CHAPTER VI

EQUILATERAL TRIANGULAR HOLE UNDER UNIFORM NORMAL PRESSURE

In this Chapter, the second order solution to the problem of an equilateral-triangular hole in a homogeneous, isotropic, compressible, infinite medium under uniform normal pressure at its internal boundary has been obtained. The solution of the problem for the linear elasticity case has also been obtained.

The second order solution of the problem, obtained here, significantly differs from that of the linear elasticity case. It reveals that the results of the second order theory are to a greater extent influenced by the elastic properties of the material than those for the infinitesimal case. The solution found here gives the potential functions which determine the second order elastic field everywhere in the
medium. It establishes that the value of the hoop-stress remains unaffected by the Poisson's ratio in infinitesimal case. It reveals that in the second order elasticity the Poisson's ratio affects the hoop-stress and develops shear stress additionally.

The complex variable method is again used to solve the problem. The complex variable method involves the expression of the two sets of potential functions \( \{ \omega(z), \omega(z) \} \) and \( \{ \Delta(z), \Delta(z) \} \) in terms of the coordinates in the undeformed state. These two sets of potential functions determine the complete elastic field with the help of the formulas stated in Chapter III. It is important to note in this context that the results corresponding to the first term only in equations (38) and (99) pertain to the linear elasticity case. If we retain the first two terms in equations (38) and (99) and interpret the corresponding results, we obtain the conclusions related to second order elasticity.

Consider an infinite medium of homogeneous, isotropic, compressible material with an equilateral triangular hole in the undeformed state. Let a uniform normal pressure of intensity \( P \) act on the boundary of the hole.

Let
\[
3_1 = x_1 + i \nu_1
\]

and
\[
3_1 = \frac{1}{3} x_1 + i \nu_1
\]

It is known \((5)\) that the mapping function
\[
\Delta = \omega(z) = A \left( \frac{1}{3} + \frac{5}{3} \right), \quad (154)
\]
nape the region exterior to an equilateral triangle onto the region \\
\mid \mathbf{s}_1 \mid < 1 \quad \text{and the boundary of the triangle transforms into the} \\\nboundary of the unit circle \mid \mathbf{s}_1 \mid = 1. \quad \text{Here } A \text{ is a real constant} \\\nwhich determines the dimensions of the equilateral triangle. \\

It is important to note that as the point \( \mathbf{z} \) traces the equilateral 
triangle in the \( s \) -plane in the counterclockwise sense, the corresponding 
point \( \mathbf{z} \) traces the circle \( \mathbf{c} \) in the clockwise sense in the \( s \) -plane. We 
distinguish a point on the boundary of the circle \( \mathbf{c} \) in the \( s \) -plane by \( s \) 
and note that \( s = e^{-\imath \theta} \), where \( \theta \) is measured from the real \( s \) -axis in 
the anti-clockwise direction. 

In view of transformation (164), the potential functions \( \{ \varphi, \omega \} \) 
and \( \{ \Delta, \delta \} \) will be functions of \( \mathbf{s}_1 \) in this case also and involve the 
following changes in the notation: 

\[
\begin{align*}
\varphi(s_1) &= \varphi \left\{ \, A \left( \frac{1}{s_1} + \frac{1}{3} \right) \right\} = \varphi(s_1), \\
\omega(s_1) &= \omega \left\{ \, A \left( \frac{1}{s_1} + \frac{1}{3} \right) \right\} = \omega(s_1), \\
\Delta(s_1) &= \Delta \left\{ \, A \left( \frac{1}{s_1} + \frac{1}{3} \right) \right\} = \Delta(s_1), \\
\delta(s_1) &= \delta \left\{ \, A \left( \frac{1}{s_1} + \frac{1}{3} \right) \right\} = \delta(s_1)
\end{align*}
\]

(165)

The potential functions \( \{ \varphi(s_1), \omega(s_1) \} \) pertain to the linear 
elasticity case and their expressions are obtained by the following manner:

With the help of equations (31) and (33) we have
\[ \Phi^0(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \Phi^0' + \frac{\psi(\sigma)}{\psi'(\sigma)} \psi^0 = F^0(\sigma), \]  

where

\[ \omega_1(\sigma) = \frac{1}{2} \Phi^0(\sigma), \]
\[ \omega_1'(\sigma) = \frac{1}{2} \psi^0(\sigma) \]

and

\[ F^0(\sigma) = \phi \int_{A} (x_n + \psi Y_n) \, d\sigma + \text{constant} \]
\[ = \phi \int (x + \psi y) \, d\sigma + \text{constant} \]  

Here \((x, y)\) represents the resultant vector of the force applied from the side of the positive normal to an arbitrary arc containing the fixed point \(A\) and the variable point \(P\).

The constant in equation (168) vanishes if \(A P\) is the simple arc of a closed curve. Hence in this case we have

\[ F^0(\sigma) = \phi \int_{C} (x_n + \psi Y_n) \, d\sigma, \]  

where \(C\) is the boundary of the unit circle in the \(xy\)-plane.

Putting

\[ x_n = -\phi \cos(n, n), \]
\[ y_n = -\phi \cos(n, n), \]

where \(n\) is the direction of the normal to the equilateral triangle, and using transformation (164) we obtain
\[ F^0(\sigma) = -PA\left(\frac{1}{\sigma} + \frac{\sigma^2}{3}\right) \]  

assuming that the stresses at infinity vanish.

To evaluate \( \varphi(\sigma), \psi(\sigma) \) the equation (166) may be integrated in the following manner:

The functions \( \varphi(\sigma), \psi(\sigma) \), being analytic within unit circle \( C \), \( \varphi^0(\sigma) \) and \( \psi^0(\sigma) \) may be expressed in terms of infinite series as follows:

\[ \varphi^0(\sigma) = \sum_{k=1}^{\infty} a_k \sigma^k, \]
\[ \psi^0(\sigma) = \sum_{k=1}^{\infty} b_k \sigma^k. \]

assuming \( \varphi^0(0) = 0 \).

Inserting these values of \( \varphi^0(\sigma), \psi^0(\sigma) \) in (166), multiplying both sides by \( \frac{d\sigma}{\sigma - \sigma_1} \) and integrating round the unit circle \( C \), we have

\[ \frac{1}{2\pi i} \int_C \frac{\varphi^0(\sigma)}{\sigma - \sigma_1} d\sigma \]
\[ + \frac{1}{2\pi i} \int_C \frac{(3 + \sigma^3)}{(2 - 3\sigma^3)} \frac{\varphi^0(\sigma)}{\sigma - \sigma_1} d\sigma \]
\[ + \frac{1}{2\pi i} \int_C \frac{\psi^0(\sigma)}{\sigma - \sigma_1} d\sigma \]
\[ = \frac{1}{2\pi i} \int_C \frac{F^0(\sigma)}{\sigma - \sigma_1} d\sigma \]  

(172)
Since the integrals in the case of second and third terms on the left hand side of the above equation are of the form \( \sum C_{-K} \sigma^{-K} \) and \( \sum \nu_{-K} \sigma^{-K} \), respectively the integrals of those functions are zero.

Thus, the equation (172) yields

\[
\Phi^0(s_1) + \overline{\Phi^0(0)} = \frac{1}{2\pi i} \int_C \frac{\Phi^0(\sigma)}{\sigma-s_1} \, d\sigma
\]

(173)

The constant \( \overline{\Phi^0(0)} \) is evaluated by putting \( s_1 = 0 \) in (173). Hence

\[
\Phi^0(s_1) = \frac{1}{2\pi i} \int_C \frac{\Phi^0(\sigma)}{\sigma-s_1} \, d\sigma - \frac{1}{2\pi i} \int_C \frac{\Phi^0(\sigma)}{\sigma} \, d\sigma
\]

or

\[
\Phi^0(s_1) = \frac{s_1}{2\pi i} \int_C \frac{\Phi^0(\sigma)}{\sigma(\sigma-s_1)} \, d\sigma
\]

(174)

where \( C \) is the contour of the unit circle lying entirely within the deformed state of the body.

After substituting the value of \( \Phi^0(\sigma) \) from (170) and integrating round the contour \( C \), the value of \( \Phi^0(s_1) \) is found to be as follows:

\[
\Phi^0(s_1) = -\frac{PA_{s_1}^2}{3}
\]

(175)

To evaluate \( \Phi^0(s_1) \), the complex conjugate of (166) may be
considered and proceeding in a similar manner, the value of \( \psi^0(s_{1}) \) may be found to be

\[
\psi^0(s_{1}) = -\left( \frac{3s_{1}^3 + 1}{2s_{1}^3 - 3} \right) \phi^0(s_{1}) + \frac{1}{2\pi} \int_{c} \frac{F_{\alpha}(\sigma)}{\sigma - s_{1}} \, d\sigma
\]

(176)

Writing down \( \phi^0(s_{1}) \) from (175) and evaluating the integral on the right hand side of (176), \( \psi^0(s_{1}) \) is found to be

\[
\psi^0(s_{1}) = \frac{11PA}{3(2s_{1}^3 - 3)}
\]

(177)

Now with the help of equations (167), (175) and (177) the potential functions \( \{\omega, \omega'\} \) are given by

\[
\begin{aligned}
\omega(s_{1}) &= -\frac{PA}{6}s_{1}^2, \\
\omega'(s_{1}) &= \frac{11PA}{6(2s_{1}^3 - 3)}
\end{aligned}
\]

(178)

Taking \( \epsilon = \frac{1}{\lambda} \), the equation (100) at the inner boundary takes the following form:

\[
\frac{2\Phi_i}{\partial \bar{s}} = \Delta(s_{1}) + 5 \Delta'(\bar{s}) + \delta'(\bar{s}) + \frac{\delta}{\bar{s}} \Omega'(\bar{s}) \Omega(s_{1}) \Omega(\bar{s})
\]

+ \frac{\delta_{2}}{\bar{s}} \Omega'(\bar{s}) \Omega'(s_{1}) \Omega(s_{1}) \Omega(\bar{s})

+ \frac{B_{2} \delta_{3}}{\bar{s}} \left\{ \Omega'(\bar{s}) \right\}^2

= 0
\]

(179)
after taking the constants $h_1$ and $h_2$ to be zero for the first boundary value problem.

The equation (179) for the inner boundary is reduced to the following form by the same procedure as suggested in Chapter IV:

$$\Delta_1(\sigma) + \left( \frac{3 + 3\sigma^4}{2 - 3\sigma^3} \right) \Delta_1'(\sigma) + \delta_1'(\sigma) = F(\sigma)$$

where

$$F(\sigma) = -\left\{ \left( \sigma - 1 \right) \frac{\alpha(\sigma, \bar{\sigma})}{\alpha} + \frac{6\bar{\sigma}}{\alpha} \right\}$$

$$+ \frac{A_3}{\alpha} \left( \frac{1}{\sigma} + \frac{\sigma^2}{3} \right) \left( \alpha' \left( \sigma \right) \right)^2$$

or

$$F(\sigma) = -\rho^2 \alpha \left\{ \frac{5(\sigma - 1)\sigma^3(\sigma - 3\sigma^3) + 4(3 + \sigma^4)h_2}{12\sigma(2 - 3\sigma^3)^2} \right\}$$

$$+ \frac{A_3}{\alpha} \left\{ \frac{7(\sigma - 1)(3 - 4\sigma^3 + 3\sigma^4) + 6\bar{\sigma}}{(1 - \sigma^3 - 3)} \right\}$$

$$- \frac{6\sigma(2 - 3\sigma^3)}{6\sigma(2 - 3\sigma^3)}$$

(131)

The expressions of the second set of potential functions $\{\Delta_1', \delta_1'\}$ are obtained on solving the following integro-differential equations obtained from (130) by the same procedure as suggested in Chapter IV:
\[
\Delta_1'(5_1) = \frac{5_1}{2\pi^2} \int_\mathbb{C} \frac{F(\sigma)}{\sigma(\sigma-5_1)} \, d\sigma
\]  

(102)

and

\[
\Delta_1'(5_1) = -\frac{(35_1^3+1)}{(25_1^3-3)} \Delta_1'(5_1) \\
+ \frac{1}{2\pi^2} \int_\mathbb{C} \frac{F(\sigma)}{\sigma-5_1} \, d\sigma
\]

(103)

After substituting the value of \( F(\sigma) \) from (101) in (102) and integrating round the contour \( C \), we obtain

\[
\Delta_1'(5_1) = \frac{\rho^2 A (y-1) 5_1 (25_1^3-3)}{6 (25_1^3-3)}
\]

(104)

Writing down \( \Delta_1'(5_1) \) from (104) in (103) and evaluating the integral on the right hand side we obtain

\[
\delta_1'(5_1) = -\frac{\rho^2 A s_1}{12} \left( (\gamma-1) \left[ \frac{2 (35_1^3-2\xi)}{(25_1^3-3)} + \frac{5 (25_1^3-3)}{(25_1^3-3)^2} \right] \\
- \frac{4 (2\xi+1) 5_1^3 (1+35_1^3)}{(25_1^3-3)^2} \\
+ 2 (1+35_1^3) \left[ 42 \xi^6 + 3 (2-5\xi) 5_1^3 + 18 \right] \right) \\
\frac{1}{(25_1^3-3)^3}
\]

(105)
\[ + \frac{2}{x} \beta_2 \left\{ \frac{-1}{(2 \sigma^3 - 3)} \left( 3 \sigma^3 R + \sigma \right) + \frac{2}{(2 \sigma^3 - 3)} \left( 1 + 3 \sigma^3 R \right) \right\} \]
\[- \gamma (2 \sigma + 1) \frac{3 \sigma^3 (1 + 3 \sigma^3 R)}{(2 \sigma^3 - 3)^2} \]

\[ (195) \]

The functions \( \{ \alpha_1, \omega_1 \} \) and \( \{ \Delta_1, \delta_1 \} \) being known from (178), (194), (195) respectively, the displacement and stress fields may be calculated at any point in the medium.

The displacement function \( D(\xi, \xi') \) at the inner boundary is given by the following expression obtainable by the similar procedure as suggested in Chapter IV for obtaining the value of the displacement function \( D \):

\[
D(\sigma, \sigma') = \frac{P A}{6 \mu} \left( \frac{3}{6} - 2 \sigma \sigma' \right) \]
\[+ \frac{P^2 A \beta_2}{12 \mu^2} \left[ \left\{ \frac{1}{\sigma} - \frac{2}{3} \frac{3 \sigma^3 - 3}{3 \sigma^3 (2 - 3 \sigma^3)} \right\} \left[ \left\{ \frac{R^2}{2} \log \left( \frac{\sigma^2 - 1}{\sigma^2 + \sigma R + R^2} \right) + \frac{R^2}{2} \tan^{-1} \left( \frac{2 \sigma + R}{\sqrt{3}} \right) \right\} \right]
\]
\[+ \frac{P^2 A \beta_2}{4 \mu^2} \left[ \frac{\sigma^2}{3} + \frac{R}{6} \log \left( \frac{1 - 2 \sigma R + R^2 \sigma^2}{1 + \sigma R + R^2 \sigma^2} \right) \right] \]
\[+ \frac{\sqrt{3} R}{3} \tan^{-1} \left( \frac{1 + 2 \sigma R}{\sqrt{3}} \right) \right] \right],
\]

\[ (196) \]
where

\[ R^3 = \frac{2}{3} \]

As given by equation (68), the displacement function \( D(\xi, \mu) \) is of the form

\[ D = u_1 + \iota u_2 \]  \hspace{1cm} (107)

where

\[ u_1 = \frac{PA}{6\mu} \left( 3\cos \theta - 2\cos 2\theta \right) \]

\[ + \rho^2 A \frac{1}{8 \pi} \left\{ \begin{array}{c}
72 \cos \theta - 2(9 \cos 4\theta + 4 \cos 2\theta - 12 \cos \theta) \\
(13 - 12 \cos 3\theta)
\end{array} \right. \]

\[ + \frac{11 \rho^2 A}{\pi} \left[ \begin{array}{c}
(3 + 3R)^{-1} - (2 + 3R) \cos \theta - 9(2 - 3R)^2 \sin^2 \theta \\
(3 + 2R + 2(2 + 3R) \cos \theta + 3R^2 (1 + 2 \cos \theta))^2
\end{array} \right] 
\]

\[ + 22 \frac{5 \rho^2 A}{\pi} \left[ \begin{array}{c}
J_3 \left( 2 \sin \theta + R \right) \\
R^2 - 2 - 2R \cos \theta
\end{array} \right] \]

\[ + \rho^2 A \frac{1}{4 \pi} \left\{ \begin{array}{c}
16 \cos 2\theta \\
18 \mu^2
\end{array} \right. \]

\[ + \rho \log \left[ \begin{array}{c}
(3 + 2R)^{-1} - (2 + 3R) \cos \theta - 9(2 - 3R)^2 \sin^2 \theta \\
(3 + 2R + 2(2 + 3R) \cos \theta + 3R^2 (1 + 2 \cos \theta))^2
\end{array} \right] 
\]

\[ \left[ \begin{array}{c}
(3 + 2R + 2(2 + 3R) \cos \theta + 3R^2 (1 + 2 \cos \theta))^2
\end{array} \right] \]
\[ u_2 = \frac{PA}{6\mu} (3 \mu_0 + 2 \mu_2 \theta) \]
\[ + \frac{p^2 A b_1}{32\\mu^2} \left( 36 \mu_0 + 12 (12 \mu_0 + 4 \mu_2 \theta - 9 \mu_4 \theta) \right) \]
\[ + \frac{11 R^2}{4} \tan^{-1} \left( \frac{3 (2 - 3 R) \mu_0 \theta}{\{(3 + 2 R) - (2 + 3 R) \cos \theta \}^2 - 6 R^2 (1 - \cos 2 \theta)} \right) \]
\[ + \frac{5}{3} \tan^{-1} \left( \frac{-\frac{5}{3} R \mu_0 \theta}{(1 + R \cos \theta + R^2)} \right) \]
\[ - \frac{p^2 A b_2}{24\\mu^2} \left( 3 \mu_2 \theta \right) \]
\[ + R \tan^{-1} \left( \frac{3 (2 - 3 R) \mu_0 \theta}{\{(3 + 2 R) - (2 + 3 R) \cos \theta \}^2 - 6 R^2 (1 - \cos 2 \theta)} \right) \]
\[ - R \frac{5}{3} \tan^{-1} \left( \frac{-\frac{5}{3} R \mu_0 \theta}{(1 + R \cos \theta + R^2)} \right) \]
where $R^3 = \frac{2}{3}$.

The stress components $\sigma^{\alpha\beta}$ may be obtained by equations (137) and (138) following the procedure suggested for the case in Chapter IV. Substituting the values of functions $u$, $\omega$, $\Delta$, $\zeta$ and their derivatives as functions of $\beta$, in equations (137) and (138) from equations (178), (194), (195) and putting $h_0 = x$ and $c = \frac{1}{\mu}$ from equations (91), (120) respectively, we obtain

$$T'' = -\rho \frac{s_1^4}{(2 - 3 s_1^2)^2} \left\{ \begin{array}{c} 10 \\ \mu \frac{(y-1)}{(2 s_1^2 - 3)} \left( \frac{(5 - s_1^3 + 5 s_1^6)}{(2 s_1^2 - 3)} - 4 x \right) \\ + 4 c_3 - 4 c_1 \frac{2 s_1^3 - 3}{2 s_1^3 (2 - 3 s_1^2)} \end{array} \right\}$$

(190)
\[ T_{12} = \frac{\rho}{(2 - 3\varepsilon_i^3)(2\varepsilon_i^3 - 3)} X \left[ 4 \left( 3 - 4\varepsilon_i^3 + 3\varepsilon_i^6 \right) + \frac{\rho\varepsilon_i^3}{\lambda} \left( \frac{5\varepsilon_i^4 - 67\varepsilon_i^3 + 65\varepsilon_i^6}{\left( 2\varepsilon_i^3 - 3 \right) \left( 2 - 3\varepsilon_i^3 \right)} \right) - 4\varepsilon_i \right] \]

\[ \left[ + \frac{4\varepsilon_i^3 - 4\varepsilon_i}{\varepsilon_i^3} \left( 2\varepsilon_i^3 - 3 \right) \right] \]

As stated earlier due to non-availability of computer facility numerical comparisons of the elastic fields for the linear and second order cases could not be given.