CHAPTER I

INTRODUCTION

PART A

1.1 History and Motivation

The abstract tendency in analysis which developed into what is now known as "Functional Analysis", began at the turn of the century with the work of Volterra, Fredholm, Hilbert, Fréchet, Riesz and Banach, to mention some of the principal figures. F. Riesz was first to formulate the Normed linear axioms (see [126]). However an abstract and full treatment to this subject was given by S. Banach in his thesis (see [10]). His book [11], which was published in 1932 was tremendously influential and signified the beginning of the systematic study on Normed linear spaces. In last three decades the research activity in this area grew considerably. As a result, Banach space theory gained very much in depth as well in scope. However not enough work seems to have been done dealing with the interplay between functional analysis and the theory of analytic functions. The reason behind it might be the fact that functional analysis techniques are essentially of real variable character. But there are parts of the theory which blend beautifully with the concepts and methods of functional analysis. As a matter of fact, the techniques of functional analysis often lend clarity and
elegance to the proofs of classical theorems and thereby make the results available in more general setting. A testimony to these facts are the books of Hoffman [52], Porcelli [113], Duren [31] and Hille [48] etc. A major portion of complex variable theory i.e. the theory of power series has been generalized, systematically, in its vector-valued form by Hille and Phillips [51]. But there are several notions in complex analysis theory which still await their generalization to vector-valued form. Classical Dirichlet series is a glaring example to it. In this work we have developed the concept of Vector-Valued Dirichlet Series (VVDS) and studied various spaces and algebras of functions represented by them.

1.2 Classical Dirichlet Series

A series of the form,

\[ \sum_{n=1}^{\infty} a_n e^{s \lambda_n}, \quad s = \sigma + it, \quad (\sigma, t \text{ real variables}), \]

is called a Dirichlet series. This series in its most original form \[ \sum_{n=1}^{\infty} a_n n^{-s} \] was first introduced by Dirichlet for his studies in the number theory. Dirichlet and Dedekind considered only real values of the variable 's' and obtained
many important theorems. The first result involving the complex values of \( s \) were obtained by Cahen [20] who determined the nature of region of convergence of the series (1.2.1). Later, Littlewood [37] succeeded in showing that the Dirichlet series could be useful in the study of entire functions, while Esterman [33] used Dirichlet series in the study of meromorphic functions. The growth of the sum function \( f(s) \), of the series (1.2.1), seems to have been first studied by Doetsch [28]. However, a systematic study of the growth of \( f(s) \), when \( f(s) \) is an entire function was made by Ritt [129], who, in fact, considered the Dirichlet series with positive exponents, instead of negative ones, just to have an analogy with the study of the growth of entire functions represented by Taylor series. In the further development of the theory of Dirichlet series, significant contributions were made by Sugimura [147], Izumi [61] and Mandelbrojt [102]. But a vast enrichment to this field with new and fruitful ideas came in the wake of the works of Tanaka [148], Azpeitia [7-8,9], Rahman [119-120], Dagene [23] et

Below we give a brief review of some of these aspects of the theory of Dirichlet series which are relevant for our study. We shall throughout consider Dirichlet series with positive exponents and take the liberty of interpreting the results of all those workers who have considered Dirichlet series with negative exponents in our terminology.
Consider the Dirichlet series

\[(1.2.3) \quad f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda_n},\]

where \(\{\lambda_n\}\) satisfy (1.2.2), \(s = \sigma + it\) (\(\sigma, t\) real variables) and \(\{a_n\} \subset \mathbb{C}\) is a sequence of complex numbers. It is known [102] that the series (1.2.3) converges in a left half plane and that the sum function \(f(s)\) of the series (1.2.3) is holomorphic in its half plane of convergence. If \(\sigma_c\) and \(\sigma_a\) denote, respectively the abscissa of convergence and the abscissa of absolute convergence of the series (1.2.3) then [102], it follows that

\[(1.2.4) \quad 0 < \sigma_c - \sigma_a \leq \limsup_{n \to \infty} \frac{\log n}{\lambda_n} = D\]

and

\[(1.2.5) \quad \sigma_c = -\limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n}.\]

Thus if \(D < \infty\) and \(\sigma_c = \infty\), \(f(s)\) represents an entire function and by (1.2.4), \(\sigma_a = \infty\) so that the series (1.2.3) converges absolutely at every point of the finite complex plane.

Further if \(D = 0\), then it is clear from the above that

\[(1.2.6) \quad \sigma_c = \sigma_a = -\limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n}.\]

The well known Cauchy formula for the coefficients of a Taylor series has also been extended. Thus [102] if \(f\) is given by (1.2.3) with \(\sigma_a > -\infty\), then
\[ a_n e^{\lambda n \sigma_1} = \lim_{T \to \infty} \int_{t_0}^{T} f(\sigma_1 + it) e^{-\lambda n it} dt \quad (n \geq 1) \]

where \( t_0 \) is arbitrary, \( \sigma_1 < \sigma \), otherwise arbitrary, and \( f(\sigma_1 + it) \) is the values of the function on the vertical line, for a fixed \( \sigma = \sigma_1 \). Also, Mandelbrojt [102] has proved that there are constants \( d, c_1, c_2 \) depending only on \( \lambda_n \) with

\[ |a_n| < c_2 M(s_1) \exp(\lambda_n(c_1 - R(s_1))), \quad n \in \mathbb{N} \quad (\text{the set of natural numbers}) \]

where \( M(s_1) = \text{lub} \ |f(s)| \) for arbitrary \( s_1 \in \mathbb{C} \).

### 1.3 Growth Properties of Classical Dirichlet Series

Set, for the non constant function \( f \) given by the Dirichlet series (1.2.3)

\[ M(\sigma) = M(\sigma; f) = \text{lub} \ |f(\sigma + it)|, \quad \text{for } \sigma < \sigma_0 < \infty < k < \infty \]

\( M(\sigma) \) is called the maximum modulus of \( f \) for \( \text{Re}(s) = \sigma \).

Doetsch [28] showed that \( \log M(\sigma) \) is an increasing function of \( \sigma \) and that if \( f \) is entire then \( M(\sigma) \to \infty \) as \( \sigma \to \infty \).

To estimate the precise growth of \( M(\sigma; f) \) when \( f \) is entire, the concept of order has been introduced by Ritt [127]. Thus if \( f \) is entire and \( f \) given by (1.2.3), then it is said to be of order \( \rho \), \( (0 \leq \rho \leq \infty) \) if and only if

\[ \rho = \lim_{\sigma \to \infty} \sup \frac{\log \log M(\sigma)}{\sigma} \]
If $0 < \rho < \infty$, the type 'T' of $f$ is defined as

$$T = \lim_{\sigma \to \infty} \sup_{0 \leq T \leq \infty} \frac{\log M(\sigma)}{\exp(\sigma \rho)},$$

(1.3.3)

The entire function $f$ is said to have growth $\{\rho, T\}$ if its order does not exceed $\rho$ and type does not exceed $T$ if it is of order $\rho$. The coefficient characterization for $\rho$ and $T$ may be found in [158]. Thus if $f$, as given by (1.2.3) is entire and $D$ given by (1.2.4) is finite, then the order $\rho$ of $f$ is given by

$$\rho = \lim_{n \to \infty} \frac{\lambda_n \log \lambda_n}{n \log |a_n|} - 1.$$

(1.3.4)

Further if $D = 0$ and $f$ defined by (1.2.3) is of order $\rho$, $(0 < \rho < \infty)$ and type $T$, $(0 \leq T \leq \infty)$ then

$$T = \lim_{n \to \infty} \frac{\lambda_n |a_n|^\rho}{e^\rho - |a_n|^{\rho/\lambda_n}}.$$

(1.3.5)

Some other important results in connection with the growth of an entire function given by a Dirichlet series, may be found in Tanaka [148], Srivastava [144, 145], Srivastava and Gupta [146], Rahman [149], Juneja [63], Juneja and Singh [67], Juneja and Kapoor [66], Reddy [124], and Srivastava [139, 140] etc.

To estimate the growth of functions, given by (1.2.3) which are not entire but analytic in $\Re(s) < 0$, Dagene introduced the notion of order etc. However, a systematic study in this direction has been made by Krishna Nandan [106], who not only extended the notions of order and type to functions...
analytic in a half plane, but also contributed various other results in this connection. Thus the function $f$ given by the Dirichlet series (1.2.3) which converges absolutely in the left half plane $\text{Re}(s) = \sigma < \alpha$, and satisfies

\[(1.3.6) \quad \lim_{n \to \infty} \sup \frac{n}{\lambda_n} = D' < \infty\]

is said to be of order '$\rho_0$' in $\sigma < \alpha$ if

\[(1.3.7) \quad \rho_0 = \lim_{\sigma \to \alpha} \sup \frac{\log \log M(\sigma)}{(1-e^{\sigma-\alpha})^{-\theta}}\]

Further if $0 < \rho_0 < \infty$, $f$ is said to be of type 'To' if

\[(1.3.8) \quad T_0 = \lim_{\sigma \to \alpha} \sup \frac{\log M(\sigma)}{(1-e^{\sigma-\alpha})^{-\theta}}, \quad (0 < T < \infty)\]

where $M(\sigma)$, for $\sigma < \alpha$, is given by (1.3.1).

The coefficient characterization for $\rho_0$ and $T_0$ have also been obtained. Thus if $f$ is given by the Dirichlet series (1.2.3) which converges absolutely in $\sigma < \alpha$ and satisfies (1.3.6) is of order $\rho_0$, then

\[(1.3.9) \quad \frac{\rho_0}{\rho_0 + 1} = \lim_{n \to \infty} \frac{\log^+ \{\alpha \lambda_n + \log |a_n|\}}{\log \lambda_n}\]

where left hand side is to be interpreted as 1 when $\rho_0 = \infty$, and

\[(1.3.10) \quad \log^+ x = \begin{cases} \log x & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}\]

For $\rho_0$ finite and nonzero, type $T_0$ is given by the relation

\[(1.3.8) \quad T_0 = \lim_{\sigma \to \alpha} \sup \frac{\log M(\sigma)}{(1-e^{\sigma-\alpha})^{-\theta}}, \quad (0 < T < \infty)\]

where $M(\sigma)$, for $\sigma < \alpha$, is given by (1.3.1).
\[
(1.3.11) \quad \frac{(1+\rho_0)^{1+\rho_0}}{\rho_0} T_0 = \limsup_{n \to \infty} \frac{\log^+|a_n| e^{\alpha n}}{\lambda_n^{1+\rho_0}}
\]

If left hand side is 0 or \(\infty\) then \(f\) is called of growth \(\{\rho_0, 0\}\) and conversely.

As a matter of fact, the growth study of a function represented by Dirichlet series runs almost parallel to the corresponding study made for functions represented by a power series.

Various workers have contributed to the study of different aspects of the growth of entire or analytic functions represented by power series. We mention a few of them such as Sato [131], Shah and Ishaq [134], Seremeta [133], Juneja, Kapoor and Bajpai [70, 71], Bogda and Shankar [16] etc.

1.4 Spaces of Entire and Analytic Functions

Suitable topologies on the class of entire functions represented by Dirichlet series have been defined to make it into a linear topological space. Similar attempts have also been made to topologise its various sub-classes. These studies run parallel to the corresponding studies made earlier by Iyer [56 to 60], Arsove [4,5], Krishnamurthy [81] etc.

It may be recalled that Iyer in 1948, introduced on the space \(\Gamma\), of all entire functions, a topology, convergence in which is equivalent to uniform convergence on compact sets. Iyer [56,57,58] obtained a variety of nice results for the space \(\Gamma\), including those on Bases etc. Iyer's concept of proper
bases was modified by Arsove [4,5] who obtained a characterization for such bases in the space $\Gamma$. He also attempted in [6] to generalize the renowned Paley–Weiner theorem in metric spaces. Markushevich [104] on the other hand considered the class $F_R$ of all single valued, analytic functions in the disk $|z| < R$, $0 < R < \infty$, and defined an invariant metric on it, so that $F_R$ becomes a linear metric space. Krishnamurthy [81] made a systematic study of proper bases, continuous linear functionals, continuous linear transformations etc. on different subspaces of entire functions. In 1973, Ekblaw [32] also studied the several subspaces of a space of entire functions for their several properties. Sisarcick [136,137] gave a characterization for scalar homomorphism and metrically bounded linear functionals on $\Gamma$. Because of the vast importance of the space $\Gamma$, Patwardhan [109] studied its bornological aspects. Recently P.D. Srivastava in his dissertation [141] considered a complete linear metric space $(X,d)$ with a Schauder base satisfying a number of suitably chosen conditions. His results for the space $(X,d)$ include, as a special case a number of known results on different spaces and subspaces of entire functions considered by earlier workers.

The study of the spaces and subspaces of entire functions represented by Dirichlet series was initiated by A. Hussain and Kamthan [55] and pursued by Kamthan [72], Kamthan and Gautam [73 to 77] in a series of papers. In fact this study
runs parallel to the study made by Iyer, Arsove and Krishnamurthy for the corresponding spaces of entire functions defined by Taylor series.

We shall state now some of their results since they are relevant for our subsequent deliberations.

Let $X$ be the space of all functions $f$, defined by Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it \quad (\sigma, t, \text{real variables}),$$

where the sequence $\{ \lambda_n \}$ satisfying the conditions

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots \quad \lambda_n \to \infty \quad (n \to \infty)$$

and

$$\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = D = 0,$$

is fixed throughout and $\{ a_n \} \subset C$ satisfying

$$\sigma_c = \sigma_a = - \limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n} = \infty.$$

It is clear from (1.4.4) that the members of $X$ are entire functions. It is easy to see that under the usual operations of addition of functions and multiplication by complex numbers, $X$ forms a linear space. For $f \in X$ and $\sigma \in \mathbb{R}$ (set of real numbers), let

$$M(\sigma, f) = \sup_{-\infty < t < \infty} |f(\sigma + it)|$$
Then \( \{ M(\sigma, \ldots) : \sigma \in \mathbb{R} \} \) defines a family of semi-norms (in fact norms) on \( X \). Let \( \tau_1 \) be the locally convex Hausdorff topology on \( X \) generated by this family of semi-norms. This topology may also be given by the invariant metric \( d_1 \) defined by the Fréchet combination

\[
d_1(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{M(\sigma_k, f-g)}{1 + M(\sigma_k, f-g)},
\]

where \( \sigma_1 < \sigma_2 < \sigma_3 \ldots < \sigma_n \to \infty \) as \( n \to \infty \) and \( f, g \in X \).

Next for each \( f \in X \), define,

\[
p(\sigma, f) = \sum_{n=1}^{\infty} |a_n| e^{\sigma_n}
\]

Then \( \{ p(\sigma, \ldots) : \sigma \in \mathbb{R} \} \) again defines a family of semi-norms (indeed norms) on \( X \), which generates another locally convex Hausdorff topology \( \tau_2 \) on \( X \), which may also be given by an invariant metric \( d_2 \) defined by

\[
d_2(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p(\sigma_k, f-g)}{1 + p(\sigma_k, f-g)},
\]

where \( \{ \sigma_n \} \) is an increasing sequence of real numbers tending to \( \infty \). Because of the spaces \( (X, \tau_1) \) and \( (X, \tau_2) \) being non-normable, attempts have been made to define norm topologies on certain subclasses of \( X \). Thus R.K. Srivastava \cite{142} considered a class \( \mathcal{D} \) of entire Dirichlet series with fixed sequence \( \{ \lambda_n \} \) of exponents which satisfies (1.2.2), defined as

\[
\mathcal{D} = \{ f, f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda_n} : e^{n \lambda_n} |a_n| < \infty, n \geq 1 \}.
\]

Then the space \( \mathcal{D} \) with the pointwise addition and scalar
multiplication and norm defined by

\[(1.4.8) \quad ||f|| = \sup e^{n\lambda_n} |a_n|, \ f \in D.\]

becomes a Banach space. In fact the properties of $D$ studied by him are analogous of the results of Sen [132], Das and Bose [24]. In his further study [143] he characterized the multipliers for $H^p$ and $l_p$ into $D$ and vice versa.

Recently, a space $\mathcal{Q}_0$ of all those Dirichlet functions $f$, \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n} \) has been considered [68] in which \( |a_n/\alpha_n| \to 0 \) \((n \to \infty)\) for \( \{\alpha_n\}\) satisfying certain conditions. After defining a suitable norm \( ||.|| \) on $\mathcal{Q}_0$, a characterization of continuous linear functionals, continuous matrix transformations from $\mathcal{Q}_0$ into itself has been obtained. In fact it has been proved that \( (\mathcal{Q}_0 ||.||) \) is non-uniformly convex Banach space which is separable also. Further if \( A = \{\xi_n, k\}_{n,k \in \mathbb{N}} \) be an infinite matrix of complex entries then it transforms from $\mathcal{Q}_0$ to itself if \( \xi_{nk} \to 0 \) \((n \to \infty, k \text{ fixed})\) and

\[
M = \sup_n \left| \sum_{k=1}^{\infty} \frac{\xi_{nk}}{\alpha_n} \alpha_k \right| < \infty, \text{ with } ||A|| = M \text{ and conversely.}
\]

1.5 \textbf{Algebras of Entire and Analytic Functions}

Attempts have also been made to embed the space of entire functions or its subspaces with additional algebraic structures. Thus Iyer [59] defines multiplication in $\Gamma$ by two ways and showed that $\Gamma$ is a commutative topological algebra w.r.t. both of these multiplication. Iyer [60]
studied the closed ideals in one of the two algebras referred to above, while Henriksen [45, 46] elaborately worked out the ideal structure in the other one. Sen [132] considered a subspace $\Gamma_1^{\circ}$ of $\Gamma$ consisting of all $f$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $|a_n|$ is bounded and defined a multiplication in it, thus making it a commutative Banach algebra with identity $e^z$. Das and Bose [24] studied some more algebraic and topological properties of $\Gamma_1^{\circ}$. Field and Sisarcick [34] made a deeper study of $\Gamma_1^{\circ}$ by obtaining characterization of regular and singular elements, spectrum of elements and form of scalar homomorphism etc. in $\Gamma_1^{\circ}$.

Analogously, subspaces of entire Dirichlet series have also been equipped with additional algebraic structures. Thus Chakraborty [21] introduced the class $\mathcal{Q}$ of all entire Dirichlet series having same sequence $\{\lambda_n\}$ of exponents, satisfying (1.2.2). After defining addition (+) and multiplication by scalars in $\mathcal{Q}$ in the usual manner he also defined star multiplication (*) in $\mathcal{Q}$. Thus for $f, g \in \mathcal{Q}$, $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$, $g(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$, $f \ast g$ is given by

$$
(1.5.1) \quad (f \ast g) (s) = \sum_{n=1}^{\infty} a_n b_n e^{s\lambda_n}
$$

After showing that $\mathcal{Q}$ is closed w.r.t. the two composition '+$'$ and '$\ast$', Chakraborty [21] showed that $\mathcal{Q}$ is a commutative Banach-algebra without identity.

Since $\mathcal{Q}$ was without identity, therefore in an attempt to generalize the results of Chakraborty [21],
Juneja and Srivastava [69] considered the class $\mathcal{Q}_u$ of all functions $f$ of the form $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ where $\{\lambda_n\}$ satisfies (1.2.2) with $|a_n/\alpha_n|$ bounded and $u$, given by $u(s) = \sum_{n=1}^{\infty} \alpha_n e^{s\lambda_n}$ is a fixed Dirichlet series s.t. $\alpha_n \neq 0$ for all $n$ and $\sum u$, the abscissa of absolute convergence of $u$ satisfies $\sigma_u \geq \alpha$, $(-\infty < \alpha < \infty)$. Then clearly $\mathcal{Q}_u$ is a Normed linear space w.r.t. the usual operation of addition and scalar multiplication with norm $|| \cdot ||$ defined as

\[(1.5.2) \quad ||f|| = \sup_u |a_n/\alpha_n|, \quad f \in \mathcal{Q}_u.
\]

Now a multiplication in $\mathcal{Q}$ of the following type

\[(1.5.3) \quad (f.g)(s) = \sum_{n=1}^{\infty} \frac{a_n b_n}{\alpha_n} s^{\lambda_n}
\]

makes the Normed linear space $\mathcal{Q}_u$ (in fact Banach space) a commutative Banach algebra with identity $u$. Then they made a detailed study of the space $\mathcal{Q}_u$ such as characterization of regular and singular elements, topological zero devisors, spectral radius and self adjoint and unitary elements etc.

Further interesting resulting, studying the various aspects of the theory of Banach algebra, $B^*$-algebra, Fréchet algebra and related topics may be found in [48, 62, 80, 90, 160], [53, 101], [54].

1.6. **Sequence Spaces (Scalar Valued)**

Many of the spaces of entire or analytic functions defined by power series or Dirichlet series may be thought as
sequence spaces since topology in these spaces is being generated with the help of the sequence of the coefficients in the series defining the function. Thus, any development in the theory of sequence spaces is bound to have an impact on the study of spaces of analytic functions and vice versa.

Now-a-days, the theory of sequence spaces, which had its birth in the early work of Köthe and Toeplitz, stands as separate discipline. The books by Kothe [79], Cooke [22], Hardy [44], Maddox [94], Lindenstrauss and Tzafrir [86] bear an eloquent testimony to this fact. These books deal with the elementary treatment of convergence notions in some familiar sequence spaces and transformation thereof related to infinite matrices. Matrix transformation on several known spaces of sequences as well as on some special spaces of functions have been a subject of central investigation by many mathematicians like Brando [19], Rao [122], Raphael [121] and Ramanujan [123] etc. Mention must also be made of the special work of Maddox, who in a series of papers has carefully studied the matrix transformation on sequence spaces along with other properties of the spaces. Lascarides in [84] carried over the work of Maddox and studied various other aspects of sequence spaces like $c_0(p), c_0(1/k)$, $1(p)$ etc. In his process of investigation, he found that Iyer's [56] interesting theorem concerning the equivalence of notion of strong and weak convergence in $\Gamma = c_0(1/k)$ is true for a more general class of spaces, namely for the space
is a sequence of strictly positive number tending to zero. He further characterized certain classes of matrix transformation and showed that certain theorems proved by Rao [122, Thm. 2] for the space of entire functions \( f \) are particular cases of his theorems. Thus he obtained that for \( p = (p_k) \) in \( l_\infty \),

\[ A = \{ \xi_{nk} \}_{n,k \in \mathbb{N}} \subset [c_0(p), l_\infty(p)] \], the class of all infinite complex matrices transforming \( c_0(p) \) into \( l_\infty(p) \) (in the usual definition of matrix transformation), if and only if there exists an absolute constant \( B > 1 \) such that

\[ \sup_{n} \sum_{k=1}^{\infty} |\xi_{nk}|B^{-r_k} p_n < \infty \]  

where \( r_k = p_k^{-1} \)

for each \( k \).

The strong Cesaro summability of order 1 and index \( p \) is denoted as \( [c, 1, p] \). We say that \( \{x_k\} \to 1 [c, 1, p] \) or \( \{x_k\} \) is summable \( [c, 1, p] \) to 'l' if and only if

\[ \sum_{k=1}^{n} |x_k - 1|^p = o(n) \]

Clearly 'l' is unique whenever it exists [95]. Summability \( [c, 1, p] \) is perhaps of special interest as being probably the only method of attaching a conventional sum to a divergent series which has been seriously investigated and which is not given either by a Toeplitz method or by any other analogous transformation. Maddox [94] studied the space \( W_p \) of all
strongly Cesàro summable complex sequences of order 1 and index p for its various properties. Thus he showed that

\[(1.6.3) \quad W_p = \{ x = \{x_k\} \subseteq C \mid x \text{ is summable } [c,1,p] \text{ to zero } \}

is a linear space w.r.t. usual operation of addition of sequences and scalar multiplication. Further he defined two equivalent norms on it as follows.

\[(1.6.4) \quad \|x\| = \sup_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \quad \text{or} \quad \|x\| = \sup_r \left( \frac{1}{2^r} \sum_{r} |x_k|^p \right)^{1/p}
\]

\[(1.6.5) \quad \|x\| = \sup_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right), \quad \text{or} \quad \|x\| = \sup_r \left( \frac{1}{2^r} \sum_{r} |x_k|^p \right)
\]

respectively for \(p \geq 1\) and \(0 < p < 1\), where \(\Sigma_r\) denote the sum over the range \(2^r \leq k < 2^{r+1}\) and observed that \(W_p\) is a complete Normed linear space for \(p \geq 1\) while for \(0 < p < 1\) it is a complete p-normed space. He also characterized all complex matrices \(A = [\xi_{nk}]_{n,k \in N}\) which map \(W_p\) in 'c' (the space of convergent sequences).

Maddox [96] in his another attempt to generalize the famous Kuttner's theorem considered the space \(W_p(X)\) of all \(X\)-valued (\(X\) is a Banach space) sequences which are strongly Cesàro summable to zero of \(X\). As above he proved here also that \(W_p(X)\) is a complete p-normed space with the
\[ ||x|| = \sup_r \left( \frac{1}{2^r} \Sigma_r ||x_k||^p \right), \quad x = \{x_k\} \in W_p(X) \]

for \( 0 < p < 1 \), and it is a complete linear space for \( 1 \leq p < \infty \) w.r.t. the norm,

\[ ||x|| = \sup_r \left( \frac{1}{2^r} \Sigma_r ||x_k||^p \right)^{1/p}, \quad x = \{x_k\} \in W_p(X) \]

where \( r \geq 0 \) and \( \Sigma_r \) denote the sum over \( 2^r \leq k < 2^{r+1} \).

In his same attempt he also characterized the class \( [W_p(X), C(Y)] \) of all infinite matrices \( A = [A_{nk}]_{n,k \in \mathbb{N}} \) of bounded linear operations from the Banach space \( X \) into the Banach space of all convergent \( Y \)-valued sequences \( C(Y) \). In fact he proved that, for \( 1 \leq p < \infty \), \( A \in [W_p(X), C(Y)] \) if and only if there exist \( \lim_{n} A_{nk} \) (for each \( k \)), \( \lim_{n} \Sigma_{k=1}^{\infty} A_{nk} \) and the two number \( M_1 \) and \( M_2 \) given by

\[ M_1 = \sup_{r=0}^{\infty} \Sigma_{r} 2^r \left( \Sigma_{r} ||A_{nk}||^q \right)^{1/q} \]

and

\[ M_2 = \sup_{r=0}^{\infty} \Sigma_{r} 2^r \left( \Sigma_{r} ||A_{nk}^* - A_{k}^*||^q \right)^{1/q} \]

where \( \frac{1}{p} + \frac{1}{q} = 1 \), suprema being over all \( n \geq 1 \) and all \( \varnothing \in S_1^* \), the unit sphere of \( Y^* \) (dual of \( Y \)) and \( A^* \), \( A_{nk}^* \) denote the adjoint operators on \( X^* \) (dual of \( X \)) corresponding to \( A \) and \( A_{nk} \) on \( X \), must be finite. Similarly for \( 0 < p < 1 \), \( A \in [W_p(X), C(Y)] \) if and only if \( \lim_{n} A_{nk} = A_k \) (for each \( k \)) exists and

\[ M'_1 = \sup_{r=0}^{\infty} \Sigma_{r} 2^{r/p} \max_{r} ||A_{nk}^* \varnothing||, \quad M'_2 = \sup_{r=0}^{\infty} \Sigma_{r} 2^{r/p} \max_{r} ||(A_{nk}^* - A_k^*) \varnothing|| \]

are finite. The case when \( p = 1 \), \( W_p \) is denoted as \( \|c_1\| \), the
space of Cesàro summable complex sequences. The functional analytic studies of this space has been made in [18, 94, 95]. But a generalization of the techniques involved in these studies of $|\sigma_1|$ was given by Freedman and his group in [35]. Further interesting result in this direction of study may also be found in [99], [135] and [12] etc.

1.7 Vector-Valued Functions and their Spaces

By vector-valued functions (VVF), we mean a mapping $f$ from $C$ into either a Banach space $E$ over the complex field $C$ or into a space $L(E,F)$ of linear bounded transformation from the Banach space $E$ to Banach space $F$.

The theory of vector-valued function (VVF) is much more recent than the theory of scalar valued functions. In fact, very little was done with the theory of VVF (Banach space-valued function) until 1938. In 1938, B.J. Pettis [111, 112] introduced the concept of weak integration while Dunford [30] proved the important result that weakly holomorphic functions with values in a complex Banach space are strongly holomorphic. Almost simultaneously the important paper of I.M. Gelfand [39] dealing with abstract functions and linear operators also appeared. Bochner and Taylor [15] characterized the form of linear transformations on some abstractly-valued function spaces. Later in 1951, Grothendieck [41] extended the theory of holomorphic functions from the domain of scalars to an arbitrary
locally convex space by using extensively Cauchy's integral formula. T.H. Hilderbrandt and A.E. Taylor, the initiators of this line of thought have described the history of vector-valued holomorphic functions in [47] and [151] respectively.

There is also a theory of analyticity in commutative Banach-algebra due to E.R. Lorch [88]. The interesting theory of analytic functions on vectors to vectors started by M. Frechet was further developed under weak assumptions by R. Gateaux [38]. Analytic function of infinitely many variables were considered by D. Hilbert and F. Riesz. Further interesting work along different aspects of this field can be found in Wiener [155], I. Gel'fand [39], Zorn [161-163] Taylor [149,150], R.S. Philliphs [113, 114], E. Hille [49, 50], and Bishop [14] etc.

The very aspect of the theory of vector-valued function defined by a power series has been given an exhaustive and systematic treatment by Hille and Philliphs in their treatise [51]. In this treatise they not only generalize the theory of power series into vector-valued ones but also showed that a substantial portion of the elements of complex function theory extends to Banach spaces. Grothendieck [41] initiated the study in this direction. Lalitha [82] considered the space \( \Gamma \) of all entire function \( f, f(z) = \sum_{n=0}^{\infty} a_n z^n \) with \( a_n \)'s belonging to a given Banach space \( B \). She extended the results of Iyer [56, 57] to vector-valued entire functions.
Thus she showed that is a complete linear metric space with the metric defined by paranorm \( p \), given by

\[(1.7.1) \quad p(\alpha) = \text{lub} (\|a_0\|, \|a_n\|^{1/n}, n \geq 1)\]

where \( \alpha, \alpha(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to \( \Gamma \). The topology of \( \Gamma \) is the least upper bound of the family of norm topologies defined by the norms

\[(1.7.2) \quad \|\alpha, R\| = \sum_{n=0}^{\infty} |a_n| R^n, \quad (R > 0)\]

The continuous linear functional on \( \Gamma \), when \( B \) is complex plane is given by \( \phi(\alpha) = \sum_{n=0}^{\infty} c_n a_n \) where \( \{c_n\}^{1/n} \) is bounded. This form is also preserved in her space. G.D. Taylor [153] considered the space of power series whose coefficients lie in a Hilbert space. He studied in quite detail the multiplier problems in the space \( D^{(1)}_\alpha \), defined as

\[(1.7.3) \quad D^{(1)}_\alpha = \{f, f(z) = \sum_{n=0}^{\infty} a_n z^n : a_n \in H, H \text{ is a given Hilbert space}\},\]

while Somasundara in [138] studied the Hilbert space of entire function as a particular case of Taylor [153]. On the other hand Pietsch [117] has given a generalization to the Köthe theory of sequence spaces, thus giving rise to vector-valued sequence spaces (VVSS). But as a matter of fact Phoung Cac [115, 116] initiated the investigation in VVSS which appeared more closely to the study of Köthe and Toeplitz of scalar-valued sequence spaces (SVSS). On the lines of Phoung Cac,
but in a slightly more general setting, are the works of Gregory [40], Garling [37] and De Grande-De Kimpe, N. [26] dealing the different aspects of the theory of VVSS. Leonard [85] while attempting to solve some question related about the random-Nikodyin property in $l_p(X)$ studied the Banach sequence spaces. The classical theory of transformation on SVSS by infinite matrices of complex numbers got a decisive break by the work of A. Robinson [128], Melvin and Melvin [105], when they considered the action of infinite matrices of linear operators from a Banach space on sequence of elements of that space. Robinson [128] infact proved a generalization of celebrated theorem of Toeplitz and Silverman regarding matrix transformation. Thus, if $\{x_k\}$ be a sequence in a complete linear metric space $X$, the transform of $\{x_k\}$ into a sequence $\{y_k\}$ in the same space by the following formula

$$y_n = \sum_{k=1}^{\infty} A_{nk} x_k, \quad n = 1, 2, 3, \ldots$$

where $A = [A_{nk}]_{n,k \in \mathbb{N}}$ being an infinite matrix of linear bounded operators on $X$ is applicable to every convergent sequence $\{x_k\}$ in $X$, in the sense that the limit of the transformed sequence $\{y_k\}$ coincide with the limit of the original sequence, it is necessary and sufficient that (i) $\lim_{n} A_{nk} = 0$ (zero operator of $X$) with $\lim_{n} A_n = I$ (identity operator)

(ii) $\sum_{k=1}^{\infty} A_{nk} = A_n$ exists for each $k$ in such a way that

$$\lim \sup_{n} \| \sum_{k=1}^{\infty} A_{nk} x_k \| \leq M < \infty.$$

$$\|x_k\| \leq 1 \quad k=1$$
The study of Robinson was further carried over by Lorentz and Macphail [89], Ramanujan [123] and Maddox [99] etc. Some more interesting matter on VVSS and matrix transformations thereon can be found in [97, 98], [42], [43] and [107]. The extension of Köthe theory to function spaces was carried over by Persson [110], Dieudonné [27] and Macdonald [91,92,93] etc.

Hence we see that the study of spaces and algebras of holomorphic functions has been vastly enriched in different direction by a number of mathematicians who have made important and significant contribution to it, by their fruitful and new ideas. However, we in this present introductory chapter confined ourselves to only those aspects of the theory in which we have attempted to make our contribution. Thus keeping in view the modern trend in mathematics, of abstraction, which is playing a prominent role in connection with many problems, especially of analysis, the present thesis is devoted to vector-valued Dirichlet series (VVDS) and their spaces etc. As classical Dirichlet series are considered a natural generalization of power series, so the concept of VVDS may also be thought of an abstract generalization of vector valued power series.

The present thesis consists of six chapters including the first introductory chapter. After introducing the concept of a Vector-Valued Dirichlet Series (VVDS) in chapter II,
its convergence properties like weak and strong convergence, absolute convergence, unconditional weak and strong convergence etc. have been studied. Various abscissas of convergence thus arising have been shown to be related to one another. After investigating the region of analyticity of a VVDS, the concepts of measurement of growth in terms of order and type for both an entire and a non-entire VVDS have been introduced and their characterizations in terms of coefficients and exponents have been obtained.

In Chapter IIIrd, we study \( \mathcal{Q}_u \), the space of all vector-valued function \( f \), \( f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \) being a VVDS such that \( \| a_n \alpha_n^{-1} \| \) is bounded where \( u(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \) is a fixed Dirichlet series. \( \mathcal{Q}_u \) has been shown to be a Banach-algebra, characterization of regular and singular elements, topological zero divisors, self adjoint and unitary elements have been obtained. It is also shown that \( \mathcal{Q}_u \) with a suitable involution induced from the involution on underlying space \( \mathcal{E} \) is a \( \mathcal{B}^* \)-algebra. Finally a partial ordering on \( \mathcal{Q}_u \) is introduced thereby making it a Banach lattice.

Chapter IV is a brief study of certain sub-algebras of \( \mathcal{Q}_u \). For these sub-algebras which are without identity, a characterization of quasi-regular elements, topological zero divisors, approximate identity etc is obtained. By introducing another norm, a two norm space is obtained which has been shown to be \( \gamma \)-complete.
Multipliers from $\Omega_u$ to $l_p(E)$ or $H^p(E)$ and vice-versa have also been characterized in this chapter.

Chapter V has been devoted to obtaining characterization of matrix transformations of bounded linear operators on various spaces like $\Omega_u$, $\Omega_o$, $\Omega_1$ and $\Omega_c$ that have been introduced and studied in previous chapters. A number of well known classical results follow as particular cases from the results of this chapter.

Chapter VI, the last chapter deals with study of various topological structures on certain classes of vector-valued functions represented by VVDS. On the class of entire VVDS three linear metric topologies have been described and shown to be equivalent. This space which is not normable is complete and thus a Frechet space. After showing that convergence in this space is equivalent to convergence on compact subsets of $\mathbb{C}$, a characterization of continuous linear functional has been obtained. A Hilbert space structure on a class of vector-valued entire functions is also constructed and some interesting properties including study of multipliers on it have been obtained. A space of VVDS whose coefficients are Cesaro summable has been defined in this section. This space which is a $p$-normed space is complete such that all its elements have abscissa of absolute convergence $\geq \sigma_a^u = \alpha$. Finally matrix transformation from this space to $\Omega_c$ has been characterized.
PART B

In this part, we present without proof a review of certain definitions and known results from Normed linear spaces (NIS), Banach spaces, Theory of vector-valued functions, Banach Algebras and infinite matrices of operators as a prelude to our work in subsequent chapters.

1.8 NIS, Banach Space, Hilbert space, Sequence and Series in a Banach Space.

Definition 1.8.1 [156, p. 52] - A paranorm is a real function 'p' defined on a linear space X and satisfying the condition (a) through (e) for all x, y ∈ X

(a) \( p(0) = 0 \)
(b) \( p(x) \geq 0 \)
(c) \( p(-x) = p(x) \)
(d) \( p(x+y) \leq p(x) + p(y) \)
(e) If \( \{t_n\} \) is a sequence of complex numbers with \( t_n \to t \) and \( \{x_n\} \) is a sequence of vectors converging to x, then \( p(t_n x_n - t_x) \to 0 \) (x → ∞).

Further if \( p(x) = 0 \Rightarrow x = 0 \), p is called total.

Remark 1.8.1 [156, p. 53, 54] - Followings are the well known facts about the paranorm p on a linear space X,

(a) The set \( (p < \varepsilon) \) is absorbing for each \( \varepsilon > 0 \)
(b) \( p(x-y) \geq |p(x) - p(y)|, \) for all \( x, y \in X \)
(c) \( p(nx) \leq np(x) \) and \( p\left(\frac{x}{n}\right) \geq \frac{p(x)}{n}, \) where \( x \in X \) and \( n \in \mathbb{N} \) (set of natural numbers)
(d) If \{t_n\} is a bounded sequence of complex numbers and 
p(a_n) \to 0 (n \to \infty) then \( p(t_n a_n) \to 0 \) as \( n \to \infty \).

(e) If there exists a number \( M \) such that \( q(x) \leq M p(x) \) 
then \( p \) is stronger than \( q \). The converse is false.

**Theorem 1.8.1** [156, p. 54] - Let \( \{p_n\} \) be a sequence of 
paranorms on \( X \), let,

\[
p(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1 + p_k(x)}
\]

(Prechet combination of \( p_n \) 's).

Then,
(a) \( p \) is the **weakest** paranorm which is stronger than each \( p_k \).
(b) \( p(x_n) \to 0 \) (n \to \infty) if and only if \( p_k(x_n) \to 0 \) (n \to \infty) 
for each \( k \).

**Definition 1.8.2** [156, p. 56] - A **semi-norm** \(| \cdot \|\) is a real 
function defined on a linear space \( X \), satisfying for all \( x, y \in X \) and \( t \in C \).

(a) \(|x| \geq 0\)
(b) \(|tx| = |t||x|\)
(c) \(|x+y| \leq ||x|| + ||y||\)

A norm is a total seminorm i.e. a semi-norm in which \(|x| = 0 \) 
\( \Rightarrow x = 0 \). A semi-norm is essentially a paranorm therefore facts 
included in Remark (1.8.1) and theorem (1.8.1) continue to hold 
for a semi-norm.
A Banach Space is a Normed linear space (NLS) which is complete under the metric given by the norm on it. The metric topology induced by the norm in a Banach space $X$ is called norm or strong topology for $X$.

**Definition 1.8.3** [17, p. 18] - Two norms $||x||_1$ and $||x||_2$ on a linear space $X$ are equivalent if they determine the same topology on $X$, and that this holds iff there exists positive constants $M_1$ and $M_2$ such that

$$||x||_1 \leq M_1 ||x||_2;$$

$$||x||_2 \leq M_2 ||x||_1$$

for all $x \in X$.

**Definition 1.8.4** [156, p. 61] - A metric 'd' for a linear space $X$ is said to be invariant if there exists a paranorm 'p' such that

$$d(x,y) = p(x-y)$$

Clearly an invariant metric 'd' satisfies the equation

$$d(x+z, y+z) = d(x,y)$$

for $x, y$ and $z \in X$.

**Definition 1.8.5** [156, p. 107] - A normed linear space (NLS) $X$ is called rotund if and only if $x \neq y$ and $||x|| = ||y|| \leq 1$ implies $||x+y|| < 2$.

**Definition 1.8.6** [156, p. 61] - A complete linear metric space is called a Fréchet space.

**Theorem 1.8.2** [156, p. 84] - An absolutely convergent series in a Fréchet space is convergent.
Definition 1.8.7 [156, p. 86] A Schauder basis for a linear metric space $X$ is a sequence $\{x_n\}$ in $X$ such that for any $x \in X$, there exists a unique sequence $\{t_n\}$ of scalars such that

$$x = \sum_{n=1}^{\infty} t_n x_n$$

where the convergence take place in the metric topology of $X$.

Definition 1.8.8 [30, p. 74] A Banach space $(x, \|\cdot\|)$ is uniformly convex, $||x_n|| = 1, ||y_n|| = 1, ||x_n + y_n|| - 2$ implies $||x_n - y_n|| - 0$. For some more general information about Normed linear space see [25] and [30].

Theorem 1.8.3 [137, Thm.2.7] Let $X$ be a vector-space over $C$, 'd' an invariant metric on $X$. Let $\{\|\cdot\|_p\}_{p=1}^{\infty}$ be a sequence of norms on $X$ such that

(1) $p \leq q$ implies $\|x\|_p \leq \|x\|_q$ for all $x \in X$;

(ii) For each $\varepsilon > 0$, there exists $p$ and $\delta > 0$ such that for all $x \in X$, $\|x\|_p < \delta$ implies $d(x, 0) < \varepsilon$, and

(iii) the metric topology is finer than each norm topology.

The following are then true.

(a) If $S$ is a subspace of $X$ with metric $d$, $S^* = \bigcup_{p=1}^{\infty} (S, \|\cdot\|_p)^*$

(b) $F(\infty \to \infty) = \bigcap_{q=1}^{\infty} \bigcup_{p=1}^{\infty} F(p \to q)$

where $F(\infty \to \infty)$ denote the set of linear functions from $(X, d)$ into itself that are continuous. $F(p \to q)$ the continuous linear functions from $(X, \|\cdot\|_p)$ into $(X, \|\cdot\|_q)$ and $S^*$ is the dual of $(S, d)$.
Remark 1.8.2 [137] — Let X be a vector space over $\mathbb{C}$ and $\{||f||_p\}_{p=1}^{\infty}$ a sequence of norms on X satisfying $||f||_p \leq ||f||_q$ for all $f \in X$ if $p \leq q$. Define an invariant metric 'd' in X by

$$d(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{||f-g||_k}{1+||f-g||_k}$$

(Fréchet combination).

Then (ii) and (iii) of Theorem (1.8.3) are satisfied.

Definition 1.8.9 [48] — A linear transformation $T$ from a Banach space X to Y is bounded if there exists a constant $M$ such that for all $x \in X$,

$$||Tx|| \leq M ||x||$$

The least value of $M$ for which the inequality is true is called the norm of $T$ written as $||T||$. Also the norm, if it exists, may be defined directly by

$$||T|| = \sup \{||Tx|| : ||x|| = 1, x \in X\}$$

Remark 1.8.3 [48] — All bounded linear transformations with domain, the Banach space X and range the Banach space Y form the set $L(X,Y)$. We know further that $L(X,Y)$ is a Normed linear space (in fact Banach space) under the norm $||T||$.

Theorem 1.8.4 [48] — Let $\{T_n\}$ be a sequence of linear bounded transformations in $L(X,Y)$ such that

(i) $||T_n|| \leq M$ for all $n$, where $M$ is some positive constant

(ii) $\lim_{n \to \infty} T_n(x)$ exists for every $x$ in a set $S$, dense in an open sphere $G$. Then
(a) \( \lim_{n \to \infty} T_n(x) \) exists for all \( x \) and the limit defines an element \( T \) of \( L(X,Y) \) with \( ||T|| \leq \lim_{n \to \infty} ||T_n|| \).

This theorem is also called the Banach-Steinhaus theorem.

**Remark 1.8.4** [52, p. 8] Let \( X \) be a Banach space, one important property of continuous linear functionals on \( X \) is the Hahn-Banach extension theorem. If \( \phi \) is a bounded linear functional on a subspace \( Y \) of \( X \), then \( \phi \) can be extended to a linear functional on \( X \) with precisely the same bound (norm) as \( \phi \).

**Definition 1.8.10** [95] If \( X \) is a complex linear space, then a functional \( ||;|| \) is a \( p \)-norm if there exists \( p > 0 \) such that

(a) \( ||tx|| = |t|^p ||x|| \)
(b) \( ||x|| = 0 \iff x = 0 \)
(c) \( ||x+y|| \leq ||x|| + ||y|| \), for every \( x, y \in X \) and every scalar \( t \in \mathbb{C} \). Clearly a 1-norm is an ordinary norm.

Now let \( Y \) be a NIS and \( T \) be a linear transformation on the \( p \)-normed space \( X \) into \( Y \), then it is easy to see that \( T \) is continuous on \( X \) if and only if there exists a positive constant \( M \) such that

\[ ||Tx|| \leq M ||x||^{1/p} \] on \( X \)

The norm on the left is the norm in \( Y \) and that on the right a \( p \)-norm in \( X \). It is also easy to check that Banach-Steinhaus theorem holds in \( p \)-normed spaces.
Theorem 1.8.5 [95] - If \( \{T_n\} \) is a sequence of continuous linear transformation on a p-normed space \( X \) into a NIS \( Y \) such that
\[
\limsup_{n \to \infty} ||T_n(x)|| < \infty
\]
on a second category set in \( X \), then
\[
\sup_n ||T_n|| < \infty.
\]

Remark 1.8.5 [95] - Let \((X, d)\) be a metric space, a set \( A \) in \( X \) is said to be nowhere-dense if its closure \( \overline{A} \) contains no sphere or equivalently if \( \overline{A} \) has no interior point. A subset \( A \) of \( X \) is said to be of first category in \( X \) if it is the union of countably many nowhere sets in \( X \) otherwise, \( A \) is said to be of the second category in \( X \).

Definition 1.8.11 [103, p. 6] - A (real or complex) Hilbert space \( H \) over the field \((\mathbb{R} \text{ or } \mathbb{C})\) is a linear space together with a scalar-valued function \( (,); \), called the inner product, defined for each pair of points \( x, y \) in \( H \). This function satisfies the following conditions.

(a) \((x,x) \geq 0; \ (x,x) = 0 \) if and only if \( x = 0, x \in H \);
(b) \((x+y, z) = (x,z) + (y,z), x,y,z \in H \);
(c) \((tx, y) = t(x,y), t \in \mathbb{C} \text{ and } x, y \in H \);
(d) \((x,y) = (y,x), x, y \in H \text{ and}
(e) \( H \) is complete with respect to the metric defined by the norm \( ||x|| = (x,x)^{1/2}, x \in H. \)
Remark 1.8.6 [130, p. 83]

By the definition (1.8.11), an easy computation establishes the identity

\[ ||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2 \]

This is known as the parallelogram law.

Definition 1.8.12 [103, p.12] - A sequence \( x_n \) in a Banach space \( X \) is said to be weakly convergent if \( \lim_{n \to \infty} x^*(x_n) \) exists for each \( x^* \in X^* \) (the dual of \( X \)), it is said to be weakly convergent to a point \( x \in X \) if \( \lim_{n \to \infty} x^*(x_n) = x^*(x) \) for every \( x^* \in X^* \) and converges strongly to \( x \in X \) if \( \lim ||x_n - x|| = 0 \).

Definition 1.8.13 [48, p.230] - Let \( X \) be a given Banach space. Let \( \sum_{k=1}^{\infty} x_k \) be a series in \( X \) with its sequence of partial sum \( S_n = \sum_{k=1}^{n} x_k \) for \( n = 1, 2, 3, \ldots \). Then the infinite series \( \sum_{k=1}^{\infty} x_k \) is said to converge weakly (strongly) to \( s \in X \) if the sequence \( \{ S_n \} \) converges weakly (strongly) to \( s \), in that case series is called convergent and may be written as

\[ \sum_{k=1}^{\infty} x_k = s \]

Further, the series is absolutely convergent if \( \sum_{k=1}^{\infty} ||x_k|| \) converges, unconditionally weakly (strongly) convergent if every subseries is weakly (strongly) convergent.

Theorem 1.8.6 [48, p. 230] - If \( \sum_{k=1}^{\infty} x_k \) converges weakly then \( ||S_n|| \) is uniformly bounded.
Theorem 1.8.7 [51] - If \( \sum_{k=1}^{\infty} x_k \) is weakly unconditionally convergent then
\[
\lim_{N \to \infty} \sum_{k=1}^{N} |x^*(x)| = 0 \text{ uniformly on the unit sphere } S_1^* \text{ of } X^*.
\]

Theorem 1.8.8 [86, p. 15] - Let \( X \) be a Banach space and \( \sum_{k=1}^{\infty} x_k \) a convergent series in \( X \). The following statements are equivalent:

(a) \( \sum_{k=1}^{\infty} x_k \) is unconditionally convergent
(b) \( \sum_{i=1}^{\infty} x_{k_i} \) converges for every choice of \( k_1 < k_2 < k_3 < \ldots \)
(c) \( \sum_{k=1}^{\infty} x_{\pi(k)} \) converges for every permutation \( \pi \) of integers.
(d) \( \sum_{k=1}^{\infty} \theta_{k} x_k \) converges for every choice of \( \theta_n = \pm 1 \).
(e) For every \( \varepsilon > 0 \), there exists an integer \( N \) so that
\[
\| \sum_{i \in \sigma} x_i \| < \varepsilon \text{ for every finite set of integers } \sigma \text{ which satisfies } \min \{ i \in \sigma \} > N.
\]

Following is the important theorem of Orlicz-Pettis which claims the equivalence of weak and strong unconditional convergence of series in a Banach spaces.

Theorem 1.8.9 [103, p. 27] - If a series converges unconditionally in the weak topology of \( X \), then it is unconditionally convergent in the strong topology of \( X \).

Remark 1.8.7 [103] - It is clear that every absolutely convergent series in a Banach space is unconditionally
convergent, the converse is however not true in an infinite dimensional Banach spaces as the famous Dvoretzky-Rogers theorem [29] states the existence of an unconditional series in every infinite dimensional Banach space which is not absolutely convergent.

Theorem 1.8.10 [103, p. 27] - (Dvoretzky-Rogers) - The unconditionally convergent series coincide with the absolutely convergent series in a Banach space \( X \) if and only if \( X \) is finite dimensional.

1.9 Analysis in Banach Spaces

Definition 1.9.1 [48, p. 226] - A vector function \( x(s) \) which is defined on a set \( S \) of \( \mathbb{R} \) with values in a Banach space \( X \) is

(a) **weakly continuous** at \( s = s_0 \in S \) if

\[
\lim_{s \to s_0} \left| x^*(x(s)) - x^*(x(s_0)) \right| = 0 \quad \text{for every } x^* \in X^*
\]

(b) **strongly continuous** at \( s = s_0 \in S \) if

\[
\lim_{s \to s_0} \| x(s) - x(s_0) \| = 0.
\]

Definition 1.9.2 [48, p. 227] - A vector function \( x(s) \) from the interval \((a, b)\) to the Banach space \( X \) is **weakly** (strongly) differentiable at \( s = s_0 \) if there is an element \( x'(s_0) \in X \) such that the difference quotient \( h^{-1} [x(s_0 + h) - x(s_0)] \) tends weakly (strongly) to \( x'(s_0) \) as \( h \to 0 \). We call \( x'(s_0) \) the weak or (strong) derivative of \( X(s) \) at \( s = s_0 \).
Theorem 1.9.1 [48,p.227] If the weak derivative of \( x(s) \) exists and is zero everywhere in \((a,b)\), then \( x(s) \) is a constant.

Remark 1.9.1 [48,p.227] For a series \( \sum x_k(s) \), \( a \leq s \leq b \) with elements in \( X \) may have convergence properties holding uniformly with respect to \( s \). Definition of uniformly weak (strong) convergence etc. can be formulated in usual way.

We consider two Banach spaces \( X \) and \( Y \), which may coincide, with their duals \( X^* \) and \( Y^* \). Let \( s \) be a complex variable, \( D \) a domain in the complex \( s \)-plane and \( x = f(s) \) a mapping from \( D \) into \( X \). Clearly it is a vector-valued function. What is to be understood by saying that \( f(s) \) is holomorphic in \( D \) goes as follows.

Definition 1.9.3 [48,p.251] We say that \( f(s) \) is holomorphic in \( D \) (notations as above) if \( x^*(f)(s) \) is holomorphic in \( D \) in the sense of Cauchy for every choice of \( x \in X \), \( x^* \in X^* \).

Theorem 1.9.2 [51,p.93] If \( f(s) \) is holomorphic in \( D \), then \( f(s) \) is strongly continuous and strongly differentiable in \( D \), uniformly with respect to \( s \) in any compact subset of \( D \).

Remark 1.9.2 [51,p.94] Let \( \gamma \) be a rectifiable curve in the complex \( \xi \)-plane given by \( \xi = \xi(t) \), \( 0 \leq t \leq t_0 \), where \( \xi(t) \) is continuous and of bounded variation (see Definition 7.3.3 of [48]). If \( f(\xi) \) is any strongly continuous function on \( \gamma \) to a Banach space \( X \), then the integral

\[
\int_0^{t_0} x(\xi(t)) \, d\xi(t) = \int_\gamma x(\xi) \, d\xi
\]
exists by Theorem 3.3.2 of and for any linear bounded functional \( x^* \in X^* \),

\[
x^* \left[ \int \gamma x(\zeta) d\zeta \right] = \int \gamma x^*(x(\zeta)) d\zeta \quad \text{holds.}
\]

**Theorem 1.9.3** [48, p.253-254] Suppose that \( f(s) \) is \( X \)-holomorphic in the sense of Definition given in (Def. 1.9.3) in a domain \( D \) of the complex plane. Suppose that \( \gamma \) is a simple closed rectifiable oriented curve in \( D \) such that \( f(s) \) is holomorphic inside and on \( \gamma \). Then

(a) \[
\int_{\gamma} f(t) \, dt = 0
\]

And, for any \( s \) inside \( \gamma \)

(b) \[
f(s) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(t)}{t-s} \, dt
\]

and

(c) \[
f^{(n)}(s) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(t) dt}{(t-s)^{n+1}} \text{ for all } n \in \mathbb{N}.
\]

where \( f^{(r)}(s) \) denotes \( r \)th strong derivative of \( f(s) \).

**Theorem 1.9.4** [48, p.255] If \( f(s) \) and \( g(s) \) are \( X \)-holomorphic in \( D \) and if a sequence \( \{s_n\} \) exists with distinct points and at least one cluster point in \( D \) such that \( f(s_n) = g(s_n) \), \( \forall n \in \mathbb{N} \), then \( f(s) = g(s) \) for all \( s \) in \( D \).

**Theorem 1.9.5** [48, p.254] If \( f(s) \) is \( X \)-holomorphic in the closed disk \( |t-s_0| \leq r \) and if \( \max |f(s_0 + re^{i\theta})| = M \), then
\[ ||f^n(s_0)|| \leq Mr^{-n}|n| \]

**Theorem 1.9.6.** [48, p. 256] If \( f(s) \) is \( X \)-holomorphic inside and on the simple closed rectifiable curve \( \gamma \) and if
\[
\max ||f(t)|| = M, \text{ then for every } s \text{ inside } \gamma
\]
\[ ||f(s)|| \leq M \]

with equality for one such 's' if and only if there is equality for all.

**Theorem 1.9.7** [48, p. 258] - Let the sequence \( \{f_n\} \) be made up of functions which are \( X \)-holomorphic in a domain \( D \). Suppose that the sequence converges uniformly on compact subsets of \( D \) to a limit \( f(s) \) then \( f \) is \( X \)-holomorphic in \( D \).

**Theorem** (Generalized Vitali Theorem) 1.9.8 [48, p. 259] Suppose that \( \{f_n\} \) is a sequence of functions \( X \)-holomorphic in a domain \( D \) where \[ ||f_n(s)|| \leq M \] for all \( n \) and all \( s \). Suppose there exists a sequence \( \{s_k\} \) in \( D \) with a cluster point \( s_0 \) in \( D \) such that \( \lim_{n \to \infty} f_n(s_n) \) exists for all \( k \). Then \( \lim_{n \to \infty} f_n(s) = f(s) \) exists for all \( s \) in \( D \), the limit \( f(s) \) is \( X \)-holomorphic in \( D \) and the convergence is uniform on compact subsets.

**Definition 1.9.4** [3] - If a linear space \( X \) is provided with two norms \( ||.||_1 \) and \( ||.|| \), then \( ||.||_1 \) is called finer than \( ||.|| \) if \( ||f_n||_1 \to 0 \) implies \( ||f_n|| \to 0 \) (or \( ||.||_1 \) is coarser than \( ||.||_1 \)) for a sequence \( \{f_n\} \) in \( X \).
Definition 1.9.5 [3] A two norm space is a linear space $X$ provided with two norms $\| \cdot \|_1$ and $\| \cdot \|$ such that $\| \cdot \|_1$ is finer than $\| \cdot \|$.

Definition 1.9.6 [3] The sequence $\{x_n\}$ in the two norm space $(X, \| \cdot \|_1, \| \cdot \|)$ is termed $\gamma$-convergent to $x_0 \in X$ (written as $x_n \xrightarrow{\gamma} x_0$) if $\sup_n \|x_n\|_1 < \infty$ and $\lim_{n \to \infty} \|x_n - x_0\| = 0$.

Definition 1.9.6a [3] A two norm space is called $\gamma$-complete if it is sequentially complete for the convergence, i.e., if it satisfies the condition

(a) If $(x_{p_n} - x_{q_n}) \xrightarrow{\gamma} 0$ as $p_n \to \infty$ and $q_n \to \infty$, then $x_0 \in X$ exists such that $x_n \xrightarrow{\gamma} x_0$.

Definition 1.9.7 [3] A functional $\psi$ on $(X, \| \cdot \|_1, \| \cdot \|)$ is said to be $\gamma$-linear if it is additive and $f_n \xrightarrow{\gamma} f$ implies $\psi(f_n) \to \psi(f)$ as $n \to \infty$, where $\{f_n\}$ is a sequence in $X$.

Definition 1.9.8 [3] A two norm space $(X, \| \cdot \|_1, \| \cdot \|)$ is called normal if $\lim_{n \to \infty} \|f_n - f\| = 0$ implies $\|f\|_1 \leq \liminf_{n \to \infty} \|f_n\|_1$.

Definition 1.9.9 [108] Let $(X, \| \cdot \|_1, \| \cdot \|)$ be a two norm space. If the set $S_0 = \{f \in X : \|f\|_1 \leq 1\}$ with distance $d(f, g) = \|f - g\|$ is complete space, then it is a Saks-space.

1.10 Banach Algebra

Definition 1.10.1 [63, p.3] A linear space $X$ over the complex field $\mathbb{C}$ is said to be an algebra if it is equipped with a binary operation, referred to as multiplication and denoted by
juxta position, from $X \ast X \to X$ such that for $x, y, z \in X$ and $\alpha \in \mathbb{C}$.

(i) $x(yz) = (xy)z$
(ii) $x(y+z) = xy+xz$ ; $(y+z)x = yx+xz$
(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

$X$ is said to be **commutative algebra** if $X$ is an algebra and

(iv) $xy = yx$ ; $x, y \in X$

where $X$ is said to be an **algebra with identity** 'e' if $X$ is an algebra and there exists some element $e \in X$ such that

(v) $ex = xe = x$ for every $x \in X$.

It is evident that if $X$ is an algebra with identity, then the identity element is unique.

**Definition 1.10.2** [a3, p.4] A normed linear space $(X, || \cdot ||)$ over $\mathbb{C}$ is said to be a **Normed-algebra** if $X$ is an algebra and

$$||xy|| \leq ||x|| \cdot ||y|| \quad \forall x, y \in X.$$ 

A normed algebra $X$ is said to be a **Banach-algebra** if the normed space $(X, || \cdot ||)$ is a Banach space. $X_1 \subseteq X$ is said to be a **sub-algebra** if $X_1$ is a subspace of $X$ such that $x, y \in X_1$ whenever $x, y \in X_1$.

**Theorem 1.10.1** [83, p.4] A closed sub-algebra of a Banach algebra is a Banach algebra.

**Definition 1.10.3** [83, p.5] Let $X$ be an algebra with identify $e$. An element $x \in X$, is said to have a **left (right) inverse** if there exists some $y \in X$, such that $yx = e$ ($xy = e$),
whereas \( x \in X \) is said to be inverse of \( y \in X \) if \( xy = yx = e \).

If \( y \in X \) has an inverse then \( y \) is said to be regular or invertible otherwise it is singular.

**Definition 1.10.4** [83, p.13] Let \( X \) be an algebra. An element \( x \in X \) is said to have a right (left) quasi-inverse if there exists some \( y \in X \) such that \( xoy = x+y - xy = 0 \), \( yox = y+x-xy=0 \) and \( x \) is said to have quasi-inverse if there exists some \( y \in X \) such that \( yox = xoy = 0 \). \( x \in X \) is quasi-invertible if it has quasi-inverse.

**Definition 1.10.5** [83, p.40] Let \( (X, \| \cdot \|) \) be a normed algebra then \( x \in X \) is said to be a left (right) topological zero divisor if there exists a sequence \( \{y_k\} \) in \( X \) such that \( \|y_k\| = 1 \), for each \( k \in \mathbb{N} \) and \( \lim_{k \to \infty} \|xy_k\| = 0 \)

(\( \lim_{k \to \infty} \|y_kx\| = 0 \)), and \( x \) is said to be topological zero divisor if and only if it is left as well as right topological zero divisor.

**Definition 1.10.6** [83, p.54] Let \( X \) be a Banach-algebra, let \( x \in X \), if \( X \) has an identity \( e \), then the spectrum of \( x \), denoted by \( \sigma(x) \), is the set of all \( \lambda \in \mathbb{C} \) such that \( (x - \lambda e) \) is singular; if \( X \) is without identity, then \( \sigma(X) \) is the set of all nonzero \( \lambda \in \mathbb{C} \) such that \( \frac{X}{\lambda} \) is quasi-singular together with \( \lambda = 0 \).

**Definition 1.10.7** [83, p.58] Let \( X \) be a Banach algebra. If \( x \in X \), then we set
\[ v(x) = \sup_{\lambda \in \sigma(x)} |\lambda|, \text{ then } v(x) \text{ is spectral radius of } x. \]

**Theorem 1.10.2** [83, p. 58] Let \( X \) be a Banach algebra. If \( x \in X \), then
\[
v(x) = \lim_{n \to \infty} \frac{||x^n||^{1/n}}{n}
\]

**Theorem 1.10.3** [125, p. 10] - The limit \( \nu(X) = \lim_{n \to \infty} \frac{||x^n||^{1/n}}{n} \) exists for each \( x \in X \) and has the following properties:

(i) \( \nu(X) = \inf_n ||x^n||^{1/n} \)

(ii) \( 0 \leq \nu(x) \leq ||x|| \)

(iii) \( \nu(\alpha x) = |\alpha| \nu(x), \alpha \in \mathbb{C} \)

(iv) \( \nu(xy) = \nu(yx) \) and \( \nu(x^k) = (\nu(x))^k, k = 1, 2, 3, \ldots \)

(v) If \( xy = yx \), then \( \nu(xy) \leq \nu(x) \nu(y) \) and \( \nu(x+y) \leq \nu(x) + \nu(y) \).

**Definition 1.10.8** [83, p.109] Let \( X \) be a commutative Banach algebra. A net \( \{e_\alpha\} \) in \( X \) is said to be an approximate identity if

(i) \( \sup_{\alpha} ||e_\alpha|| < \infty \)

(ii) \( \lim_{\alpha} ||e_\alpha - x|| = 0 \) \( \forall x \in X. \)

**Definition 1.10.9** [83, p. 273] Let \( X \) be an algebra. Then \( X \) is said to be an algebra with involution, if there exists a mapping \( * : X \to X \) such that for any \( x, y \in X \) and \( \alpha \in \mathbb{C} \),
we have

(i) \((x+y)^* = x^* + y^*\)
(ii) \((ax)^* = \overline{ax}\)
(iii) \((xy)^* = y^*x^*\)
(iv) \((x^*)^* = x^{**} = x\).

A Banach algebra \(X\) with an involution \(*\) is said to be a Banach \(*\)-algebra, a Banach\(*\)-algebra is said to be \(B^*\)-algebra if

\[ ||x^*x|| = ||x||^2, \quad x \in X \]

Remark 1.10.1 \([125, \text{p. 130}]\) - An involution \(*\) is isometry if

\[ ||x^*|| = ||x||. \]

Definition 1.10.10 \([17, \text{p. 63}]\) - Let \(X\) be a Banach \(*\)-algebra. An element \(x \in X\) is self adjoint with respect to the involution \(*\) if \(x^* = x\). The set of all self adjoint elements of \(X\) is denoted by \(\text{sym}(X)\).

Definition 1.10.11 \([17, \text{p. 66}]\) - Let \(X\) be a star algebra with unit (identity) \(1\). An element \(x \in X\) is unitary if \(xx^* = x^*x = 1\).

Definition 1.10.12 \([17, \text{p. 2}]\) - Let \(X, Y\) be two algebras over the same field. A homomorphism of \(X\) into \(Y\) is a linear mapping \(\phi\) such that

\[ \phi(xy) = \phi(x) \phi(y), \quad x, y \in X. \]

Definition 1.10.13 \([17, \text{p. 65}]\) - Let \((X^*), (Y, \ast)\) be star algebras. A star homomorphism \(\phi\) is a homomorphism \(\phi\) of \(X\) into \(Y\) such that

\[ \phi(x^*) = (\phi(x))^*, \quad x \in X. \]
1.11 Partially Ordered Set

Definition 1.11.1 [13] - A partially ordered set (Poset) \( P \) is a set in which a binary relation \( x \leq y \) is defined, which satisfies for all \( x, y, z \in P \), the following conditions

(i) \( x \leq x \), \( \forall x \in P \) (Reflexive)
(ii) If \( x \leq y \) and \( y \leq x \), then \( x = y \) (Anti-symmetry)
(iii) If \( x \leq y \) and \( y \leq z \), then \( x \leq z \) (transitivity)

Definition 1.11.2 [13, p. 6] - An upperbound (lowerbound) of a subset \( P_1 \) of a poset \( P \) is an element \( a \in P \) such that \( x \leq a(a \leq x) \) for every \( x \in P_1 \). An element \( z \in P \) is least upper bound (lub) of \( P_1 \) if \( z \) is an upper bound of \( P_1 \) and if \( z_1 \) is any upper bound of \( P_1 \) then \( z \leq z_1 \). Similarly, greatest lower bound (glb) can be defined.

Definition 1.11.3 [13, p. 6] - A lattice is a poset any two of whose element have a glb denoted by \( x \land y \) and a lub denoted by \( x \lor y \).

Definition 1.11.4 [13, p. 7]. A sublattice of a lattice \( L \) is a subset \( L_1 \) of \( L \) such that \( a \in L_1 \), \( b \in L_1 \) imply \( a \land b \in L_1 \) and \( a \lor b \in L_1 \).

Definition 1.11.5 [13, p. 13] - A lattice \( L \) is said to be modular if it satisfies the modular identity, viz, If \( x \leq z \), then \( x \lor (y \land z) = (x \lor y) \land z \) for every \( y \in L \).

Definition 1.11.6 [13, p. 28] A pgroup \( G_1 \) is a group which is also a poset and in which every group translation is isotone.
i.e. if $x \leq y$; $x, y \in G_1$ then $a + x + b \leq a + y + b$ for all $a, b \in G_1$. If the pogroup is also a lattice, it is called a lattice-ordered group or $l$-group.

**Definition 1.11.7** [13, p. 290] An element 'a' of a pogroup $G_1$ is said to be incomparably smaller than a given element $b \in G_1$ when $a + a + a + \ldots a$ (n summand) = $na \leq b$ for every integer $n$.

A pogroup $G_1$ is called Archimedean when $a$ is incomparably smaller than $b$ implies $a = 0$.

**Definition 1.11.8** [13, p. 366] - A Banach lattice $L$ is a real Banach space $(L, || ||)$ which is also a lattice under $\leq$, satisfying the following conditions.

$$|x| \leq |y| \Rightarrow ||x|| \leq ||y||, \ x, y \in L \text{ where } |x| = xV(-x).$$

1.12 Topological Vector Spaces (TVS)

**Definition 1.12.1** [129, p. 7] Suppose $\tau$ is a topology on a vector space $X$ such that

(a) Every point of $X$ is a closed set, and

(b) The vector space operations are continuous with respect to $\tau$.

Then $\tau$ is called a vector topology (or Hausdorff topology) on $X$ and $(X, \tau)$ is a Topological vector space (TVS).

**Definition 1.12.2** [129, p. 8] A local base of a TVS $X$ is a collection $B$ of neighbourhoods of null element such that every neighbourhood of null element contains a member of $B$. 
X is **locally convex** if there exists a local base \( B \) whose members are convex. X is **metrizable** if its topology \( \tau \) is induced by a complete invariant metric 'd'.

X is a **Frechet space** if X is a locally convex \( F \)-space.

For some more information on TVS, see [154 J, [78 J and [79 J.

1.13 **Cesaro Summability etc.**

**Definition 1.13.1** [100, p. 4] - A sequence \( \{x_n\} \in C \) is said to be \( (C,1) \) (or Cesaro) summable to 'l' if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k = l \quad \text{and it is called strongly Cesaro summable to } l \quad \text{if and only if}
\]

\[
\lim_{n \to \infty} \sum_{k=1}^{n} |x_k - l|^p = 0 \quad \text{where } p \text{ is index such that } 0 < p < \infty.
\]

**Remark 1.13.1** [95 J - 1 is unique whenever it exists.

If \( X, Y \) are Banach spaces, we denote by \( L(X,Y) \) the Banach-algebra of bounded linear operators on \( X \) into \( Y \) with the usual operator norm.

The continuous dual of \( Y \), i.e. the space of continuous linear functionals on \( Y \), is \( L(Y,C) \), and is written as \( Y^* \). For each \( T \in L(X, Y) \), we denote the adjoint of \( T \) by \( T^* \), where \( T^* \) is defined by \( (\emptyset, Tx) = (T^*\emptyset, x) \), for \( \emptyset \in Y^* \) and all \( x \in X \).

We shall also write

\[
S_1^* = \{ \emptyset \in Y^* : ||f|| \leq 1 \} \quad \text{and}
\]

\[
S_1 = \{ x \in X : ||x|| \leq 1 \}.
\]
We make use in our last chapter, the well known theorem of Hahn-Banach which states that for every \( y \in Y \) there exists \( \phi \in S_1^* \) such that \( \| y \| = \phi(y) \).

The following concept was introduced by Robinson [128] and was termed the group norm by Lorentz and Macphail [99].

**Definition 1.13.2** [100, p. 5] - Let \( \{T_k\} \) be a sequence in \( L(X,Y) \). Then the group norm of \( \{T_k\} \) is

\[
\| (T_k) \| = \sup \left\{ \| \sum_{k=1}^{n} T_k x_k \| : n \in \mathbb{N} \right\}
\]

where the supremum is over all \( n \in \mathbb{N} \), and all \( x_k \in S_1 \).

**Remark 1.13.2** [100] - Group norm may not be finite.

**Theorem 1.13.1(1)** [100, p. 5] - If \( \{A_k\} \subset S(C^*) \), the sequence space of \( C^* \)-valued sequences. Then \( A_k \) may be identified with the complex number \( a_k \) and

\[
\| (A_k) \| = \sum_{k=1}^{\infty} |a_k|
\]

whence the group norm is finite if and only if \( \{a_k\} \in l_1 \).

(ii) If \( \{T_k\} \) is a sequence in \( L(X,Y) \) and we write \( R_n = (T_n, T_{n+1}, T_{n+2}, \ldots, T_{n+r}, \ldots) \) then

(a) \( \| T_m \| \leq \| R_n \| \) for all \( m \geq n \).

(b) \( \| R_{n+p} \| \leq \| R_n \| \) for all \( n \in \mathbb{N} \).

(c) \( \| \sum_{k=n}^{n+p} T_k x_k \| \leq \| R_n \| \max \{ \| x_k \| : n \leq k \leq n+p \} \)

for any \( x_k \) and all \( n \in \mathbb{N} \), and all non-negative integers \( p \).
(iii) If \( \{T_k\} \subseteq L(X,Y) \) then \( \sum_{k=1}^{\infty} \|T_k\| < \infty \) implies \( \|T_k\| < \infty \). Also \( \|T_k\| < \infty \) implies \( \sup_k \|T_k\| < \infty \).

(iv) If \( Z \) denote the set of all sequence \( T = \{T_k\} \) such that each group norm \( \|T\| \) is finite then \( Z \) becomes a Banach space, with the natural operations under the norm \( \|T\| \).

1.14 Notations

In this section, we explain some of the notations used in this work.

- **\( \mathbb{N} \)** - Set of natural numbers
- **\( \mathbb{R} \)** - Set of Real numbers
- **\( \mathbb{C} \)** - Set of Complex numbers
- **\( \mathbb{R}(s) \)** - Real part of the complex number \( s = \sigma + it \)
- **\( \mathbb{R}(\sigma) \)** - Denotes the half plane \( \sigma \leq \alpha \).
- **\( \mathcal{C}_0 \)** - The space of Null sequences \( x = [x_n] \) with natural norm \( \|x\| = \sup |x_n| \)
- **\( \mathcal{C} \)** - The space of convergent sequences with sup norm.
- **\( m = l_\infty \)** - The space of bounded sequences with sup norm.
- **\( l_p \)** - The space of all sequences \( x \) such that \( \sum_{k=1}^{\infty} |x_k|^p < \infty \) and for \( p \geq 1 \),
  \( \|x\| = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \), \( 0 < p < 1 \), \( \sum_{k=1}^{\infty} |x_k|^p = \|x\| \) (\( ! ! \) paranorm)
Let $E$ be a Banach space over the real or complex field, the Banach sequence spaces $C_0(E)$, $l_p(E)$ and $m(E) \equiv l_\infty(E)$ are defined analogously. The class $H^p(E)$ is defined as the space of all $f : U \rightarrow E$, ($U$ is the unit disk of complex plane) such that

$$M_p[f,r] = \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p \right)^{1/p}$$

is bounded with

$$\|f\|_p = \sup_r M_p[f,r].$$