STRESS INTENSITY FACTORS FOR TWO GRIFFITH CRACKS OPENED BY A SYMMETRICAL SYSTEM OF BODY FORCES IN A STRESS-FREE STRIP

The problem of determining the stress and the displacement fields in the neighbourhood of two Griffith cracks is reduced to the second kind of Fredholm integral equation by using Fourier transforms. The solution of this integral equation is obtained by expanding the unknown function in terms of \((a^{-1})\) — a being half of the width of the strip. The partial closure of the crack is also considered. The numerical results for special point body force are shown graphically.

8.1. INTRODUCTION

So many problems of crack opening due to forces applied at crack faces are solved in the literature \([1]\). The crack opening due to general system of body forces in infinite isotropic and homogeneous solids has been solved with Fourier transforms by Sneddon and Tweed \([2,3]\). The similar problems were solved for finite boundaries (rigidly lubricated) by Parihar and Kushwaha \([4]\) and Kushwaha \([5]\). In the present chapter we are extending the analysis of \([8]\) to the title problem.

Thus we are solving the problem of cracks \(y = c\), \(b < x < c\), \(-c < x < -b\) opened by symmetrical system of body forces \([X,Y]\) in homogeneous, isotropic stress-free strip
Fig 8.1 Geometry of the problem
having symmetrically placed cracks normal to edges (see figure 8.1). Splitting into two displacement boundary value problems, namely, Problem I and Problem II, we solve these separately. Therefore, of concern is the problem,

\[
\begin{align*}
\sigma_{xx}(\pm a, y) &= \sigma_{xy}(\pm a, y) = 0, \quad 0 < |y| < \infty, \\
\sigma_{xy}(x, 0) &= \theta, \quad 0 \leq |x| \leq a, \\
\sigma_{yy}(x, 0) &= 0, \quad b < |x| < c, \\
u_y(x, 0) &= 0, \quad c < |x| \leq a, \\
\end{align*}
\]

where \((\sigma_{xx}, \sigma_{xy}, \sigma_{yy})\) and \((u_x, u_y)\) are the components of stress-tensor and of displacement vector, respectively. Following the analysis of [4.1], we get.

**Problem I**: is the solution of equations of equilibrium in the presence of body forces \([X,Y]\) and the following boundary conditions

\[
\begin{align*}
\sigma_{xy}^{(1)}(x, 0) &= 0, \quad 0 \leq |x| \leq a, \\
\sigma_{xy}^{(1)}(\pm a, y) &= 0, \quad 0 \leq |y| < \infty, \\
u_y^{(1)}(x, 0) &= 0, \quad 0 \leq |x| \leq a,
\end{align*}
\]

and the problem two as

**Problem II**: is the solution of equations of equilibrium in the absence of \([X,Y]\) and the following boundary conditions:
\[ \sigma^{(1)}_{xx}(\pm a,y) + \sigma^{(2)}_{xx}(\pm a,y) = 0 \quad 0 \leq |y| < \infty, \quad (8.1.8) \]
\[ \sigma^{(2)}_{xy}(x,0) = 0, \quad 0 \leq |x| \leq a, \]
\[ \sigma^{(2)}_{xy}(\pm a,y) = 0, \quad 0 \leq |y| \leq \infty \]
\[ \sigma^{(1)}_{yy}(x,0) + \sigma^{(3)}_{yy}(x,0) = 0, \quad b < |x| < c, \quad (8.1.10) \]
\[ u^{(2)}_{y}(x,0) = 0, \quad c < |x| \leq a, \quad (8.1.11) \]

where the subscript (1) and superscript (2) over functions represent the functions obtained for problem I and problem II respectively. Upto section 4 of the chapter, the condition

\[ u^{(2)}_{y}(x,0) > 0, \quad b < |x| < c, \quad (8.1.12) \]

(see Burniston [6]) is being observed. The following notations for transform are being used

\[ f_{sc}(\alpha_n, \gamma) = \int_{0}^{a} \left( \cos \frac{\alpha_n x}{\sin \alpha_n} \right) dx \int_{0}^{\infty} f(x,y) \left( \cos \frac{\gamma y}{\sin \gamma} \right) dy, \]

with the usual inversion formulae and \( \alpha_n = n\pi/a = nq. \)

The outlay of the paper is as follows: section 8.2 deals with the formulation of the problem. Section 8.3 deals with the solution of Fredholm integral equation along with the general expressions for physical quantities like, crack shape, normal component of stress at \( y = 0 \) and then the stress-intensity factors. Section 8.4 gives one example. Section 8.5 deals with the condition of partial closing of the crack at the centre.
8.2. FORMULATION

**Problem I**: To solve this problem with the boundary conditions (2.1.5)-(2.1.7) we take appropriate Fourier transform of equations of equilibrium and of stress-strain relations, and substitute for transformed stress components in the transformed equations of equilibrium and invert we get as

\[
\begin{align*}
    u_x^{(1)}(x,y) &= \nu \sum_{n=1}^{\infty} \int_0^{\infty} \sin(\alpha_n x) \left[w_1 X_{sc} - w_2 Y_{cs}\right] \cos \gamma ds, \\
    u_y^{(1)}(x,y) &= \frac{1}{2} u_{yc}^{(1)}(\alpha_n, y) + \sum_{n=1}^{\infty} u_{yc}^{(1)}(\alpha_n, y) \cos \alpha_n x, \\
    u_{yc}^{(1)}(\alpha_n, y) &= -\nu \int_0^{\infty} \left[w_2 X_{sc} - w_3 Y_{cs}\right] ds \sin \gamma,
\end{align*}
\]

and

\[
\nu = \Phi(1+\eta) \beta^{-1} (\pi \alpha^2), \quad \beta^2 = 2(1-\eta)/(1-2\eta),
\]

\[
\begin{align*}
    w_1 &= \frac{\alpha_n^2 + \beta \gamma^2}{(\alpha_n^2 + \gamma^2)^2}, \quad w_2 = \frac{\beta(\beta^2-1) \alpha_n \gamma}{(\alpha_n^2 + \gamma^2)^2}, \quad w_3 = \frac{\beta^2 \alpha_n^2 + \gamma^2}{(\alpha_n^2 + \gamma^2)^2},
\end{align*}
\]

where \( \rho \) and \( \eta \) are mass density and Poisson ratio of the medium, respectively. These expressions are same as given in Chapter VII, but for self containment of this Chapter these are given here.

**Problem II**: The solution of problem II is obtained through the similar method of Sneddon and Srivastav [7] and written as
$$u^{(2)}_x(x,y) = \frac{2(1+\eta)}{aE} \left[ \sum_{n=1}^{\infty} \frac{\sin \alpha_n x}{\alpha_n} (1-\eta) \phi_{2,yy} + \eta a^2 \phi_1 + \int_0^\infty \left\{ (1-\eta) \phi_{2,xx} + (\eta-2) \frac{2}{S} \phi_{2,x} \right\} \frac{\cos \frac{\pi y}{b}}{S^2} \, ds \right],$$

$$(8.2.6)$$

$$u^{(2)}_y(x,y) = \frac{1}{2} u^{(2)}_{yc}(0,y) + \sum_{n=1}^{\infty} u^{(2)}(\alpha_n y) \cos \alpha_n x + \frac{2(1+\eta)}{\pi E}$$

$$\int_0^\infty \left\{ (1-\eta) \phi_{2,xx} + \eta \frac{2}{S} \phi_{2,x} \right\} \frac{\sin \frac{\pi y}{b}}{S} \, ds, \quad (8.2.7)$$

with

$$u^{(2)}(\alpha_n y) = \frac{2(1+\eta)}{aE \alpha_n^2} \left[ (1-\eta) \phi_{1,y} + (\eta+2) \frac{\alpha_n^2}{\alpha_n^2} \phi_{2,y} \right], \quad (8.2.8)$$

and

$$\phi_1 = A_n (1+\alpha_n y) e^{-\alpha_n y},$$

$$\phi_2 = A(s) \left[ \cosh(sx) - \tanh(sa) \sinh(sx) \right], \quad (8.2.9)$$

where $[A_n]$ and $A(s)$ are arbitrary constants to be determined, and we calculate stress components due to elasticity problem

$$\sigma^{(2)}_{yy}(x,y) = \frac{1}{2} \sigma^{(2)}_{yy}(0,y) + \sum_{n=1}^{\infty} \sigma^{(2)}_{yy}(\alpha_n y) \cos \alpha_n x$$

$$+ \int_0^\infty \sigma_{yye}(x,s) \cos(sy) \, ds, \quad (8.2.10)$$

$$\sigma^{(2)}_{yye}(x,s) = s^2 A(s) \left[ \cosh(sx) - \tanh(sa) \sinh(sx) \right] \cos(sy) \, ds, \quad (8.2.11)$$

$$\sigma^{(2)}_{yye}(\alpha_n y) = A_n \left[ 1 + \alpha_n y \right] e^{-\alpha_n y}, \quad (8.2.12)$$
From the expressions (9.2.6) - (9.2.9) we see that the boundary conditions (9.1.9) are satisfied identically. The boundary condition (9.1.8) and equations (9.2.10) - (9.2.12) give

\[ A(s) = \left[ - \frac{4}{a} \sum_{n=1}^{\infty} A_n \frac{\beta}{n^2} \left( \frac{\beta}{n^2} \right)^2 \right. \left. + \frac{2}{s} \sum_{n=1}^{\infty} P_n(s) \cosh \left( \frac{s}{2} \right) \right], \quad (9.2.13) \]

where

\[ P_n(s) = \frac{1}{2} \sigma^{(1)}_{xxc}(a, s) + \sum_{n=1}^{\infty} (-1)^n \sigma^{(1)}_{xxc}(n, s), \quad (9.2.14) \]

where \( \sigma^{(1)}_{xxc}(x, y) \) is defined by (8.2.7). Thus we are left with one arbitrary constant and the mixed boundary conditions (9.1.10) - (9.1.11). These boundary conditions yield the following system of triple series relations:

\[ \frac{A}{2} + \sum_{n=1}^{\infty} a_n A_n \cos \alpha_n x = 0, \quad 0 \leq x \leq b, \quad c \leq x \leq a, \quad (9.2.15) \]

\[ \sum_{n=1}^{\infty} \alpha_n^2 A_n \cos \alpha_n x = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^n \alpha_n^3 A_n \int_{0}^{\infty} \left[ \cosh s x - \tanh s a \sinh s x \right] \left( \alpha_n^2 + s^2 \right)^{-2} ds + \int_{0}^{\infty} P_n(s) \cosh s(a-x) ds \]

\[ - \frac{3}{2} \sigma_{yy}^{(1)}(x, 0), \quad b < x < c, \quad (9.2.16) \]

**Reduction to Fredholm Integral Equation**

The solution of triple series relations (9.2.15) - (9.2.16) is obtained by the method of Parihar [8] and given as

\[ g(t) + \frac{4}{a^2} \int_{b}^{c} g(y) K(t, y) dy = \frac{2g}{Q(t)} \left[ \int_{b}^{c} \frac{\sin(qx)}{G(x, t)} Q(x) F(x) dx \right] + D, \quad (9.2.17) \]
where

\[ F(x) = - \frac{a}{2} \sigma_y^{(1)}(x,0) + \int_0^\infty P_1(\xi) \cosh \xi(a-x) \, dy, \quad (9.2.18) \]

\[ K(y,t) = [\Theta(t)]^{-1} \int_b^c \frac{\Theta(x) \sin(qx)}{G(x,t)} \sum_{n=1}^\infty (-1)^n a_n \sin \alpha_n x \]

\[ \int_0^\infty \frac{2}{a} \cosh \xi(a-x) \left[ \frac{2}{a} + \alpha_n^2 \right]^{\frac{1}{2}} d\xi, \quad (9.2.19) \]

\[ \Theta(x) = \left[ |G(b,x)G(x,c)| \right]^{1/2}, G(x,y) = \cos(qx) - \cos(qy) \quad (9.2.20) \]

and

\[ a_n^2 A_n = \int_b^c g(t) \sin \alpha_n t \, dt, \quad A_0 = \int_b^c g(t) \, dt, \quad (9.2.21) \]

\[ \int_b^c g(t) \, dt = 0. \quad (9.2.22) \]

\( D \) is an arbitrary constant which will be determined through (9.2.22).

3. SOLUTION OF FREDHOLM INTEGRAL EQUATION

We first approximate the singular kernel \( K(y,t) \). Having expanded the hyperbolic functions in terms of exponentials and then using the relation (see [3]) (3.4.1), we get \( F(x,y) \) as given by equation (3.4.3).

Substituting the value of \( F(x,y) \) into Kernel \( K(y,t) \) and evaluating some integrals, we get

\[ K(y,t) = \frac{\pi}{2a} \sum_{\ell=0}^m \sum_{m=0}^\infty \frac{(m+1)}{(2\ell+1)^{m+2}} \frac{1}{\Theta(t)} \int_b^c \frac{\sin(qx)}{G(x,t)} \]

\[ \left( \frac{x+y}{a} \right)^m \, dx \right] dx + \frac{\pi}{a} \sum_{m=0}^\infty \frac{1}{\Theta(t)} \int_b^c \frac{\sin(qx)}{G(x,t)} \]
\[
\begin{align*}
\int_{b}^{c} \sin(qx) \Theta(x) F(x) dx & \int_{b}^{c} \frac{dt}{\Theta(t) G(x,t)} \\
+ \frac{4D_{t}}{n} G(0,c) F(\frac{\lambda}{2}, \lambda) & T_{1} \\
+ \frac{n^{2}}{3q} [G(b,c) G(0,c) F(\frac{\lambda}{2}, \lambda) + G(0,b) E(\frac{\lambda}{2}, \lambda)] T_{1} \\
+ \frac{4}{n^{2}} \int_{b}^{c} \frac{dt}{\Theta(t)} \int_{b}^{c} \frac{\sin(qx) \Theta(x)}{G(x,t)} [- \frac{n^{2}}{12} (x+y) + \\
+(x^{2}+y^{2}) [D_{6}(1) + D_{7}(1)] - 2xy [D_{6}(1) - D_{7}(1)]] dx dy \\
+ \frac{D_{t}}{nq} \int_{b}^{c} \frac{dt}{\Theta(t)} \int_{b}^{c} g_{1}^{(y)} dy + \frac{12}{n} \int_{b}^{c} \frac{dt}{\Theta(t)} \int_{b}^{c} g_{1}^{(y)} dy \\
\int_{b}^{c} \frac{\sin(qx)}{G(x,t)} \left[ \sum_{\lambda = 0}^{\infty} \frac{(x+y)^{2}}{(2\lambda+1)} \right] dx dy + \frac{2D_{t}}{q} F(\frac{\lambda}{2}, \lambda) G(0,c) \\
\int_{b}^{c} g_{1}^{(y)} dy + \frac{4}{n^{2}} \int_{b}^{c} \frac{dt}{\Theta(t)} \int_{b}^{c} g_{1}^{(y)} dy \left[ \int_{b}^{c} \frac{\sin(qx) \Theta(x)}{G(x,t)} \\
[- \frac{n^{2}}{12} (x+y) + (x^{2}+y^{2}) [D_{6}(1) + D_{7}(1)] - 2xy [D_{6}(1) - D_{7}(1)] dx] \\
\int_{b}^{c} \frac{dt}{\Theta(t)} \int_{b}^{c} g_{1}^{(y)} dy \int_{b}^{c} \sin(qx) \Theta(x) \\
\left[ -2 \sum_{\lambda = 0}^{\infty} \frac{(x+y)^{3}}{(2\lambda+1)} + (x-y)^{4} D_{6}(2) + (x+y)^{4} D_{7}(2) \right] dx dy \\
+ \frac{D_{t}}{q} G(0,c) F(\frac{\lambda}{2}, \lambda) \int_{b}^{c} g_{1}^{(y)} dy + \frac{n}{12q} F(\frac{\lambda}{2}, \lambda) \\
\end{align*}
\]
\[ G_0, c) \int_b^c G_4'(y) \, dy - \frac{20}{\pi^2} \int_b^c \frac{dt}{Q(t)} \int_b^c g_0'(y) \, dy \]

\[ \int_b^c \frac{\sin(gx) \Theta(x)}{G(x,t)} \left[ \sum_{l=0}^{\infty} \frac{(x+y)^4}{(2l+1)^5} \right] \, dx \right] \, dy + \]

\[ + \frac{8}{\pi^2} \int_b^c \frac{dt}{Q(t)} \int_b^c g_3'(y) \, dy \int_b^c \frac{\sin(gx) \Theta(x)}{G(x,t)} \]

\[ \left[ - \frac{x^2}{12} (x+y) + (x-y)^2 D_6(1) + (x+y)^2 D_7(1) \right] \, dx \, dy \]

and

\[ T_1 = \int_b^c \frac{dt}{Q(t)} \int_b^c \frac{\sin(gx) \Theta(x) F(x)}{G(x,t)} \, dx \]

\[ = \int_b^c \frac{g_0'(t)}{Q(t)} \, dt \]  

\[ g_3'(y) = \frac{1}{\Theta(y)} \int_b^c g_3'(y) \, dy, \]  

\[ g_4'(y) = \frac{G(b,c) + G(y,c)}{\Theta(y)} \frac{\pi^2}{12} \int_b^c g_0'(y) \, dy, \]

\[ g_5'(y) = \frac{1}{\Theta(y)} \int_b^c g_5'(t) \, dt + \int_b^c \frac{\sin(gx) \Theta(x)}{G(x,t)} \]

\[ \left[ - \frac{x^2}{12}(x,t) + (x+t)^2 D_7(1) + (x-t)^2 D_6(1) \right] \, dx, \]

\[ D_4 = G(o,c) F(\pi/2, \lambda_1) + \frac{4D_4}{\pi} G^2(o,c) F^2(\pi/2, \lambda_1) + \frac{2}{3q} [G(b,c) G(o,c) F(\pi/2, \lambda_1) + G(o,b) E(\pi/2, \lambda_1)] \]
\[
G(o,c)F(\pi/2, \lambda_1) - \frac{4}{\pi^2} \int_b^c \frac{dt}{\Theta(t)} \int_b^c \frac{\sin(ax)\Theta(x)}{G(x,t)} G(o,c) \\
F(\lambda/2, \lambda_1) \left[-(x+t)^2 + (x-t)^2 D_6(1) + (x+t)^2 D_7(1) \right] dx \\
+ \frac{4D_1^2}{\pi q} G(oc) F^3(\lambda/2, \lambda_1) - \frac{12}{\pi} \int_b^c \frac{dt}{\Theta(t)} \int_b^c \frac{dy}{\Theta(y)} \\
\int_b^c \frac{\sin(ax)\Theta(x)}{G(x,t)} \left[ \sum_{l=0}^{\infty} \frac{(x+y)^2}{(2l+1)} dx dy \right] + \frac{4}{q^2} D_1 \frac{\pi^2}{12} I_1 \\
[G(b,c) G(o,c) F(\lambda/2, \lambda_1) + G(o,b) E(\lambda/2, \lambda_1)] G(o,c) \\
F(\lambda/2, \lambda_1) - \frac{16}{\pi^2} D_1 I_1 \int_b^c \frac{dt}{\Theta(t)} \int_b^c \frac{dt}{\Theta(y)} \int_b^c \frac{\sin(ax)\Theta(x)}{G(x,t)} \\
\left[- \frac{(x+y)^2}{12} + (x-y)^2 D_6(1) + (x+y)^2 D_7(1) \right] dx dy \\
+ \frac{D_1}{3q^2} \left[G(b,c) G(o,c) F(\lambda/2, \lambda_1) + G(o,b) E(\lambda/2, \lambda_1) \right] \\
I_1 G(o,c) F(\pi/2, \lambda_1) - \frac{16D_1}{\pi^3} I_1 G(o,c) F(\pi/2, \lambda_1) \int_b^c \frac{dt}{\Theta(t)} \\
\int_b^c \frac{dy}{\Theta(y)} \int_b^c \frac{\sin(ax)\Theta(x)}{G(x,t)} \left[- \frac{\pi^2}{12}(x+y) + (x-y)^2 D_6(1) + (x+y)^2 D_7(1) \right] dx dy - \frac{8}{\pi^2} I_1 \int_b^c \frac{dt}{\Theta(t)} \int_b^c \frac{dy}{\Theta(y)} \\
\int_b^c \frac{\sin(ax)\Theta(x)}{G(x,t)} \left[-2 \sum_{l=0}^{\infty} \frac{(x+y)^3}{(2l+1)^4} + (x-y)^4 D_6(2) \right] dx dy
\]
+ \int_{-\infty}^{\infty} \sin(gx) \phi(x) \, dx \frac{\pi}{g(x,t)} \left[ \sum_{\lambda=0}^{\infty} \frac{(x+y)^{4\lambda}}{(2\lambda+1)^{5}} \right] \\
- \frac{8}{3} \int_{-\infty}^{\infty} \frac{d^t}{e(t)} \int_{-\infty}^{\infty} g_4(y) \, dy \int_{-\infty}^{\infty} \frac{\sin(gx) \phi(x) \, dx}{g(x,t)} \\
\left[-\frac{8}{3} (x+y) + (x-y)^2 D_1(1) + (x+y)^2 D_1(1)\right], (8.3.12)

where \( g_0(y), g_3(y), g_4(y), g_5(y) \) are given by (8.3.8) - (8.3.11) and \( \lambda_1 \) is given by equation (4.3.15).

**Approximation II**

[A] In this approximation we assume the constant of integration \( D_1 \) as given by equation (4.3.16) and the condition (8.2.20) as

\[ \int_{-\infty}^{\infty} g_m(t) \, dt = 0, \quad m = 0,1,2, \ldots \ldots \] (8.3.13)

where \([g_m]')s are given by (8.3.2) - (8.3.4) etc. Thus put the value of \( D, g(t) \) and of \( K(y,t) \) in equation (8.2.17) and compare the coefficients of \( a^{-m} \), we get

\[ g_0(t) = \frac{P(t)}{\phi(t)}, \] (8.3.14)

\[ g_1(t) = \frac{d_1}{\phi(t)}, \] (8.3.15)
\[
g_2(t) = \frac{d_2}{\Theta(t)}, \quad (8.3.16)
\]

\[
g_3(t) = -\frac{4D_1}{n^2 \Theta(t)} \int_b^c g_0(y) dy + \frac{d_3}{\Theta(t)} + \frac{2}{q} \frac{D_6(o) + D_7(o)}{\Theta(t)}
\]

\[
[G(b,c) + G(t,c)] \int_b^c g_1(y) dy, \quad (8.3.17)
\]

\[
P(t) = \int_b^c \frac{\sin(ax) \Theta(x) F(x)}{G(x,t)} dx + d, \quad (8.3.18)
\]

and other values of \( g_m(t), m \geq 4, \) can be obtained from (4.3.22) - (4.3.24) when \( g_0, g_1, g_2, g_3 \) will be given by equations (8.3.14) - (8.3.17). Using condition (9.3.13) we get \( D_0 = D \) (given by equation (8.3.5)).

\[
d_1 = d_2 = 0 \quad \text{gives}
\]

\[
g_1(t) = g_2(t) = 0, \quad (9.3.19)
\]

\[
d_3 = \frac{4D_1}{n^2} \int_b^c \frac{g_0(y) dy}{G(\theta,c) F(\pi/2,\gamma_1^1)}, \quad (8.3.20)
\]

\[
d_4 = \frac{n^2}{3q} [G(b,c) + \frac{G(o,b)}{G(\theta,c)} \frac{F(\pi/2,\lambda_1)}{F(\pi/2,\lambda_1^1)}] \int_b^c g_0(y) dy, \quad (8.3.21)
\]

\[
d_5 = \frac{4}{n^2} \left[ \int_b^c \frac{dt}{\Theta(t)} \int_b^c g_0(y) \int_b^c \frac{\sin(ax) \Theta(x)}{G(x,t)} dx \right] - \frac{n^2}{12} (x+y) +
\]

\[
(x-y)^2 D_6(1) + (x+y)^2 D_7(1)] dy) \text{dy}/[G(\theta,c) F(\pi/2,\lambda_1^1)] \quad (8.3.22)
\]

\[
d_6 = \frac{D_4}{nq} \int_b^c g_3(y) dy /[G(\theta,c) F(\pi/2,\lambda_1^1)]
\]
In this case we consider

\[ d_0 = d_1 = d_3 = \ldots \quad d_m = \ldots, \]  

(8.3.24)

we get \([g_m(t)]\) as given by case (4.3.19) - (4.3.24) when \(g_0(t)\) will be given by equations (8.3.14) and (8.3.18).

Thus \(d_0\) will be given by equations (8.2.20) and the value of \(g(t)\) is obtained after the approximation.

\[ d_0 = N_4 / D_4, \]  

(8.3.25)

where

\[ D_4 = D_0 + \frac{2}{q} [D_6(o) + D_7(o)] [G(b,c) G(o,c) F(\pi/2, \lambda_1)] \]

\[ + G(o,b) E(\pi/2, \lambda_1) ] G(o,c) F(\pi/2, \lambda_1) \]

\[ + \frac{4}{q} G(o,c) F(\pi/2, \lambda_1) [D_6(o) + D_7(o)] \]

\[ [G(b,c) G(o,c) F(\pi/2, \lambda_1) + G(o,b) E(\pi/2, \lambda_1)] \]

\[ + \frac{4}{q} G(o,c) F(\pi/2, \lambda_1) [G(b,c) G(o,c) F(\pi/2, \lambda_1)] \]

\[ + G(o,b) E(\pi/2, \lambda_1) [D_6(o) + D_7(o)] \]
\[
- \frac{\pi}{6} \left[ \int_b^c \frac{dt}{E(t)} \int_b^c \frac{dy}{E(y)} \int_b^c \frac{\sin(ax)}{G(x,t)} \Theta(x)(x+y) \, dx \right] \\
G(o,c)F(\pi/2,\lambda_1) - \frac{4}{\pi} \int_b^c \frac{dt}{E(t)} \int_b^c \frac{dy}{E(y)} \int_b^c \frac{\sin ax \Theta(x)}{G(x,t)} \, dx \\
\left[- \frac{\pi}{12} (x+y) + D_6(1)(x-y)^2 + D_7(1)(x+y)^2 \right] + 6G(o,c) \\
F(\pi/2,\lambda_1) + \frac{\pi^2}{3q} [G(b,c)G(o,c)F(\pi/2,\lambda_1) \\
+ G(o,b)E(\pi/2,\lambda_1)]G(o,c)F(\pi/2,\lambda_1), \\
\text{(8.3.26)}
\]

where \( N_4 \) is given by the equations (8.3.7)-(8.3.8). Thus the solution of Fredholm integral equation (9.2.17) will be given by four different cases.

**PHYSICAL QUANTITIES**

**Crack shape**: Evaluating the value of the series (8.2.15) for \( b < x < c \) through equations (8.2.21) - (8.2.22) we get

\[
u_y^{(2)}(x,o) = \frac{2(1-\nu)^2}{E_b} \int_b^c g(t) \, dt, \quad b < x < c \\
\text{(8.3.27)}
\]

where \( g(t) \) will be given by different equations for different cases of approximations.

**Normal-stress component at \( y = 0 \)**: The evaluation of \( \sigma_{yy}^{(2)}(x,o) \) through the equations (8.2.6) - (8.2.14), along with stress-strain relations, we get
Fig. 8.2 Geometry of special point body forces.
\[ \sigma_{yy}^{(2)}(x,0) = \frac{2}{a} \int b_{m} G(x,y) g(t) \, dt + \frac{2}{\pi a} \int c_{y} \, dy \]

\[ \int_{0}^{\infty} \frac{\cos P(a-x)}{\sinh \beta \pi} \left[ y \cosh \frac{\beta y}{\pi} - \pi \sinh \frac{\beta y}{\pi} \cosh \frac{\beta y}{\pi} \right] \, ds \]

\[ - 2 \int_{0}^{\infty} P_{1}(\beta \cosh y) \, ds, \quad (9.3.28) \]

It is being assumed that there is no singularity in \( \sigma_{yy}^{(2)}(x,0) \) at crack tips.

**Stress-intensity factors**: We define the stress-intensity factors at the crack tips \((b,0)\) and \((c,0)\) as

\[ K_{b} = \lim_{x \to b^{-}} \sqrt{b-x} \sigma_{yy}^{(2)}(x,0), \quad K_{c} = \lim_{x \to c^{+}} \sqrt{x-c} \sigma_{yy}^{(2)}(x,0). \quad (9.3.29) \]

Substituting from (9.3.2) in (9.3.28) and evaluating the integrals and then using the definitions (9.3.29) we get

\[ K_{b} = -m(b)[P(b) + \frac{4}{\pi a} \sum_{n=1}^{\infty} a^{-n}g_{n}(b)], \quad K_{c} = m(c)[P(c) + \frac{4}{\pi a} \sum_{n=1}^{\infty} a^{-n}g_{n}(c)], \quad (9.3.30) \]

with

\[ m(y) = \left[ 2q \sin(qy) G(b,c) \right]^{-1/2}. \quad (9.3.31) \]

**8.4. AN EXAMPLE**

To make the analysis of sections 8.1 - 8.3 clear we consider one example of point body force. The loading is defined as, (see figure 8.2),

\[ ... \]
\[ x(x,y) = 0, \quad y(x,y) = \frac{Q x}{2P} [S(y-h) - S(y+h)], \quad (8.4.1) \]

where \( Q \) is the magnitude of the loading, \( P \) is the mass density of the medium. The above loading represents a force at \((0,h)\) in positive \(y\)-direction and at \((0,-h)\) in negative \(y\)-direction.

We evaluate the components of stress at \(y=0\) by using the equations \((8.2.1) - (8.2.5)\) and the stress-strain relations.

To get \( g(t) \) through equation \((8.3.2)\) we need to evaluate the integrals for \( P(t) \) which involves \( \sigma_{yy}(x,0) \) and \( \sigma_{xx}(n_0,0) \).

Thus we get

\[ P(t) = Q[a_1 h \sin(\phi h) G(t,\phi)] \left[ 1 - \frac{G(h, t)}{R(h, t)} \right] + D + P_3(t), \quad (8.4.2) \]

\[ P_3(t) = \int_0^\infty \sin(qx) S(x) P_2(x) dx / G(x, t) \quad (8.4.3) \]

\[ P_2(x) = Q \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \beta^2 \left( \sin(S_1 + S_2 - S_3 - S_4) + a(S_1^2 + S_2^2 + S_3^2 + S_4^2) \right) \]

\[ - \frac{q n \phi^2}{\gamma} \left( \frac{1}{x + 2ma} + \frac{1}{4a - x + 2ma} \right), \quad (8.4.4) \]

\[ \varepsilon_1 = x + 2a(m+1), \quad \varepsilon_2 = 4a - x + 2a1, \quad \varepsilon_3 = \varepsilon_1 + 2a, \quad (8.4.5) \]

\[ \varepsilon_4 = \varepsilon_2 + 2a, \quad \varepsilon_4 + \frac{\varepsilon_1}{\varepsilon_2} \]

\[ T_1 = 1 - (\frac{\varepsilon_2 - 1}{\varepsilon_1} + \frac{1}{\varepsilon_2}), \quad (8.4.6) \]
Fig. 8.3 Variation of stress-intensity factor against h/c.
Fig. 8.4 Crack-shape variation with x/c.

$h/c = 0.5$
$b/c = 0.2$
$c/a = 0.9$
$c/a = 0.5$
Therefore, $g(t)$ will be given by equations (8.3.2), (8.3.3) - (8.3.23) and (8.4.3) - (8.4.6) for different cases. Substituting for $P(y)$ and $g_n(y)$ in equations (8.3.30) we get the stress-intensity factors. We have plotted the case of approximation I (B) for $\left( \frac{2a}{c} / q \right) K, (K=K_b, K_c)$, against $h/c$ for different values of $c/a = 0.5$ in figures 8.3. We took $\lambda = 0.25$. We truncated the series for $\left[ g_n(t) \right]$ at $n=10$, for $\int F(x,y)$; for $l=15$, $m=25$. Accuracy is of the order of $5\%$. In figures 8.4 we plotted $\left( \frac{2a}{c} / q \right) u_i(x,y)$ against $x/c$ for $h/c = 0.5$, and $c/a = 0.9$. For $0.5$ we took $b/c = 0.2$.

**Case II:** Under this case we consider the loading (see figures 8.5 and 8.6)

$$Y(x,y) = 0, \quad X(x,y) = \frac{P}{2\pi} \delta(y) \left( x \right) \left[ \delta(x-d) - \delta(x+d) \right]$$

$$0 < d < b, \quad (8.4.7)$$

or $$Y(x,y) = 0, \quad X(x,y) = \frac{P}{2\pi} \delta(y) \left[ \delta(x+d) - \delta(x-d) \right]$$

$$0 < d < a, \quad (8.4.8)$$

Where point forces of magnitude $P$ are acting at points $(\pm d, 0)$ in positive and negative directions of $x$ for case (8.4.7) and negative and positive directions of $x$, for case (8.4.8), respectively. We calculate $\mathcal{C}(y)(x,a)$ as in Chapter VI - VII.
Fig. 8.5 Geometry of special point body forces.
Fig. 8.6 Geometry of special point body forces.
\[ \sigma_{yy}(x,0) = -\frac{P}{\alpha \beta^2} \frac{\sin(qd)}{G(x,d)}, \quad 0 < d < b, \quad (8.4.9) \]

and

\[ \sigma_{yy}^{(u)}(x,0) = \frac{P}{\alpha \beta^2} \frac{\sin(qd)}{G(x,d)}, \quad 0 < d \leq a, \quad (8.4.10) \]

Similarly we evaluate

\[ P_2(x) = \frac{4dQ}{\alpha \beta^2} \left[ \frac{1}{2(x^2-d^2)} + \frac{\beta^2-3}{2} \sum_{m=0}^{\infty} \frac{1}{(2a+2am+2al+x)^2-d^2} \right] \quad (8.4.11) \]

and

\[ P_3(t) = \int_{b}^{c} \frac{P_2(x) \sin(qx) \, dx}{G(x,t)}, \quad (8.4.12) \]

\[ P(t) = \frac{Q}{\beta^2} \frac{\sin(qd)}{G(t,d)} + D - P_3(t), \quad (8.4.13) \]

where \( D \) is constant to be determined for different cases of approximations.

**8.5. PARTIAL CLOSING**

Now we consider the problem of finding the stress field in the neighbourhood of the Griffith crack \( 0 \leq x \leq c \) \((y=0)\) in the stress free-strip \( 0 \leq |x| < a \) \((0 \leq |y| < \infty)\) which is acted upon by a uniform tension \( T \) at infinity normal to \( x \)-axis and a system of body forces such that the crack faces some where near the centre of the crack.

The corresponding problem for rigidly lubricated strip has been solved by Parihar and Kushwaha [4].
The formation of the above boundary value problem is exactly the same as in sections 8.1 - 8.2 with the change in boundary condition (8.1.10). Thus

\[ \sigma_{yy}^{(1)}(x,0) + \sigma_{yy}^{(2)}(x,0) = -T, \quad b \leq |x| < c, \quad (8.5.1) \]

where \( b \) in this case is an unknown parameter to be determined by the conditions of finiteness of stress \( \sigma_{yy}(x,0) \) at \( (b,0) \).

The solution could be obtained through the equations (8.2.17) - (8.2.22) with the change in \( F(x) \) as

\[ F(x) = \frac{\pi}{2} \left[ \sigma_{yy}^{(1)}(x,0) + T \right] + \int_{0}^{\infty} P_{1} (\xi) \cosh \xi (a-x) \, d\xi, \quad (8.5.2) \]

Thus the condition of finiteness of \( \sigma_{yy}(x,0) \) at \( (b,0) \) gives \( K_{b} \) to be zero at \( (b,0) \). Therefore we get

\[ P(b) + \frac{2}{\pi a^{2}} \sum_{n=1}^{\infty} g_{n}(b) a^{-n} = 0 \quad (8.5.3) \]

with

\[ P(b) = \int_{b}^{c} \sin (qx) \sqrt{\frac{G(x,c)}{G(b,x)}} \left[ \sigma_{yy}^{(1)}(x,0) + T - a^{-1} \int_{0}^{\infty} P_{1}(\xi) \cosh \xi (a-x) \, d\xi \right] \, dx \quad (8.5.4) \]

where \( D \) will be obtained from the condition (8.2.22). To illustrate the use of the general formula one special case is being considered.
Fig. 8.7 Geometry of special point body force.
Fig. 8.8 Variation of point body force against b/c.
We consider the case in which the crack is opened by constant uniform tension $T$ at infinity and closed partially due to the point body forces, see figure 7, specified by equation

$$Y(x, y) = -\frac{S_2 S_1(x)}{2\pi} [\delta(y-h) - \delta(y+h)], \ X(x, y) = 0. \quad (8.5.5)$$

The function $F(x)$ can easily be evaluated, as in section 8.4 with the change in $Q$ by $(-S)$. We have plotted $S/(cT)$ against $b/c$ for different values of $c/a$ and of $h/c = 0.5, 1.0$ in figure 8, from equations (8.5.3) – (8.5.4).
REFERENCES


