Chapter VI
Chapter-VI

STUDY OF FIXED POINT IN MEASURE OF NON-COMPACTNESS OF METRIC SPACE

This chapter consists of one section. In this section two theorems are proved in measure of non-compactness in metric space about convergence point or fixed point.

§1. FIXED POINT OR CONVERGENCE POINT IN MEASURE OF NON-COMPACTNESS.

Let \((X, d)\) be a complete metric space and \(B\) the family of bounded subsets of \(X\). A map \(\phi : B \rightarrow [0, \infty)\) is called a Measure of Non-compactness (MNC) defined on \(X\) if it satisfies the following properties.

(1.1a) Regularity : \(\phi (B) = 0 \iff B\) is a pre-compact set under closure

\[ \phi (B) = \phi (\overline{B}), \forall B \in B. \]

(1.1b) Semi-additive : \(\phi (B_1 \cup B_2) = \text{Max} \{\phi (B_1), \phi (B_2)\}\)

\[ \forall B_1 \text{ and } B_2 \in B. \]

According to Kuratowski (1991) and Hausdorff (1991); let \((X,d)\) be a complete metric space and \(B\) the family of bounded subsets of \(X\). For every \(B \in B\) Measure of non-compactness (MNC) \(\kappa\) about Kuratowski and \(\chi\) about Hausdorff is as follows:

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\( (1.2) \quad \kappa (B) = \inf \{ \varepsilon > 0 : B \text{ can be covered by finitely many sets of diameter } \leq \varepsilon \} \)

\( (1.3) \quad \chi (B) = \inf \{ \varepsilon > 0 : B. \text{ can be covered by finitely many balls of radius } \leq \varepsilon \} \)

Let \((X, d)\) be a complete metric space. The set \(B\) is said to be \(r\)-separated if \(d(x,y) \geq 0, \forall x,y \in B, x \neq y.\) The set \(B\) will be called an \(r\)-separation of \(X\) i.e. let \(B\) be the family of bounded sets on a metric space \((X,d)\). For every \(B \in B\) we define.

\( (1.4a) \quad \tau (B) = \sup \{ r > 0 ; B \in B \text{ has an infinite } r\text{-separation} \}. \) OR equivalently.

\( (1.4b) \quad \tau (B) = \inf \{ r > 0 ; B \text{ does not have an infinite } r\text{-separation} \}. \)

**Theorem:** (1.1) Let \(X\) is a complete metric space and \(\varphi\) is the measure of noncompactness (MNC). Let \(M\) is the family of totally bounded subsets of \(X\). Let \(f\) is a mapping defined on \(M\). Also let \(M\) is a \(\varphi\)-separation of \(M\). Then \(F\) has a fixed point.

**Proof:** Let \(\{x_n\}\) is a sequence in \(M\). Let us consider an arbitrary sequence \(\{x_{n,i}\}\) in \(\{x_n\}\) of \(M\). Since \(M\) is totally bounded there exists a finite class of open sphere whose union contains \(M\) and we can extract a subsequence \(\{x_{n,i,j}\}\) of \(\{x_{n,i}\}\).

Another application of the total boundedness we have a subsequence \(\{x_{n+1,i}\}\) whose union contains \(M\).

Thus we get a subsequence \(\{x_{n,i}\}, \{x_{n+1,i}\}, \{x_{n+2,i}\}, ............ \{x_{n_{ij}}\}\) of \(\{x_n\}\) in \(M\). \(\forall i,j \in N.\)
Now we define the mapping \( f \) in \( M \) as follows.

(1.5) \[
\{x_{n_i}\} = \{x_{n_{i,j}}\}; \text{ for } i = 1, 2, 3, \ldots \ldots \text{ and }
\]

(1.6) \[
\{x_{n_i}\} \in \{x_{n}\}; \{x_{n_{i,j}}\} \in \{x_{n_j}\}.
\]

Since \( M \) is \( \varphi \)-separated then for \( \varepsilon > 0 \), we have from zorn's Lemma (\( \varphi(M) + \varepsilon \))-separated subset \( S_1 \) (say) of \( \{x_n\} \) which is finite according to the definition of \( \varphi \). \( S_1 \) is finite and for \( \{x_{n_{i,j}}\} \in \{x_n\} \) let us consider an open ball \( B_1 \{f\{x_{n_{i,j}}\}, \varphi(M) + \varepsilon\} = B_1 \{x_{n_{i,j}}; \varphi(M) + \varepsilon\} \) which contains an infinite subset of \( S_1 \). Let us take it as \( G_1 \). Now consider a sphere concentric to \( G_1 \), with radius one half of \( (\varphi(M) + \varepsilon) \) i.e. of \( \frac{\varphi(M) + \varepsilon}{2} \). Since \( M \) is \( \varphi \)-separated by zorn's Lemma we can find out a \( \frac{\varphi(M) + \varepsilon}{2} \)-separated finite subset \( S_2 \) of \( \{x_{n_{i,j}}\} \) and the ball is \( B_2 \{f\{x_{n_{i,j}}; \frac{\varphi(M) + \varepsilon}{2}\} = B_2 \{x_{n_{i,j}}; \frac{\varphi(M) + \varepsilon}{2}\} \) which contains an infinite subset \( G_2 \). Again consider a concentric to \( G_2 \) an open sphere with radius \( \frac{\varphi(M) + \varepsilon}{4} \) i.e. half of \( \frac{\varphi(M) + \varepsilon}{2} \).

Another application of Zorn's Lemma we get \( \frac{\varphi(M) + \varepsilon}{4} \)-separated maximal subset \( S_3 \) of \( \{x_{n_{i,j}}\} \) and a ball \( B_3 \{f\{x_{n_{i,j}}; \frac{\varphi(M) + \varepsilon}{4}\} = B_3 \{x_{n_{i,j}}; \frac{\varphi(M) + \varepsilon}{4}\} \). Continuing in this same process we obtain that \( S_n \) a maximal \( \frac{\varphi(M) + \varepsilon}{2^{n-1}} \)-separated subset of \( \{x_{n_{i,j}}\} \) and \( B_n \{x_{n_{i,j}}; \frac{\varphi(M) + \varepsilon}{2^{n-1}}\} \).
Thus we get $S_n$ and $G_n$ are decreasing subset of $M$. Consequently $B_n$ are also decreasing ball whose union is $M$. Applying the same process upto infinity we get,

\[(1.7)\]

$$B_n\{f_{x_{n,i}^i} ; \frac{\phi(M) + \epsilon}{2^{-i}}\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$  

i.e. $f(x_{n,i})$ is a cauchy sequence which is convergent as $X$ is complete metric space and $M$ is totally bounded. Let it converges to $u$.

\[(1.8)\]

$$\lim f(x_{n,n}) = \lim \{x_{n+1,n}\} = u.$$  

$n \rightarrow \infty \quad n \rightarrow \infty$

We know that a sequence and its subsequence converges to same point. Therefore $f(x_n) \rightarrow u$ as $n \rightarrow \infty$.

Hence $f$ has a convergence or fixed point $u$ in $M$.

**Theorem:** (1.2) Let $X$ be a complete Metric space and $M$ is the family of totally bounded subset of $X$. Let $\phi$ is the measure of Noncompactness of $X$. Let $f$ is a mapping defined on $M$ such that,

\[(1.9)\]

$$d(fx, fy) \leq \phi(M) + \alpha_1 d(x, y) + \alpha_2 d(x, fy) + \alpha_3 d(fx, y).$$

$\forall \ x, y \in X$. Then $f$ has a fixed point in $X$. where $\alpha_1, \alpha_2, \alpha_3 \epsilon(0, 1)$,

$$0 < 1 - \alpha_3 < 1, \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} > 0.$$

**Proof:** Let $\{x_n\}$ is a sequence in $M$. Let $\{x_{n,i}\}$ be an arbitrary subsequence of $\{x_n\}$. Since $X$ is complete and $M$ is totally bounded we can find out a finite class of subsequences $\{x_{n,j}\}$ all of whose points lies in $\{x_{n,i}\}$. Another application of
total boundedness and completeness of $X$ shows similarly that $\{x_{n_i}\}$ has a subsequence $\{x_{m_j}\}$ all of whose points lies in $\{x_{n_i}\}$. Continuing in this way we get the successive subsequence in the following array.

$$ \begin{pmatrix} x_{n_1,1} & x_{n_1,2} & x_{n_1,3} & \cdots & x_{n_1,n} & \in \{x_n\} \\ x_{n_2,1} & x_{n_2,2} & x_{n_2,3} & \cdots & x_{n_2,n} & \in \{x_{n_j}\} \\ x_{n_3,1} & x_{n_3,2} & x_{n_3,3} & \cdots & x_{n_3,n} & \in \{x_{n_j}\} \\ & & & \cdots & \cdots & \\ x_{n_{n_j},1} & x_{n_{n_j},2} & x_{n_{n_j},3} & \cdots & x_{n_{n_j},n} & \in \{x_{n_{n_j-1}}\} \end{pmatrix} $$

(1.10)

from the above set of subsequence let us extract the diagonal element $\{x_{n_1,1}, x_{n_2,2}, x_{n_3,3}, \ldots \ldots \ldots \ldots x_{n_n,n}\}$ which is a diagonal subsequence of $\{x_n\}$.

Now we show that this diagonal subsequences convergent and converges to a point.

Let us define the mapping $f$ on $M$ such that

$$ f(x_{n_i}) = x_{n_{i+1,i}}, \ i \in \mathbb{N} $$

(1.11) 

Now from the inequality (1.9) we obtain,

$$ d(x_{n_2}, x_{n_3}) = d(fx_{n_1}, fx_{n_2}). $$

(1.12) 

$$ \leq \varphi(M) + \alpha_1 d(x_{n_1}, x_{n_2}) $$

$$ + \alpha_2 d(x_{n_2}, fx_{n_2}) + \alpha_3 d(fx_{n_1}, x_{n_2}). $$

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= \phi(M) + \alpha_1 d \left( x_{n_1}, x_{n_2} \right) \\
+ \alpha_2 d \left( x_{n_3}, x_{n_4} \right) + \alpha_3 d \left( x_{n_5}, x_{n_6} \right) \\
= \phi(M) + \alpha_1 d \left( x_{n_1}, x_{n_2} \right) \\
+ \alpha_2 d \left( x_{n_3}, x_{n_4} \right) \\
\leq \phi(M) + \alpha_1 d \left( x_{n_1}, x_{n_2} \right) \\
+ \alpha_2 \{ d \left( x_{n_3}, x_{n_4} \right) + d \left( x_{n_5}, x_{n_6} \right) \} \\
\leq \phi(M) + (\alpha_1 + \alpha_2) d \left( x_{n_1}, x_{n_2} \right) + \alpha_2 d \left( x_{n_3}, x_{n_4} \right) \\
i.e. \left( 1 - \alpha_2 \right) d \left( x_{n_5}, x_{n_6} \right) \leq \phi(M) + (\alpha_1 + \alpha_2) d \left( x_{n_1}, x_{n_2} \right) \\
i.e.

\begin{equation}
1.13 \quad d \left( x_{n_5}, x_{n_6} \right) \leq \frac{\phi(M)}{1 - \alpha_2} + \frac{\alpha_1}{1 - \alpha_2} d \left( x_{n_1}, x_{n_2} \right)
\end{equation}

Again

d \left( x_{n_3}, x_{n_4} \right) = d \left( f_{x_{n_3}}, f_{x_{n_4}} \right)

\begin{equation}
1.14 \quad \leq \phi(M) + \alpha_1 d \left( x_{n_1}, x_{n_2} \right) \\
+ \alpha_2 d \left( x_{n_3}, x_{n_4} \right) + \alpha_3 d \left( f_{x_{n_3}}, x_{n_4} \right) \\
= \phi(M) + \alpha_1 d \left( x_{n_1}, x_{n_2} \right) \\
+ \alpha_2 d \left( x_{n_3}, x_{n_4} \right) + \alpha_3 d \left( x_{n_3}, x_{n_4} \right) \\
\leq \phi(M) + \alpha_1 d \left( x_{n_1}, x_{n_2} \right)
\end{equation}
\[ d(x, y) = d(x, w) + d(w, y) \]

\[ = \varphi(M) + (\alpha_1 + \alpha_2) d(x_{n_2}, x_{n_3}) + \alpha_2 d(x_{n_4}, x_{n_5}) \]

i.e.

\[ d(x_{n_3}, x_{n_4}) \leq \frac{\varphi(M) + \alpha_1 + \alpha_2}{1 - \alpha_2} \]

\[ \leq \frac{\varphi(M) + \alpha_1 + \alpha_2}{1 - \alpha_2} \left\{ \varphi(M) + \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} d(x_{n_1}, x_{n_2}) \right\} \]

from (1.13).

i.e.

\[ d(x_{n_3}, x_{n_4}) \leq \frac{\varphi(M) + \alpha_1 + \alpha_2}{1 - \alpha_2} \varphi(M) + \left( \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} \right)^2 d(x_{n_1}, x_{n_2}) \]

Similarly,

\[ d(x_{n_4}, x_{n_5}) \leq d(f x_{n_3}, f x_{n_4}) \]

\[ \leq \varphi(M) + \alpha_1 d(x_{n_4}, x_{n_5}) + \alpha_2 d(x_{n_5}, x_{n_6}) + \alpha_3 d(x_{n_6}, x_{n_3}) \]

\[ \Rightarrow d(x_{n_4}, x_{n_5}) \leq \frac{\varphi(M) + \alpha_1 + \alpha_2}{1 - \alpha_2} d(x_{n_3}, x_{n_4}) \]

i.e.

\[ d(x_{n_4}, x_{n_5}) \leq \frac{\varphi(M) + \alpha_1 + \alpha_2}{1 - \alpha_2} \varphi(M) + \left( \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} \right)^2 d(x_{n_1}, x_{n_2}) \]

\[ \therefore 135 \]
Continuing in this way we obtain for \( x = x_{n_1} \); \( y = x_{n_1, n_2} \):

\[
d(x_{n_1, n_1}, x_{n_1, n_2})
\]

\[
\leq \frac{\varphi(M)}{1 - \alpha_1} + \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} \varphi(M) + \frac{(\alpha_1 + \alpha_2)^2}{(1 - \alpha_2)^3} \varphi(M)
\]

\[
+ \ldots + \frac{(\alpha_1 + \alpha_2)^n}{(1 - \alpha_2)^n} \varphi(M) + \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_2}\right)^n d(x_{n_1}, x_{n_2})
\]

\[
+ \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_2}\right)^n d(x_{n_1}, x_{n_2})
\]

i.e.

\[
d(x_{n_1, n_1}, x_{n_1, n_2})
\]

\[
\leq \frac{\varphi(M)}{1 - \alpha_1} \left\{1 + \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} + \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_2}\right)^2 + \ldots + \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_2}\right)^n\right\}
\]

\[
+ \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_2}\right)^n d(x_{n_1}, x_{n_2})
\]

Let

\[
S_n = 1 + \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} + \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_2}\right)^2 + \ldots + \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_2}\right)^n
\]

Then

\[
S_n = \frac{\left(\frac{1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^n - 1}{\frac{1 - \alpha_2}{\alpha_1 + \alpha_2} - 1}; \text{ if } \frac{1 - \alpha_2}{\alpha_1 + \alpha_2} > 1
\]

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Otherwise

\[
1.22 \quad S_n = \frac{1 - \left(1 - \frac{\alpha_2}{\alpha_1 + \alpha_2} \right)^n}{1 - \frac{1 - \alpha_2}{\alpha_1 + \alpha_2}}; \text{ if } \frac{1 - \alpha_2}{\alpha_1 + \alpha_2} < 1
\]

Thus in either cases \( S_n \) is a finite number which belongs to \([0, \infty)\). Again \( d(x_n, x_m) \geq 0 \), then for \( 0 \leq 1 - \alpha_2 \leq 1 \), \( \left( \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} \right)^n \) \( d(x_n, x_m) \) is a finite number in \([0, \infty)\) and \( \frac{\varphi(M)}{1 - \alpha_2} S_n \) \( \in [0, \infty) \) as \( \varphi : M \to [0, \infty) \). Hence \( d(x_n, x_{n+l,m}) \to \) a finite number belongs to \([0, \infty)\)

Thus obviously \( d(x_n, x_{n+l,m}) \) is convergent. Let \( m > n \) a positive number such that

\[
1.23 \quad d(x_{n,m}, x_{n,m}) \leq \sum_{i=1}^{n} d(x_{n+i,m}, x_{n+i,m})
\]

Which converges to zero as \( n, m \to \infty \). Hence \( \{x_n\} \) is a convergent sequence of \( \{x_0\} \). Let it converges of the point \( u \). As any sequence and its subsequences converges to the same point; then,

\[
1.24 \quad \lim_{n \to \infty} \{x_n\} = u
\]
i.e.

1.25 $u = \lim_{n \to \infty} \{x_{n,n}\} = \lim_{n \to \infty} \{x_n\}$

And

1.26 $fu = \lim_{n \to \infty} f(x_{n,n}) = \lim_{n \to \infty} \{x_{n+1,n+1}\}$

$= u$

Hence $f$ has a convergence point or fixed point $u$ in $X$.  

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