t) = \alpha [I(t)]^\beta$, where $I(t)$ is the inventory level, $\alpha > 0$, and $0 < \beta < 1$). Mandal and Phaujdar (1989) then incorporated deteriorating items with linearly stock-dependent demand. Datta and Pal (1990) modified the model of Baker and Urban (1988) by assuming that the stock-dependent
demand rate was down to a given level of inventory, beyond which it is a constant. By their assumption, not all customers are attracted to purchase goods by the huge stock. When the stock level declines to a certain level, customers arrive to purchase goods because of its goodwill, good quality or facilities. They presented an EOQ model in which the demand rate was dependent on the instantaneous stock amount displayed until a given level of inventory $L$ is achieved. After this given inventory level, the demand rate becomes constant (i.e., $D(t) = \alpha[I(t)]^\beta$ if $I(t) > L$ and $D(t) = \alpha L^\beta$ if $0 \leq I(t) \leq L$). Bar-Lev et al. (1994) developed an extension of the inventory-level-dependent demand-type EOQ model with random yield. Some of the other related works in this area are: Dutta et al. (1998), Dye (2002), Chuang (2003), Zhou and Yang (2005), Alfares (2007), Goyal and Chang (2009) and Yang et al. (2010).

It is observed that the prices of everything going up over the years. Inflation is a rise in the general level of prices of goods and services in an economy over a period of time. Inflation can also be described as a decline in the real value of money, a loss of purchasing power in the medium of exchange. So it is more realistic to consider the inflation in our model. In this paper most of the factors which effect the inventory management are considered. Many researchers have done the work on supply chain by
considering a constant holding cost, which is not actually considerable in case of perishable items like food products.

This chapter deals a supply chain inventory model for decaying items with time & stock dependent demand rate. To make the study more practical, the effect of permissible delay in payment and inflation has been taken into account. In real life situation, the storage cost varies with time. The storage time can be classified into different ranges, each with its distinctive unit holding cost. Thus, two times dependent holding cost step functions are considered that is retroactive and incremental holding cost. Shortages are allowed in inventory with partial backlogging. We provided the solution procedure for finding the total cost. Further we use a numerical example to illustrate the model and sensitivity analysis.

3.2 ASSUMPTIONS AND NOTATIONS

The inventory model is developed on the basis of the following assumptions and notations.

3.2.1 ASSUMPTIONS

(i) Single vendor, single buyer with one item is assumed.

(ii) Single item with constant deteriorating rate of the on hand inventory in considered.
(iii) The holding cost is varying as an increasing step function of time in storage.

(iv) Shortages are allowed with partial backlogging.

(v) There is no replacement or repair of deteriorated units.

(vi) A stationary policy where the same lot size is assumed.

(vii) The production rate is finite and is greater than the sum of all the buyers demand.

(viii) Demand rate is depending on stock & time.

(ix) Inflation is considered.

(x) Permissible delay on payment is considered.

3.2.2 NOTATIONS

3.2.2.1 The parameters related to vendor are

\( I_{v_1}(t) \)  Inventory level for vendor when \( t \) is between 0 and \( T_1 \).

\( I_{v_2}(t) \)  Inventory level for vendor when \( t \) is between 0 and \( T_2 \).

\( I_{mv} \)  Maximum inventory for vendor.

\( P_v \)  The unit production cost for vendor.

\( F_{vj} \)  Holding cost of the item in period \( j \) for vendor.

\( F_v(t) \)  Holding cost of the item at time \( t \), \( F_v(t) = F_{vj} \) if \( t_{j-1} < t < t_j \).

\( C_{sv} \)  The set up cost per production cycle for vendor.

\( VC \)  Total vendor’s cost per unit item.
3.2.2.2 The parameters related to buyer are

- $I_b(t)$: Inventory level for buyer.
- $I_{mb}$: Maximum inventory for buyer.
- $P_b$: The unit price cost for buyer.
- $F_{bj}$: Holding cost of the item in period $j$ for buyer.
- $F_b(t)$: Holding cost of the item at time $t$, $F_b(t) = F_{bj}$ if $t_{j-1} < t < t_j$.
- $C_{sb}$: The set up cost per order for buyer.
- $C_2$: Shortage cost per item for backlogged items.
- $C_3$: The unit cost of lost sales.
- $B(t)$: Denote the fraction where $t$ is the waiting time up to the next replenishment. We take $B(t) = 1/(1+\delta t)$, where the backlogging parameter $\delta$ is a positive constant.
- $BC$: Total buyer’s cost per unit item.

3.2.2.3 The other related parameters are as follows:

- $a+bl(t)+ct$: Demand rate where $a$, $b$ and $c$ are positive constants and $l(t)$ is the inventory level at time $t$.
- $T$: Time length of each cycle, where $T = T_1 + T_2$.
- $\theta$: Deterioration rate.
- $T_1$: The length of production time in each production cycle.
Chapter 3  

3.3 MATHEMATICAL MODEL

3.3.1 VENDOR’S INVENTORY SYSTEM

In this model the production starts at time zero for vendor with constant rate $P$ from the starting of each cycle i.e. at $t = 0$. Inventory level increases due to production and decreases due to demand and deterioration upto time $T_1$ and reaches at maximum value $I_{mv}$ as shown in the Fig.3.3.1.

$$I_{v1}'(t) + \theta I_{v1}(t) = P - [a + bI_{v1}(t) + ct], \quad 0 \leq t \leq T_1 \quad \ldots (3.3.1)$$

After time $T_1$ the inventory level decreases due to demand and deterioration up to time $T_2$ and reaches at the zero level at time $T_2$.

$$I_{v2}'(t) + \theta I_{v2}(t) = -[a + bI_{v2}(t) + ct], \quad 0 \leq t \leq T_2 \quad \ldots (3.3.2)$$

We have boundary conditions

$$I_{v1}(0) = 0, I_{v2}(T_2) = 0$$
Now the solutions of the above differential equations are

\[
I_{v_1}(t) = (1 - e^{-(\theta + b)t}) \left( \frac{P - a}{\theta + b} + \frac{c}{(\theta + b)^2} \right) - \frac{ct}{\theta + b}, \quad 0 \leq t \leq T_1 \quad \ldots(3.3.3)
\]

\[
I_{v_2}(t) = \left[ \frac{a}{\theta + b} - \frac{c}{(\theta + b)^2} \right] (e^{(\theta + b)(T_2 - t)} - 1) + \frac{c}{(\theta + b)} (T_2 e^{(\theta + b)(T_2 - t)} - t), \quad 0 \leq t \leq T_2 \quad \ldots(3.3.4)
\]

### 3.3.2 BUYER’S INVENTORY SYSTEM

Buyer cycle starts with the maximum inventory \( I_{mb} \) at \( t = 0 \) and this inventory gradually depletes to zero at time \( t_1' \) due to the simultaneous effect of demand and deterioration, shown by the Fig. 3.3.2.

\[
I_{b_1}(t) + \theta I_{b_1}(t) = -[a + bI_{b_1}(t) + ct], \quad 0 \leq t \leq t_1' \quad \ldots(3.3.5)
\]
After time ‘$t_1$’ partial backlogging occurs and the change in the inventory is directed by the following differential equation

$$I_{b_2}'(t) = -\frac{[a+ct]}{1+\delta}\left(\frac{T}{n}-t\right)$$ \hspace{1cm} t_1' < t \leq T/n \hspace{1cm} \text{(3.3.6)}$$

Boundary condition is $I_{b_1}(t_1') = 0 = I_{b_2}(t_1')$

Fig 3.3.2: Inventory system for buyer when shortage is allowed

Now the solutions of the above differential equations are

$$I_{b_1}(t) = \frac{a}{\theta+b} - \frac{c}{(\theta+b)^2}\left(e^{(\theta+b)(t'-t)} - 1\right) + \frac{c}{(\theta+b)}\left(t_1'e^{(\theta+b)(t'-t')} - t\right), \hspace{0.5cm} 0 \leq t \leq t_1'$$

$$\hspace{1cm} \text{(3.3.7)}$$
\[ I_{v_2}(t) = -\frac{1}{\delta}\left[ a + \frac{c}{\delta}\left( 1 + \frac{T}{n} - t'_1 \right) \right] \log \left( \frac{1 + \delta \left( \frac{T}{n} - t'_1 \right)}{1 + \delta \left( \frac{T}{n} - t \right)} \right) + \frac{c}{\delta}(t - t'_1) \quad t'_1 \leq t \leq T/n \]

... (3.3.8)

By using the boundary condition \( I_{mv} = I_{v_2}(0) \) and \( I_{mb} = I_{b_1}(0) \), we have

\[ I_{mv} = T_2(a + cT_2) + \frac{T_2^2}{2}[a(\theta + b) - c + cT_2(\theta + b)] \]  

... (3.3.9)

\[ I_{mb} = t'_1(a + ct'_1) + \frac{T_2^2}{2}[a(\theta + b) - c + cT_2(\theta + b)] \]  

... (3.3.10)

By the boundary condition, \( I_{v_1}(T_1) = I_{v_2}(0) \), we can derive the following equation:

\[ (p - a)\left( T_1 - \frac{(\theta + b)T_2^2}{2} \right) - \frac{cT_2^2}{2} = T_2(a + cT_2) + \frac{T_2^2}{2}[a(\theta + b) - c + cT_2(\theta + b)] \]

By Taylor’s series expansion and the assumption \( \theta \ll 1 \) and by neglecting some small terms, we get

\[ T_1 = \frac{(a + cT_2)T_2}{(p - a)} + \frac{T_2^2}{2(p - a)}[(\theta + b)(a + cT_2) - c] \]  

... (3.3.11)

Knowing

\[ T = T_1 + T_2 \]

We can derive

\[ T = T_2 \left[ 1 + \frac{(a + cT_2)}{(p - a)} + \frac{T_2}{2(p - a)}[(\theta + b)(a + cT_2) - c] \right] \]  

... (3.3.12)
Case I: Retroactive holding cost increase

The yearly holding cost for all the buyers and vendors are

\[ HC_b = \frac{F_b n^{\gamma'}}{T} \int_0^{T_1} I_{b1}(t)e^{-rt} dt \]

\[ = \frac{F_b n^{\gamma'}}{T} \left[ \frac{(a + ct_1')(t_1')^2}{2} - \frac{t_1'^2}{2r} \left( \theta + b(a + ct_1') - c \right) \right] \]  \hspace{1cm} \text{...(3.3.13)}

and

\[ HC_v = \frac{F_v n^{\gamma'}}{T} \left[ \int_0^{T_2} I_{v1}(t)e^{-rt} dt + e^{-rT_1} \int_0^{T_3} I_{v2}(t)e^{-rt} dt - n \int_0^{T_1} I_{b1}(t)e^{-rt} dt \right] \]

\[ = \frac{F_v n^{\gamma'}}{T} \left[ (p-a) \left( \frac{T_2^2}{2} - \frac{rT_3^3}{2} + \frac{\theta + b) rT_4^4}{4} \right) + \frac{crT_4^4}{4} + e^{-rT_1} \left( \frac{a + cT_2)T_5^2}{2} \right) \right. \]

\[ -n \left\{ \frac{(a + ct_1') t_1'^2}{2} \right. - \left( \frac{(\theta + b)(a + ct_1') - c)}{2} \right) \frac{t_1'^2}{r} \left] \right. \]  \hspace{1cm} \text{...(3.3.14)}

Case II: Incremental holding cost increase

\[ HC_b = \frac{n}{T} \left[ \int_0^{T_1} I_{b1}(t)e^{-rt} dt + F_{b2} \int_0^{T_2} I_{b1}(t)e^{-rt} dt + \ldots + F_{b_c} \int_{t_{j-1}}^{t_j} I_{b1}(t)e^{-rt} dt \right] \]

\[ = \frac{n}{T} \sum_{j=1}^{c} F_{bj} \int_{t_{j-1}}^{t_j} I_{b1}(t)e^{-rt} dt \]

\[ = \frac{n}{T} \sum_{j=1}^{c} F_{bj} \left[ (a + ct_1') \left\{ -(t_{j-1} - t_j) \frac{e^{-rT}}{r} + \frac{e^{-rT}}{r^2} + (t_{j-1} - t_{j-1}) \frac{e^{-rT}}{r} - \frac{e^{-rT}}{r^2} \right\} \right. \]

\[ + \left\{ a(\theta + b) - c + ct_1'(\theta + b) \right\} \left\{ -(t_{j-1} - t_j) \frac{e^{-rT}}{r} + 2(t_{j-1} - t_j) - 2 \frac{e^{-rT}}{r^3} \right\} \]
\[ + (t_1' - t_{j-1})^2 \frac{e^{-rt_{j-1}}}{r} - 2(t_1' - t_{j-1}) \frac{e^{-rt_{j-1}}}{r^2} + 2e^{-rt_{j-1}} \frac{1}{r^3} \] \] \[ \ldots (3.3.15) \]

\[ HC_v = \frac{1}{T} \left[ F_{v1} \int_0^{t_1} I_{v1}(t) e^{-r t} dt + e^{-r t_1} F_{v2} \int_0^{t_1} I_{v2}(t) e^{-r t} dt \right. \]

\[ - n \left[ F_{b1} \int_0^{t_1} I_{b1}(t) e^{-r t} dt + F_{b2} \int_0^{t_1} I_{b2}(t) e^{-r t} dt + \ldots + F_{bc} \int_{t_{j-1}}^{t_j} I_{b1}(t) e^{-r t} dt \right] \]

\[ HC_v = \frac{1}{T} \left[ (p - a) F_{v1} \left\{ \frac{T_2^2}{2} - \frac{r T_3^3}{2} + (\theta + b) \frac{r T_4^4}{4} \right\} + F_{v1} c r T_4^4 \frac{1}{4} + e^{-r t_1} F_{v2} (a + c T_2) T_2^2 \right] \]

\[ - n \sum_{j=1}^e F_{b1}(a + ct_1') \left\{ -(t_1' - t_j) \frac{e^{-rt_j}}{r} + e^{-rt_j} - (t_1' - t_{j-1}) \frac{e^{-rt_{j-1}}}{r} - \frac{e^{-rt_{j-1}}}{r^2} \right\} \]

\[ + \left\{ \frac{a(\theta + b) - c + c t_1' (\theta + b)}{2} \right\} \left\{ -(t_1' - t_j)^2 \frac{e^{-rt_j}}{r} + 2(t_1' - t_j) - 2 \frac{e^{-rt_j}}{r^3} \right\} \]

\[ + (t_1' - t_{j-1})^2 \frac{e^{-rt_{j-1}}}{r} - 2(t_1' - t_{j-1}) \frac{e^{-rt_{j-1}}}{r^2} + 2e^{-rt_{j-1}} \frac{1}{r^3} \] \[ \ldots (3.3.16) \]

(i) **When** \( t_1' > M \)

Interest payable per cycle per unit time is

\[ IP = \frac{n P_r I_P}{T} \int_M^{t_1'} I_{b1}(t) e^{-r t} dt \]

\[ = \frac{n P_r I_P}{T} \int_M^{t_1'} \left[ (t_1' - t)(a + c t_1') + \frac{(t_1' - t)^2}{2} \left\{ a(\theta + b) - c + c t_1' (\theta + b) \right\} \right] e^{-r t} dt \]

\[ = \frac{n P_r I_P}{T} \left[ (a + c t_1')(t_1' - M) \left\{ t_1' - M \frac{1}{2} + M^2 \frac{1}{2} \right\} \right] \]
Interest earned per cycle per unit time is

\[
IE = \frac{nPI_c}{T} \int_0^{t_1'} e^{-r}(a + bI_n + ct)dt
\]

\[
= \frac{nPI_c}{T} \left[ (a + ct_1')(t_1' - \frac{rt_1'^2}{2} + bt_1'^2) + b \left( \frac{(a + ct_1')(\theta + c)}{2} \right) \left[ 1 - \frac{2}{r} \right] t_1'^2 \right]
\]

\[
= \frac{nPI_c}{T} \left[ (a + ct_1')(t_1' - \frac{rt_1'^2}{2} + bt_1'^2) + b \left( \frac{(a + ct_1')(\theta + c)}{2} \right) \left[ 1 - \frac{2}{r} \right] t_1'^2 \right]
\]

\[
\text{(ii)} \quad \text{When } t_1' \leq M
\]

Interest earned up to time \( t_1' \) is

\[
IE = P_b I_c \int_0^{t_1'} e^{-r}(a + bI_n + ct)dt
\]

\[
= P_b I_c \left[ (a + ct_1')(t_1' - \frac{rt_1'^2}{2} + bt_1'^2) + b \left( \frac{(a + ct_1')(\theta + c)}{2} \right) \left[ 1 - \frac{2}{r} \right] t_1'^2 \right]
\]

Interest earned during \((M - t_1')\) is

\[
P_b I_c \left[ (a + ct_1')(t_1' - \frac{rt_1'^2}{2} + bt_1'^2) + b \left( \frac{(a + ct_1')(\theta + c)}{2} \right) \left[ 1 - \frac{2}{r} \right] t_1'^2 \right] \int_{t_1'}^M e^{-r} dt
\]

\[
= P_b I_c \left[ (a + ct_1')(t_1' - \frac{rt_1'^2}{2} + bt_1'^2) + b \left( \frac{(a + ct_1')(\theta + c)}{2} \right) \left[ 1 - \frac{2}{r} \right] t_1'^2 \right] \left( e^{-r} - e^{-rM} \right)
\]

Total Interest earned per unit time is

\[
IE = \frac{nPI_c}{T} \left[ (a + ct_1') \left( t_1' - \frac{rt_1'^2}{2} + bt_1'^2 \right) \right]
\]
+b \left\{ \frac{(a+ct_t')(\theta+b)-c}{2} \left[ \frac{1}{2} - \frac{2}{r} t_1'^2 \right] \left[ 1 + \frac{e^{-r_1 t}}{r} - e^{-rM} \right] \right\} \ldots(3.3.19)

In this case customer pays no interest.

The annual deteriorated costs for all buyer and vendor are:

\[ DC_b = \frac{nP_b}{T} \left[ I_{mb} - \int_0^{t_1}(a+bI_{b1}+ct)e^{-r} dt \right] \]

\[ = \frac{nP_b}{T} t_1'^2 \left[ \frac{(a(\theta+b)-c)}{2} + \frac{ar}{2} - \frac{ab}{2} + \left( \frac{1}{r} - \frac{1}{4} \right) b(\theta+b)-c \right] \ldots(3.3.20) \]

\[ DC_v = \frac{P_v}{T} \left[ PT_1 - nI_{mb} \right] \]

\[ = \frac{P_v}{T} \left[ PT_1 - n \left( t_1'(a+ct_t') + \frac{t_1'^2}{2} ((\theta+b)(a+ct_t')-c) \right) \right] \ldots(3.3.21) \]

The set up cost per year for all buyer and vendor are:

\[ SC_b = \frac{nC_{ab}}{T} \ldots(3.3.22) \]

\[ SC_v = \frac{C_{av}}{T} \ldots(3.3.23) \]

Shortage cost per cycle for the buyer

\[ AS_b = \frac{nC_2}{T} \int_{t_1'}^{T/n} -I_{b2}(t)e^{-r} dt \]

\[ = -\frac{nC_2}{\delta T} \left[ a + \frac{c}{\delta} (1+\delta \frac{T}{n}) \right] \log \left[ 1 + \delta \left( \frac{T}{n} - t_1' \right) \right] \left[ \frac{T}{n} - t_1' - \frac{1}{\delta} \left( 1 - \frac{r}{2\delta} \left( 1 + \delta \frac{T}{n} \right) \right) \right] \]

\[ \left( 1 + \delta \frac{T}{n} + \left( t_1' - \frac{r_1 t_1'^2}{2} \right) \right) + \frac{c}{\delta r} \left( \frac{T}{n} - t_1' \right) \left( 1 - e^{-\frac{T}{n}} \right) - \left\{ a + \frac{c}{\delta} (1+\delta \frac{T}{n}) \right\} \]
Chapter 3

Supply Chain Model With Stock

\[
\left(\frac{T}{n} - t_1'\right) \left[\frac{r}{4} \left(\frac{T}{n} + t_1'\right) - \left(1 - \frac{r}{2\delta} \left(1 + \delta \frac{T}{n}\right)\right)\right]
\]  ...(3.3.24)

Opportunity cost per cycle due to lost sales

\[
OC = \frac{nC_3}{T} \left[\frac{T}{n}\right] \left[1 - \frac{1}{1 + \delta \left(\frac{T}{n} - t\right)}\right] (a + ct) e^{-n} dt
\]

\[
= \frac{nC_3}{T} \left[\left(e^{-rT/n} - e^{-n}\right) \left\{-\frac{c}{r^3} - \frac{c}{r^3} - \frac{2\delta c}{r^3} - \frac{\delta a}{r^3}\right\} - \frac{\delta c T}{r^3 n} e^{-rT/n}
\right.
\]

\[
+ \frac{\delta c t_3}{r^3} e^{-n} - \frac{c\delta}{r^3} \left(\frac{T}{n} - t_1'\right) e^{-n};
\]  ...(3.3.25)

**Case I : Retroactive holding cost increase**

(i) When \( t_1' > M \)

Total Cost ( \( TC_{1a} \) ) = VC + BC

\[= HC_b + DC_b + SC_b + AS_b + OC + HC_v + DC_v + SC_v + \text{Interest payable} \]

- Interest earned

where HC, DC, SC, AS, OC, HC, DC, SC, Interest payable and Interest earned are (3.3.13), (3.3.20), (3.3.22), (3.3.24), (3.3.25), (3.3.14), (3.3.21), (3.3.23), (3.3.17) and (3.3.18) respectively.

The necessary and sufficient conditions for the total relevant cost per unit time to be minimize are

and
\[
\frac{\partial TC_{lb}(T_2, t_1')}{\partial T_2} = 0 \quad \text{and} \quad \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial t_1'^2} > 0
\]

and

\[
\left( \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial T_2^2} \right) \left( \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial t_1'^2} \right) - \left( \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial t_1'^2 \partial T_2} \right) > 0
\]

(ii) When \( t_1' \leq M \)

Total Cost (\( TC_{lb} \)) = VC + BC

\[= HC_b + DC_b + SC_b + AS_b + HC_v + DC_v + SC_v - \text{Interest earned} \]

where HC\(_b\), DC\(_b\), SC\(_b\), AS\(_b\), OC, HC\(_v\), DC\(_v\), SC\(_v\) and Interest earned are (3.3.13), (3.3.20), (3.3.22), (3.3.24), (3.3.25), (3.3.14), (3.3.21), (3.3.23) and (3.3.19) respectively.

The necessary and sufficient conditions for the total relevant cost per unit time to be minimize are

\[
\frac{\partial TC_{lb}(T_2, t_1')}{\partial t_1'} = 0 \quad \text{and} \quad \frac{\partial TC_{lb}(T_2, t_1')}{\partial T_2} = 0
\]

\[
\frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial T_2^2} > 0, \quad \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial t_1'^2} > 0 \quad \text{and}
\]

\[
\left( \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial T_2^2} \right) \left( \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial t_1'^2} \right) - \left( \frac{\partial^2 TC_{lb}(T_2, t_1')}{\partial t_1'^2 \partial T_2} \right) > 0
\]
Case II: Incremental holding cost increase

(i) When \( t_i' > M \)

Total Cost \( (TC_{2a}) = VC + BC \)

\[
= HC_b + DC_b + SC_b + AS_b + HC_v + DC_v + SC_v + \text{Interest payable} - \text{Interest earned}
\]

where \( HC_b, DC_b, SC_b, AS_b, OC, HC_v, DC_v, SC_v \), Interest payable and Interest earned are \((3.3.15), (3.3.20), (3.3.22), (3.3.24), (3.3.25), (3.3.16), (3.3.21), (3.3.23), (3.3.17)\) and \( (3.3.18) \) respectively.

The necessary and sufficient conditions for the total relevant cost per unit time to be minimize are

\[
\frac{\partial TC_{2a}(T_2, t_i')}{\partial t_i'} = 0 \quad \text{and} \quad \frac{\partial TC_{2a}(T_2, t_i')}{\partial T_2} = 0
\]

\[
\frac{\partial^2 TC_{2a}(T_2, t_i')}{\partial T_2^2} > 0, \quad \frac{\partial^2 TC_{2a}(T_2, t_i')}{\partial t_i'^2} > 0 \quad \text{and}
\]

\[
\left( \frac{\partial^2 TC_{2a}(T_2, t_i')}{\partial T_2^2} \right) \left( \frac{\partial^2 TC_{2a}(T_2, t_i')}{\partial t_i'^2} \right) - \left( \frac{\partial^2 TC_{2a}(T_2, t_i')}{\partial t_i' \partial T_2} \right) > 0
\]

(ii) When \( t_i' \leq M \)

Total Cost \( (TC_{2b}) = VC + BC \)

\[
= HC_b + DC_b + SC_b + AS_b + HC_v + DC_v + SC_v - \text{Interest earned}
\]
where $HC_b$, $DC_b$, $SC_b$, $AS_b$, $OC$, $HC_v$, $DC_v$, $SC_v$ and Interest earned are (3.3.15), (3.3.20), (3.3.22), (3.3.24), (3.3.25), (3.3.16), (3.3.21), (3.3.23) and (3.3.19) respectively.

The necessary and sufficient conditions for the total relevant cost per unit time to be minimize are

$$\frac{\partial TC_{2b}(T_2, t_1)}{\partial t_1'} = 0 \quad \text{and} \quad \frac{\partial TC_{2b}(T_2, t_1')}{\partial T_2} = 0$$

$$\frac{\partial^2 TC_{2b}(T_2, t_1')}{\partial T_2^2} > 0, \quad \frac{\partial^2 TC_{2b}(T_2, t_1')}{\partial t_1'^2} > 0 \quad \text{and}$$

$$\left(\frac{\partial^2 TC_{2b}(T_2, t_1')}{\partial T_2^2}\right)\left(\frac{\partial^2 TC_{2b}(T_2, t_1')}{\partial t_1'^2}\right) - \left(\frac{\partial^2 TC_{2b}(T_2, t_1')}{\partial t_1'\partial T_2}\right) > 0$$

These equations are solved by the mathematical software Mathematica 5.2.

### 3.4 NUMERICAL EXAMPLE

In this section, a numerical example is considered to illustrate the model as shown in Table 3.4.1 and Table 3.4.2. The following values of parameters are used in the example.

- $P = 250$ unit, $n = 2$, $a = 120$ unit, $P_v = 5$, $P_b = 6$, $C_v = 50$, $C_{vb} = 25$, $\theta = 0.01$, $b = 0.05$, $I_p = 0.09$, $I_e = 0.06$, $M = 25$ days, $C_2 = 0.5$, $r = 0.01$, $c = 0.02$, $\delta = 0.1$, $C_3 = 0.5$ and $T = 60$ days.
Table 3.4.1

<table>
<thead>
<tr>
<th></th>
<th>when ( t_i' &gt; M )</th>
<th>when ( t_i' \leq M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_2 )</td>
<td>42.6685 days</td>
<td>42.6685 days</td>
</tr>
<tr>
<td>( T_1 )</td>
<td>17.3315 days</td>
<td>17.3315 days</td>
</tr>
<tr>
<td>( t_1' )</td>
<td>25.8584 days</td>
<td>6.78251 days</td>
</tr>
<tr>
<td><strong>Total cost</strong></td>
<td>1456.54 Rs</td>
<td>1599.14 Rs</td>
</tr>
</tbody>
</table>

In the incremental holding cost if the cycle ends in period \( e \), then holding cost \( F_{b1} \) is applied to period 1, \( F_{b2} \) is applied to period 2 and so on. Here we have considered \( e = 2 \), thus for the case I when \( t_i' > M \), range of \( t_i' \) is 0 to 12.9292 days and 12.9292 to 25.8584 days and for the case II when \( t_i' \leq M \) range of \( t_i' \) is 0 to 3.39126 days and 3.39126 to 6.78251 days. We obtain the following result

Table 3.4.2

<table>
<thead>
<tr>
<th>Incremental holding cost increase</th>
<th>( F_{ij} = 0.40 )</th>
<th>( 0 &lt; t &lt; T_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 0.50 )</td>
<td>( T_1 &lt; t &lt; T_2 )</td>
</tr>
<tr>
<td>when ( t_i' &gt; M )</td>
<td>( F_{ij} = 0.90 )</td>
<td>( 0 &lt; t_i' &lt; 12.9292 )</td>
</tr>
<tr>
<td></td>
<td>( 0.80 )</td>
<td>( 12.9292 &lt; t_i' &lt; 25.85843 )</td>
</tr>
<tr>
<td>( T_2 )</td>
<td>22.1316 days</td>
<td>13.1026 days</td>
</tr>
<tr>
<td>( T_1 )</td>
<td>37.8684 days</td>
<td>46.8964 days</td>
</tr>
<tr>
<td><strong>Total cost</strong></td>
<td>4719.62 Rs.</td>
<td>1555.76 Rs.</td>
</tr>
</tbody>
</table>
We observed that retroactive increase in holding cost is more costly than incremental holding cost when shortage period is less than the credit period and when shortage period is greater than the credit period then incremental holding cost is very costly than retroactive increase in holding cost.

3.5 SENSITIVITY ANALYSIS

We now study sensitivity of the optimal solution to changes in the different parameters associated with the model. The following points are noted from the numerical study:

(i) All of $TC_{1a}$, $TC_{1b}$, $TC_{2a}$ and $TC_{2b}$ are very sensitive to changes in the value of the inflation rate $r$.

(ii) $TC_{1a}$ is moderately sensitive to changes in the value of the parameter $P$, $M$, $\theta$ and $r$. Moreover, $TC_{1a}$ is increases (decreases) with increase (decrease) in $P$, $M$, $\theta$ and $r$.

(iii) $TC_{1b}$ is low sensitive to changes in the value of the parameter $\theta$, $M$ while it is sensitive to changes in the value of the parameter $P$, $r$. Moreover, $TC_{1b}$ is increases (decreases) with increase (decrease) in $P$, $M$, and $\theta$ but it decreases (increases) with increase (decrease) in $r$. 
(iv) $TC_{2a}$ is low sensitive to changes in the value of the parameter $P$, $\theta$ while it is sensitive to changes in the value of the parameter $M$, $r$. Moreover, $TC_{2a}$ is increases (decreases) with increase (decrease) in $P$ and $\theta$ but it decreases (increases) with increase (decrease) in $r$. With the $M$ it decreases first and then increases.

(v) $TC_{2b}$ is moderately sensitive to changes in the value of the parameter $P$ and $r$ while it has low sensitivity to changes in $M$ and $\theta$. Moreover, $TC_{2b}$ is increases (decreases) with increase (decrease) in $P$, $\theta$ but it decreases (increases) with increase (decrease) in $r$ and $M$.

### Table 3.5.1

<table>
<thead>
<tr>
<th>Variation parameter</th>
<th>Total Cost</th>
<th>Percentage variation in parameters</th>
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<td>-10%</td>
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<tr>
<td>$P$</td>
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</tr>
<tr>
<td>$TC_{1a}$</td>
<td>1389.36</td>
<td>1424.53</td>
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<tr>
<td>$TC_{1b}$</td>
<td>1531.95</td>
<td>1567.12</td>
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<tr>
<td>$TC_{2a}$</td>
<td>4637.85</td>
<td>4679.2</td>
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<tr>
<td>$TC_{2b}$</td>
<td>1454.32</td>
<td>1505.09</td>
</tr>
<tr>
<td>$M$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$TC_{1a}$</td>
<td>1359.22</td>
<td>1404.35</td>
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<tr>
<td>$TC_{1b}$</td>
<td>1585.23</td>
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<td>$TC_{2a}$</td>
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<tr>
<td>$TC_{2b}$</td>
<td>1559.93</td>
<td>1557.9</td>
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<table>
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<th></th>
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<th>( \text{TC}_{1b} )</th>
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<th>( \text{TC}_{2b} )</th>
<th>( \text{TC}_{2a} )</th>
<th>( \text{TC}_{2b} )</th>
</tr>
</thead>
<tbody>
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<td>( \text{TC}_{1a} )</td>
<td>( \text{TC}_{1b} )</td>
<td>( \text{TC}_{2a} )</td>
<td>( \text{TC}_{2b} )</td>
<td>( \text{TC}_{1a} )</td>
<td>( \text{TC}_{1b} )</td>
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<td>1557.12</td>
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<td>1818.66</td>
<td>1664.47</td>
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<tr>
<td></td>
<td>( R )</td>
<td>( \text{TC}_{1a} )</td>
<td>( \text{TC}_{1b} )</td>
<td>( \text{TC}_{2a} )</td>
<td>( \text{TC}_{2b} )</td>
<td>( \text{TC}_{1a} )</td>
<td>( \text{TC}_{1b} )</td>
</tr>
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<td>1473.37</td>
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<td>1759.1</td>
</tr>
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<td>1599.14</td>
<td>1461.13</td>
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<td>1664.47</td>
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<td>1555.76</td>
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<td>1321.18</td>
<td>1818.66</td>
<td>1664.47</td>
</tr>
</tbody>
</table>
Chapter 3  
Supply Chain Model With Stock

Fig 3.5.1: Variation in TC$_{1a}$, TC$_{1b}$, TC$_{2b}$ w.r.t. ‘P’

Fig 3.5.2: Variation in TC$_{2a}$ w.r.t. ‘P’
Chapter 3

Supply Chain Model With Stock

Fig 3.5.3: Variation in $TC_{1a}$, $TC_{1b}$, $TC_{2b}$ w.r.t. ‘M’

Fig 3.5.4: Variation in $TC_{2a}$ w.r.t. ‘M’
Fig 3.5.5 : Variation in TC$_{1a}$, TC$_{1b}$, TC$_{2b}$ w.r.t. $\theta$

Fig 3.5.6 : Variation in TC$_{2a}$ w.r.t. $\theta$
Chapter 3  Supply Chain Model With Stock ........

Fig 3.5.7 : Variation in TC$_{1a}$, TC$_{1b}$, TC$_{2b}$ w.r.t. ‘r’

Fig 3.5.8 : Variation in TC$_{2a}$ w.r.t. ‘r’
3.6 CONCLUSION

In this chapter, supply chain inventory model is developed for deteriorating items with multi-variate demand, permitting shortage and time-proportional backlogging rate. In particular, the backlogging rate considered to be a decreasing function in the waiting time until the next replenishment is more realistic. It is true that the stock-out is very difficult to measure.

We considered the variable holding cost (Retroactive holding cost and Incremental holding cost). In addition to the above mentioned facts we have also considered the effects of permissible delay in payments and inflation. A numerical example and sensitivity analysis are implemented to illustrate the model. The total cost function has been minimized for optimality.

The variation in the system parameters has been clearly demonstrated with the help of graphs. This shows that the solution not only exists but is also unique.
REFERENCES


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