CHAPTER IX

A THEOREM FOR PARTICULAR TYPE OF RING

9.1 Introduction:

The usual relation in Boolean ring is extended to reduced rings A (has no nilpotent elements) when it is expressed as \( a \preceq b \) if and only if \( ab = a^2 \). Abian \([1],[2]\) has shown that the relation \( \preceq \) defined by \( x \preceq y \) iff \( xy = x^2 \), \( x^2y = xy^2 = x^3 \) is partially ordered on commutative semisimple rings. Myung and Jimenez \([33]\) extended the above results to any alternative rings. Gonzalez and Matinez \([25]\) have shown that the relation \( \preceq \) defined by \( x \preceq y \) iff \( xy = x^2 \), \( x^2y = xy^2 = x^3 \) is an order relation for a class of Jordan rings.

The aim of this chapter is to define the relation \( x \prec y \) iff \( xy = x^k \), \( x^ky = xy^k = x^{k+1} \), for \( k \) a positive integer and prove that it is reflexive and transitive for a class of rings.

9.1.1 Preliminaries: In this section we have given elementary properties and definitions related to rings which will be needed in this chapter. Let \( R \) be a commutative non-associate ring satisfying \( (xy)x^k = x(yx^k) \) and \( x, y \in R \). In the terms of associatore \( (x,y,z) = (xy)z - x(yz) \) that is to say \( (x,y,x^k) = 0 \). If \( R \) is a associative ring then it is well known that the elements of \( R \) form a ring under the same operation of addition and under the multiplication defined as \( ab = \frac{1}{2} (ab + ba) \), where \( ab \) denote the associative product of \( a \) and \( b \) in \( R \). This ring will be denoted by \( R^+ \).
The identities most used are as follow:

1. \( x^ny = 2x^{n-1}(xy) - 2x^{n-k}(xyz) + x^{n-k}(xy) \) and \( x,y \in R \).

\( x^n(yx^m) = (x^ny)x^m \) and \( x,y \in R, n,m \geq 0 \).

2. \((xy)z)u + ((xu)z)y + (yu)z)x = (xy)(zu) + (xz)(yu) + (yz)uxz\).

That is to say using the associators,

\( (xy,z,u) + (xu,z,y) + (yu,z,x) = 0 \)

and

\( \forall x,y,z,u \in R \)

\( (z,xy,u) = (z,x,u)y + (z,y,u)x \)

3. By the commutativity of \( R \) we have

\( (x,y,x) = 0 \) or equivalently \( (x,y,z) = -(z,y,x) \)

\( (x,y,z) + (y,z,x) + (z,x,y) = 0 \)

\( \forall x,y,z \in R \).

9.2 Order Relation:

Let \( R \) be a ring in which \( 2x = 0 \Rightarrow x = 0 \) \( \forall x \) in \( R \).

We define the relations

\( x \leq y \iff xy = x^k, x^ky = xy^k = x^{k+1} \) and if \( x \leq y \) then

\( (x,x,y) = 0 = (x,y,y) \).

Thus, using the identities

1. \( x^iy = x^{i+1} \)
2. \( x^iy^j = x^{i+j} \) \( \forall i,j \geq 1 \)

and the algebra generated by \( x \) and \( y \) is an associative algebra.

9.2.1 Lemma: If \( x \leq y, \forall x, y \in R \). Then \( R \) has no nilpotent elements \((\neq 0)\).

Proof: Assume \( x^k = 0 \) in \( R \) implies \( x \leq x^k \) and \( x^k = x \).
Then $x \leq 0$. However, 0 is the least element under $\leq$. Hence $x = x^k = 0$. Therefore $x = 0$.

From here it follows that $R$ always stands for a ring with no nilpotent elements.

9.2.2 Lemma: The relation $\leq$ is reflexive.

Proof: We have define the relations $x \leq y$ iff $xy = x^k$. Since $xx = x^k$ if follows from the relation that $x \leq x$. Then $\leq$ is reflexive.

Now let $R$ be a ring without non-zero elements satisfying the property (P) given by $(x,x,y) = 0$ implies $(xy,x,y) = 0$.

9.3 Theorem:

Let $R$ be a ring without non-zero nilpotent elements satisfying (P). Then the relation $\leq$ is reflexive and transitive.

Proof: Let $x \leq y$ and $y \leq z$. Then $xy = x^k$, $x^ky = xy^k = x^{k+1}$ and $yz = y^k$, $y^kz = yz^k = y^{k+1}$. Then

$$(x^k, x, z) + (2xz, x, x) = 0$$

This implies $3xz^k = xz^k + 2x(xz)$.

$$(x^k, z, y) + 2(xy, z, x) = 0$$

Since $(xy, z, x) = 0 = (x^k, z, x)$.

We have

$$(x^k, z, y) = (x^k z) y - x^k(zy) = 0$$

$$(xy, z, y) + (y^k, z, x) + (xy, z, y) = 0$$
Using 9.3(2), we get
\[(y^k, z, x) = 0 \text{ i.e., } y^k(zx) = (y^kz)x = y^kx^k = x^{k+k} \quad (3)\]
\[(y^k, x, z) + 2(yz, x, y) = 0 = (y^k, x, z) + 2(y^k, x, y)\]
Then \[(y^k, x, z) = 0 \text{ i.e., }\]
\[(y^kz)x = y^k(zx) = y^kx^k = x^{k+k} \quad (4)\]
\[x^{k+1}z = y^k(xz) = y^kx^k = x^{k+k} = (x^kz)y \quad (5)\]
by 9.3(3) and 9.3(4)
\[(xy, y, z) + (xz, y, y) + (yz, y, x) = 0\]
So, \[(x^k, y, z) + (xz, y, y) + (y^k, y, x) = 0. \text{ Since } (y^k, y, x) = 0 \]
we have \[(x^k)z - x^k(yz) + ((xz)y) - (xz)(y^k) = 0 \text{ by } 9.3(5)\]
\[(x^k)z = x^{k+1}z = x^k(yz) = x^ky^k \quad (6)\]
Thus
\[((xz)y) = (xz)y^k = x^{k+k}\]
By using 9.3(5) and 9.3(6), we get
\[(x^k)z = x^{k+1}z = y^{k+1}z = y^k(xz) = (x^kz)y = ((xz)y)y \quad (7)\]
\[(x^k, y, z) + 2(xz, y, x) = 0\]
Since
\[(x^k, y, z) = (x^k)z - x^k(yz) = x^{k+1}z - x^ky^k = x^{k+1}z - x^{k+k} = 0\]
We get \[(xz, y, x) = 0 \text{ i.e., }\]
\[((xz)y) = (xz)(yx) = (xz)y^k \quad (8)\]
As
\[x^{k+1}z = (x^k)z = x^{k+k} \quad (9)\]
we have
\[(x^{k+1}, z, y) = 0 = (x^ky, z, y).\]
But $2(x^k y, z, y) + (y^k, z, y) = 0$

Implies with the above that

$\langle y^k, z, x^k \rangle = 0$ and so

$(y^k z) x^k = y^k (zx^k) = y^{k+1} x^k = x^{2k+1}$

On the other hand

$(x^{k+1}, y, z) + (y, z, x^{k+1}) + (z, x^{k+1}, y) = 0$

and $(x^{k+1}, z, y) = -(y, z, x^{k+1}) = 0$

imply that $(x^{k+1}, y, z) = -(z, x^{k+1}, y) = -(z, x^k, y)$

$= -(z, x, y^k) - (z, y^k, y) x$

$= - y^k (zx y - x^k y^k)$

But $y^k (zx y) = (y^k (zx)) y = x^{k+1} y = (x)^{2k+1}$ by 9.3(7) hence

$(x^{k+1}, y, z) = 0$ implies

$(x^k y)(yz) = x^{k+1} (yz) = x^{k+1} y = x^{2k+1}$

Using the identities,

$x^k z = x^{k+1}$

$(x^k, z, x)^k = (x^k z) x^{k+1} + (x^k z, x, x^k)$

$= x^k (zx^{k+1}) = (x)^{2k+2}$

Since $2(x^k z, x, x^k) + (x^{k+k}, x, z) = 0$ and by 9.2.1(9)

$(x^{k+k}, x, z) = 0$ (10)

$(x^k, z, x)^2 = (x^{k+k})^2 = x^{4k}$

$(x^k, z, x)^2 = (x^k z, x)(x^k, z, x)$

$= (x^k z x) x^k (z x) - (x^k z x, x^k, z x)$

But $x^k (x^k, z, x) = (x^k x^k) z x - x^k (x^k z, x) = 0$
by 9.3(9). Then property (P) implies \((x^k z x, x^k, zx) = 0\).

Hence, we get

\[
(x^k z x)^2 = (x^k z x) x^k (z x) = x^{3k} z x = x^{4k}
\]

\[
(x^{k+k}) = x^k z x.
\]  

In fact,

\[
(x^k z x - x^{k+k})^2 = (x^k z x)^2 - 2(x^k z x)(x^{k+k})
\]

\[
= (x^k)(x^k)(x^k)(x^k) - 2(x^k)(x^k)(x^k)(x^k)
\]

\[
+ (x^k)(x^k)(x^k)(x^k)
\]

\[
= 0
\]

hence \(x^k z x = x^{k+k}\) that is to say \((x^k, x, z) = 0\)

\[
(x^k, x, z) + 2(xz, x, x) = 0
\]  

9.3(13) \Rightarrow \((xz, x, x) = 0\) and so \((x z x)x = x z x^k =

\((x^k)(y^k) = (x^k z)_y\) by 9.3(2)

\[
x^k z x = x^{2k+1}
\]  

\[
x^{2k+1} = (x^{k+k}) z = (x^k)(x^k) z =
\]

\[
= 2x^{k+1}(xz) - 2x^k(xz) + x^k(zx^k)
\]

\[
= 2x^{k+1} x^k - 2x^k x^{k+1} + x^k z x^k
\]

\[
= x^k z x^k
\]

\[
x^k z = x^{k+1}.
\]  

In fact

\[
(x^k z - x^{k+1}) = (x^k z)(x^k z) - 2(x^k z)(x^{k+1}) + (x^{k+1})(x^{k+1})
\]

\[
= (x^k z)(x^k z) - 2(x^k z)(x^k z) + (x^k z)(x^k z)
\]
\[ (x^k z)(x^k z) = 2(x^{k+1}z)x^k + (x^{k+1})(x^k) \]
\[ = (x^k z x^k)z - (x^k z, x^k, z) - (x^{k+1})(x^{k+1}) \]
\[ = (x^{2k+1})z - (x^{2k+1})z - (x^k z, x^k z) \]
\[ = 0 \text{ by 9.3(15)} \]
and 
\[ (x^k, x^k, z) = (x^k x^k)z - x^k(x^k z) \]
\[ = (x^{2k+1} - x^{2k+1}) = 0 \]
implies 
\[ (x^k z, x^k z) = 0 \text{ by (P)}. \]

Hence, we obtain 
\[ x^k z = x^{k+1} \]
\[ x^k z x = x^{k+1}. \]  
(17)

In fact
\[ (x z x - x^{k+1})^2 = (x z x)^2 - 2x^{k+1}(xzx) + (x^{k+1})(x^{k+1}) \]
\[ = xzx^k z + 2x^k zx z + x^k z x z = 0 \]
Hence, 
\[ xzx = x^{k+1} \]
\[ x^k z = x^k. \]  
(18)

In fact
\[ (xz - x^k)^2 = (xz)(xz) - 2(xz)x^k + (x^k)(x^k) \]
\[ = (x^k x^k - 2x^k x^k + x^k x^k) \]
\[ = 0 \]
because 
\[ (xz)^2 = (xz)(zx) = (xzx)z - (xz, x, z) \]
\[ = (x^{k+1}z = x^k zx = x^k x^k \]
Because 
\[ (x, x, z) = x^k z - xzx = x^{k+1} - x^{k+1} = 0 \]
by 9.3(16) and 9.3(17) and property (P) implies
\[ (xz, x, z) = 0 \]
Finally

$$x^k = x^{k+1}$$

In fact $x \leq y$ and $y \leq z$ implies $x^k \leq y^k$ and $y^k \leq z^k$.

By the above results for $x^k$, $y^k$, $z^k$, we have

$$x^k z^k = x^{k+k} \quad \text{and} \quad x^{2k} z^k = x^{3k}.$$  

Also $(x, x, z^k) + 2(z, x, xz) = 0$ and $(xz, x, z) = -(z, x, zx) = 0$ then $(x, x, z^k) = 0$. Hence, $xz^k x = x(xz^k) = x z^k - (x, x, z^k)$

$$= x z^k = x^{k+k},$$

and so $(xz^k)^2 = (xz^k)(xz^k) = (xz^k)xz^k = (xz^k, x, z^k) = (x^{k+k} z^k = x^{3k}$ because $(x, x, z^k) = 0$ implies $2x(x, x, z^k) = (x, x^k, z^k) = 0$ and so $x(x^k z^k) = x^{k+1} z^k$, i.e.

$$x^{k+1} (z^k x) = (x^{k+1} z^k) x = (x^k z^k) x = x^{2k}$$

Hence, $xz^k = x^{k+1}$. Therefore, the relation $\leq$ is reflexive and transitive.

**Note:** If $R$ is a ring without zero divisors (i.e. an integral domain) then relation $\leq$ will be symmetric.

**Proof:** $x \leq y$ iff $xy = x^k$ implies $yx = x^k$

$$x^k y = xy^k = x^{k+1}$$

and

$$yx^k = y^k x = x^{k+1}$$

Since,

$$xx = x^k$$

then

$$xx = xy \implies x = y$$

this implies

$$yx = y^k \implies y = x$$

as desired.
Special Case: If $k = 2$ then the relation defined as $x \leq y$ if and only if $xy = x^k$,

$$x^k y = xy^k = x^{k+1} \quad / \quad x, y \in R$$

will be reflexive anti-symmetric and transitive i.e. partial order for Jordan rings. Here, ring $R$ is taken to be Jordan ring. If we take alternative ring then the relation defined as above will be partial order for a class of alternative rings. If $R$ is commutative semi-simple ring then the same result holds for a class of commutative semi-simple rings.