Chapter IV
In this chapter we have proved some results on Fixed point theorem in a complete 2-Metric space. The present chapter consists of one section. In this section we have established some fixed point theorems and corollary for surjective mapping and commutating mappings in a complete 2-Metric space.

Gahler in (1963) [45] given the idea of 2-Metric spaces and fixed point theorem or contraction mapping principle. After that many authors have worked in this line.

§ 1. ON COMMON FIXED POINT IN 2-METRIC SPACE

Let X be a non-empty set consisting at least three points. A 2-Metric space on X is a mapping d from $X \times X \times X$ to the set of non-negative real numbers which satisfies the following conditions:

(1.1.a) To each pair of distinct points $x, y$ in X, there exists a point $z$ in X such that $d(x, y, z) \neq 0$
(1.1b) \[ d(x, y, z) = 0 \text{ when at least two of } x, y, z \text{ are equal.} \]

(1.1c) \[ d(x, y, z) = d(x, z, y) = d(y, z, x), \]
\[ \forall x, y, z \in X \]

(1.1d) \[ d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z), \]
\[ \forall x, y, z \in X \]

A 2-Metric 'd' on \( X \) is said to be continuous on \( X \) if it is sequentially continuous too.

A sequence \( \{x_n\} \) in a 2-Metric space \((X, d)\) is said to be convergent with \( \lim x \) in \( X \) if -

(1.2) \[ \lim_{n \to \infty} d(x_n, x, a) = 0, \forall a \in X \]

A sequence \( \{x_n\} \) in a 2-Metric space \((X, d)\) is said to be a Cauchy sequence if

(1.3) \[ \lim_{mn \to \infty} d(x_n, x_m, a) = 0, \forall x \text{ in } X \]

A 2-Metric space is said to be complete if every Cauchy sequence in it is convergent.

A non negative real function \( \phi : \mathbb{R}^* \to \mathbb{R}^* \) is called upper semi-continuous if
\[(1.4) \quad \lim_{x \to a} \text{Sup} \phi (x) \leq \phi (a), \ \forall \ a \in \mathbb{R}^+ \]

and \( \phi \) is called lower semi-continuous if

\[(1.5) \quad \lim_{x \to a} \inf \phi (x) \geq \phi (a), \ \forall \ a \in \mathbb{R}^+ \]

where \( \mathbb{R}^+ \) is the set of all non-negative real numbers.

**Theorem :** (1.1) Let \( f, g \) are surjective self mappings on a complete 2-metric space \( X \) into itself, satisfying the following conditions:

\[(1.6) \quad \phi \text{ is lower semicontinuous} - \]

\[(1.7) \quad \text{Let } a_1, a_2, a_3 \in \mathbb{R}^+ \text{ such that either } a_1 \geq \phi (a_1, a_2, a_2, a_3); \ a_1 \geq \phi (a_2, a_1, a_2, a_2) \text{ or} \]

\[a_1 \geq \phi (a_2, a_1, a_2, a_3) \text{ etc. then } a_1 \geq h a_2, \text{ where } h > 1\]

And

\[(1.8) \quad d(fx, gy, a) \geq \phi [d(x, y, a), \ d(fx, x, a) + d(gy, y, a)]\]

\[d(fx, y, a) + d(gy, x, a))],\]

\[\forall \ x, y, a \in X, \text{ then } f, g \text{ have a common fixed point.} \]

**Proof :** Let \( x_0 \) be an arbitrary point in \( X \). Since \( f \) and \( g \) are surjective then there exists \( x_1, x_2 \in X \), such that

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(1.9a) \( f_{x_1} = x_0 \), and

(1.9b) \( g_{x_2} = x_1 \)

Let \( \{x_{2n}\} \) and \( \{x_{2n+1}\} \) are two sequences in \( X \) such that

(1.10a) \( f_{x_{2n+1}} = x_{2n} \) and

(1.10b) \( g_{x_{2n+2}} = x_{2n+1} \), where \( n \in \mathbb{N} \cup \{0\} \)

From the inequality (1.8) for \( x=x_1, \ y=x_2, \ a=x_m \) we obtain,

\[
d(f_{x_1}, g_{x_2}, x_m) = (x_0, x_i, x_m) \\
\geq \phi [d(x_i, x_2, x_m), \\
d(f_{x_1}, x_1, x_m), \ d(g_{x_2}, x_2, x_m), \\
d(f_{x_1}, x_2, a) + d(g_{x_2}, x_i, x_m)] \\
= \phi [d(x_i, x_2, x_m), \\
d(x_0, x_i, x_m), \ d(x_i, x_2, x_m), \\
d(x_0, x_2, x_m) + d(x_i, x_1, x_m)] \\
\Rightarrow \ d(x_0, x_1, x_m) \geq \phi [d(x_i, x_2, x_m), \ d(x_0, x_i, x_m), \\
d(x_1, x_2, x_m), \ d(x_0, x_2, x_m)]
\]

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\[ \Rightarrow d(x_0, x_1, x_m) \geq h \, d(x_1, x_2, x_m) \]

(Using (1.7))

i.e.

\[(1.12) \quad d(x_1, x_2, x_m) \leq 1/h \, d(x_0, x_1, x_m)\]

Again,

\[d(fx_3, gx_2, x_m) = d(x_2, x_1, x_m)\]

\[(1.13) \quad \geq \phi [d(x_3, x_2, x_m), d(fx_3, x_3, x_m),

\quad d(gx_2, x_2, x_m), d(fx_3, x_2, x_m) + d(gx_2, x_3, x_m)]

\quad = \phi [d(x_2, x_3, x_m), d(x_2, x_3, x_m),

\quad d(x_1, x_2, x_m), d(x_2, x_2, x_m),

\quad + \, d(x_1, x_3, x_m)]

\quad = \phi [d(x_2, x_3, x_m), d(x_2, x_3, x_m),

\quad d(x_1, x_2, x_m), d(x_1, x_3, x_m),

\quad d(x_1, x_2, x_m)]

\Rightarrow d(x_1, x_2, x_m) \geq h \, d(x_2, x_3, x_m)

(Using (1.7))
\[ \begin{align*}
\Rightarrow \quad d (x_2, x_3, x_m) & \leq 1/h \cdot d (x_1, x_2, x_m) \\
& \leq 1/h \cdot 1/h \cdot d (x_0, x_1, x_m) \\
(\text{Using (1.12)}) \\
i.e.
\end{align*} \]

(1.14) \[ d (x_2, x_3, x_m) \leq 1/h^2 \cdot d (x_0, x_1, x_m) \]

Similarly,

\[ d (f x_3, g x_4, x_m) = d (x_2, x_3, x_m) \]

(1.15) \[ \geq h d (x_3, x_4, x_m) \]

\[ \Rightarrow \quad d (x_3, x_4, x_m) \leq 1/h \cdot d (x_2, x_3, x_m) \]

\[ \leq 1/h \cdot 1/h^2 \cdot d (x_0, x_1, x_m) \]

i.e.

(1.16) \[ d (x_3, x_4, x_m) \leq 1/h^3 \cdot d (x_0, x_1, x_m) \]

Continuing the repeated application we get in general-

(1.17) \[ d (x_{2n}, x_{2n+1}, x_m) \leq 1/h^n \cdot d (x_0, x_1, x_m) \]

which yields \[ d (x_{2n}, x_{2n+1}, x_m) \rightarrow 0 \text{ as } m > n, \]

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h>1 and n \to \infty.

Thus it follows that \{x_{2n}\} is a Cauchy sequence and let it converges to \(u\) in \(X\). Consequently \{x_{2n+1}\} also converges to \(u\).

Since \(f\) and \(g\) are surjective there exists \(v\) and \(w\) in \(X\) such that,

(1.18a) \quad fv = u, \quad \text{and}

(1.18b) \quad gw = u,

Now we claim that \(u = v = w\)

Setting \(x = x_{2n+1}, y = w\) in (1.8) we obtain,

\[
d(fx_{2n+1}, gw, a) = d(x_{2n}, gw, a)
\]

(1.19) \quad \geq \phi [d(x_{2n+1}, w, a),

\quad d(fx_{2n+1}, x_{2n+1}, a),

\quad d(gw, w, a), \quad (d(fx_{2n+1}, w, a) + d(gw, x_{2n+1}, a))]

\[
\Rightarrow d(x_{2n}, u, a) \geq \phi [d(x_{2n+1}, w, a),
\quad d(x_{2n}, x_{2n+1}, a), d(u, w, a)
\]

\[
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\]
(d (x_{2n}, w, a) + d (u, x_{2n+1}, a))]

Letting \( n \to \infty \) we obtain

\[
d (u, u, a) \geq \phi [d (u, w, a), d (u, u, a), d (u, w, a), d(u, w, a)]
\]

i.e.

\[(1.20) \quad hd (u, w, a) \leq 0 \text{ which is a contradiction as } h > 1\]

and \( d (u, w, a) \geq 0 \). which gives

\[(1.21) \quad u = w\]

Similarly we can prove,

\[(1.22) \quad u = v\]

Thus from (1.21) and (1.22) we get,

\[(1.23) \quad u = v = w\]

i.e.

\[(1.24) \quad fu = u = gu\]

Hence \( u \) is the common fixed point of \( f \) and \( g \) in \( X \).
Theorem: (1.2) Let $f$ and $g$ are surjective self-mappings on a complete 2-metric space $(X, d)$ into itself satisfies the following conditions -

(1.25) \[ \phi \text{ is lower semicontinuous} \]

(1.26) \[ \lim_{n \to \infty} \phi^n (t) = 0 \]

(1.27) Let $a_1, a_2, a_3 \in \mathbb{R}^*$ such that either

\[ \phi (a_i) \geq \min (a_1, a_2, a_2, a_3); \]

\[ \phi (a_i) \geq \min (a_1, a_2, a_2, a_3) \]

or $\phi (a_i) \geq \min (a_2, a_1, a_2, a_3)$; etc. then

$a_1 \geq a a_2$ where $a > 1$

And

(1.28) \[ \phi [d (fx, gy, a)] \geq \min [d (x, y, a), \]

\[ d (fx, x, a), d (gy, y, a), \]

\[ (d (fx, y, a) + d (gy, x, a))] \]

\[ \forall x, y, a \in X. \text{ Then } f \text{ and } g \text{ have a common fixed point.} \]
Proof: Since $f$ and $g$ are surjective mappings then there exists two points $x_1, x_2, \in X$ such that for $x_0$ fixed arbitrary point in $X$,

\begin{align}
& (1.29 \ a) \quad f x_1 = x_0, \\
& (1.29b) \quad g x_2 = x_1
\end{align}

Thus in general for the sequence $\{x_n\}$ and $\{x_{n+1}\}$ in $X$ we have,

\begin{align}
& (1.30a) \quad f x_{n+1} = x_n, \text{ and} \\
& (1.30b) \quad g x_{n+2} = x_{n+1}
\end{align}

Now using the inequality (1.28) we obtain

\[ \phi \{d(f x_1, g x_2, a)\} \geq \alpha \min \{d(x_1, x_2, a), \]

\begin{align}
& (1.31) \quad d(f x_1, x_1, a), d(g x_2, x_2, a), \\
& d(f x_1, x_2, a) + d(g x_2, x_1, a) \]

\[ = \alpha \min \{d(x_1, x_2, a), \]

\begin{align}
& d(x_0, x_1, a), d(x_1, x_2, a), \\
& d(x_0, x_2, a) + d(x_1, x_1, a) \]

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\[ \Rightarrow \quad \phi \{d(fx_0, gx_1, a)\} \geq \alpha \cdot d(x_1, x_2, a) \]

[ using (1.27) ]

i.e.

\[ d(x_1, x_2, a) \leq \frac{1}{\alpha} \phi\{d(x_0, x_1, a)\}, \]

Again for \( x = x_2, y = x_3 \) and \( \forall a \) we get,

\[ \phi \{d(fx_2, gx_3, a)\} \geq \alpha \cdot \text{Min} \{d(x_2, x_3, a), \}

\]

\[ d(fx_2, x_2, a), d(gx_3, x_3, a),\]

\[ d(fx_2, x_3, a) + d(gx_3, x_2, a) \]

\[ = \alpha \cdot \text{Min} \{d(x_2, x_3, a), d(x_1, x_2, a), \}

\]

\[ d(x_2, x_3, a), (d(x_1, x_3, a), \]

\[ + d(x_2, x_2, a))\}

\[ \Rightarrow \quad \phi \{d(x_1, x_2, a)\} = \phi \{d(fx_2, gx_3, a)\} \]

\[ \geq \alpha \cdot d(x_2, x_3, a) \]
\[ \Rightarrow \quad d(x_2, x_3, a) \leq \frac{1}{\alpha} \phi \{ d(x_1, x_2, a) \} \]

i.e.

\[ (1.34) \quad d(x_2, x_3, a) \leq \frac{1}{\alpha} \phi^2 \{ d(x_0, x_1, a) \} \]

[ Using (1.32) ]

Similarly

\[ (1.35) \quad d(x_3, x_4, a) \leq \frac{1}{\alpha} \phi^3 \{ d(x_0, x_1, a) \}, \]

Thus in general we get,

\[ (1.36) \quad d(x_n, x_{n+1}, a) \leq \frac{1}{\alpha} \phi^n \{ d(x_0, x_1, a) \} \]

Since \( \phi^n(t) \to 0 \) as \( n \to \infty \) we get,

\[ d(x_n, x_{n+1}, a) \to 0, \] which shows that \( \{x_n\} \) is a Cauchy sequence. Let it converges to \( u \) in \( X \), as \( X \) is complete 2-Metric space.

Since \( f \) and \( g \) are surjective then there exits \( w \) and \( u \) in

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X such that

\[ (1.37) \quad f_U = u \text{ and } g_V = w \]

Now using the same process as in theorem (1.1) we can prove that \( u = v = w \)

i.e.

\[ (1.38) \quad f_u = u = g_u \]

Hence \( u \) is the Common fixed point of \( f \) and \( g \) in \( X \).

**Corollary :** (1.1) Let \( f \) and \( g \) are surjective self-mappings on a complete 2-metric space. \( A \) is a mapping on complete 2-Metric space \( X \) into itself satisfies the following conditions -

\[ (1.39) \quad \phi \text{ is lower semicontinuous} \]

\[ (1.40) \quad A \text{ Commutes with } f \text{ and } g \text{ and} \]

\[ (1.41) \quad d(Ax, Ay, a) \geq \phi \max \{d(fx, gy, a), \]

\[ d(Ax, fx, a), d(Ay, gy, a), \]

\[ d(Ax, gy, a) d(Ay, fx, a) \]
∀ x, y, a ∈ X. Then f and g have a common fixed point.

**Proof:** Let x₀ be an arbitrary point in X such that

(1.42 a) \( Ax₀ = fx₁ \), and

(1.42 b) \( Ax₁ = gx₂ \), for \( x₁ \) and \( x₂ \) ∈ X

and A Commutes with f and g.

Since f and g are surjective, there exits \( y₁ \) and \( y₂ \) ∈ X such that

(1.43a) \( y₁ = fx₀ = Ax₁ \) and

(1.30b) \( y₂ = gx₁ = Ax₂ \)

Thus in general

(1.44a) \( yₙ = fxₙ₋₁ = Axₙ \) and

(1.44b) \( yₙ₊₁ = gxₙ = Axₙ₊₁ \)

Now setting \( x = xₙ \), \( y = xₙ₊₁ \), from the inequality (1.41) we obtain,
\[ d(Ax_n, Ax_{n+1}, a) = d(y_n, y_{n+1}, a) \]

(1.45) \[ \geq \phi \{ \max \{ d(fx_n, gx_{n+1}, a), \]
\[ d(Ax_n, fx_n, a), d(Ax_{n+1}, gx_{n+1}, a), \]
\[ d(Ax_n, gx_{n+1}, a) d(Ax_{n+1}, fx_n, a) \}
\]

\[ => d(y_n, y_{n+1}, a) \geq \phi \{ \max \{ d(y_{n+1}, y_{n+2}, a), \]
\[ d(y_n, y_{n+1}, a), d(y_{n+1}, y_{n+2}, a), \]
\[ d(y_n, y_{n+2}, a) d(y_{n+1}, y_{n+1}, a) \} \]

\[ => d(y_n, y_{n+1}, a) \]

(1.46) \[ \geq \phi \{ \max \{ d(y_n, y_{n+1}, a), \]
\[ d(y_{n+1}, y_{n+2}, a) \} \}

Now

\[ d(y_{n+1}, y_{n+2}, a) = d(Ax_{n+1}, Ax_{n+2}, a) \]

(1.47) \[ \geq \phi \{ \max \{ d(fx_{n+1}, gx_{n+2}, a), \]
\[ d(Ax_{n+1}, fx_{n+1}, a), d(Ax_{n+2}, gx_{n+2}, a), \]
\[ d(Ax_{n+1}, gx_{n+2}, a) d(Ax_{n+2}, fx_{n+1}, a) \} \]

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\[= \phi \left[ \max \{ d( y_{n+2}, y_{n+3}, a), \\
d( y_{n+1}, y_{n+2}, a), d( y_{n+2}, y_{n+3}, a), \\
d( y_{n+1}, y_{n+3}, a) \} \right] \]

\[= \phi \left[ \max \{ d( y_{n+2}, y_{n+3}, a), \\
d( y_{n+1}, y_{n+2}, a), d( y_{n+1}, y_{n+2}, a) \} \right] \]

i.e.

\[(1.48) \quad d( y_{n+1}, y_{n+2}, a) \]

\[\geq \phi \left[ \max \{ d( y_{n+2}, y_{n+3}, a), \\
d( y_{n+1}, y_{n+2}, a) \} \right] \]

Now substituting values of (1.48) in the inequation (1.46) we obtain,

\[d( y_{n}, y_{n+1}, a) \]

\[(1.49) \quad \geq \phi^2 \left[ \max \{ d( y_{n}, y_{n+1}, a), \\
d( y_{n+1}, y_{n+2}, a), d( y_{n+2}, y_{n+3}, a) \} \right] \]
i.e.

\[ d \left( y_n, y_{n+1}, a \right) \geq \phi^2 \left[ \max \left\{ d \left( y_{n+1}, y_{n+2}, a \right), \right. \right. \\
\left. \left. d \left( y_n, y_{n+1}, a \right), d \left( y_{n+2}, y_{n+3}, a \right) \right\} \right] \]

(1.50) \[ d \left( y_n, y_{n+1}, a \right) \geq \phi^2 \left[ \max \left\{ \frac{\max_{n \leq \sigma \leq n+2} \{ d \left( y_{n}, y_{n+1}, a \right) \}}{n + 1, \beta, n + 3} \right\} \right] \]

Continuing the above process we obtain.

(1.51) \[ d \left( y_n, y_{n+1}, a \right) \geq \phi^n \left[ \max \left\{ d \left( y_{a}, y_{b}, a \right) \right\} \right] \]

which gives \[ d \left( y_n, y_{n+1}, a \right) \to 0 \] as \( n \to \infty \)

and since \( \phi^n(t) \to 0 \) as \( n \to \infty \)

Therefore \( \{ y_n \} \) is a Cauchy sequence. Let it converges to \( u \) in \( X \). consequently \( \{ y_{n+1} \} \) also converges to \( u \) in \( X \).

i.e.

(1.52) \[ \lim_{n \to \infty} \{ y_n \} = \lim_{n \to \infty} \{ y_{n+1} \} = u \]

Since \( f \) and \( g \) are continuous mappings then,

(1.53) \[ \lim_{n \to \infty} f_{n+1} \cdot x_{n+1} = \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} A x_{n+1} \]

\[ = u = \lim_{n \to \infty} g x_{n+2}. \]

As \( f, g \) are continuous mappings and commute with \( A \),

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then \( \{f_{nx_n}\}, \{ff_{nx_n}\} \) and \( \{gx_{nx_n}\}, \{gg_{nx_n}\} \) converges to \( fu \) and \( gu \) respectively when \( n \to \infty \).

Now setting \( x = f_{x_n}, y = x_{n+1} \), in (1.41) we obtain,

\[
\begin{align*}
&d(Afx_n, Ax_{n+1}, a) \geq \phi \left( \max \left\{ d(f^2x_n, gx_{n+1}, a), \\
&\quad d(f^2x_n, x_n, a), d(gx_{n+1}, x_{n+1}, a), \\
&\quad d(f^2 x_n, x_{n+1}, a) \right\} \right) \\
&\quad \text{ Letting } n \to \infty \text{ we get } \\
&d(fu, u, a) \geq \phi \left( \max \left\{ d(fu, u, a), \\
&\quad d(fu, u, a), d(u, u, a), \\
&\quad d(fu, u, a) \right\} \right) \\
&\quad \text{ i.e. } \\
&\quad d(fu, u, a) \geq \phi \left\{ d(fu, u, a) \right\} \\
&\quad \geq d(fu, u, a), \text{ which gives a contradiction and hence,} \\
&\quad (1.56) \quad fu = u
\end{align*}
\]
Similarly, we have,

\[(1.57) \quad gu = u \text{ and} \]

\[(1.58) \quad Au = u \]

Hence \( u \) is the common fixed point of \( f, g \) and \( A \) in \( X \).