CHAPTER 3
3. BALANCED DOMINATION FOR NAMED AND SPECIAL GRAPHS

3.1. Named and Special graphs

1) The **Bidiakis cube** is a 3-regular graph with 12 vertices and 18 edges as shown in Figure 19.

![Figure 19: Bidiakis cube](image)

For the Bidiakis cube graph, $\gamma_{bd} = 5$

Here $f(V) = 22$ and $D = \{v_2, v_6, v_8, v_{10}, v_{11}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3 + 3 + 2 + 2 + 1 = 11$$
Therefore, $\gamma_{bd}=5$

2) The **Franklin graph** is a 3-regular graph with 12 vertices and 18 edges as shown in Figure 20.

![Franklin graph diagram](image)

**Figure 20: Franklin graph**

For the Franklin graph, $\gamma_{bd}=5$

Here $f(V) = 18$ and $D= \{v_2, v_6, v_8, v_{10}, v_{12}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 2 + 2 + 2 + 2 + 1 = 9$$

Therefore, $\gamma_{bd}=5$

3) The **Frucht graph** is a 3-regular graph with 12 vertices, 18 edges, and no nontrivial symmetries as shown in Figure 21.
For the Frucht graph, $\gamma_{bd}=4$

Here $f(V) = 24$ and $D = \{v_7, v_8, v_{11}, v_{12}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3 + 3 + 3 + 3 = 12$$

Therefore, $\gamma_{bd}=4$

4) The **Wagner graph** is a 3-regular graph with 8 vertices and 12 edges as shown in Figure 22, named after Klaus Wagner. It is the 8-vertex Mobius
ladder graph. Moebius ladder is a cubic circular graph with an even number ‘n’ vertices, formed from an n-cycle by adding edges connecting opposite pairs of vertices in the cycle.

![Wagner graph](image)

**Figure 22: Wagner graph**

For the Wagner graph, \( \gamma_{bd}=3 \)

Here \( f(V) = 16 \) and \( D= \{v_2, v_5, v_7\} \) is a balanced dominating set.

\[ \sum_{v \in D} f(v) = 3+3+2 = 8 \]

Therefore, \( \gamma_{bd}=3 \).
5) The **Herschel graph** is a bipartite undirected graph with 11 vertices and 18 edges as shown in Figure 23, the smallest non Hamiltonian polyhedral graph. It is named after British astronomer Alexander Stewart Herschel.

![Herschel Graph](image)

**Figure 23: Herschel graph**

For the Herschel graph, $\gamma_{bd}=5$

Here $f(V) = 16$ and $D = \{v_4, v_8, v_9, v_{10}, v_{11}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 2+2+2+1+1 = 8$$

Therefore, $\gamma_{bd}=5$
6) **Moser spindle** (also called the Moser’s spindle or Moser graph) is an undirected graph, named after mathematicians Leo Moser and his brother William with seven vertices and eleven edges as shown in Figure 24.

![Moser spindle graph](image)

**Figure 24: Moser spindle**

For the Moser spindle graph, $\gamma_{bd}=3$

Here $f(V) = 10$ and $D = \{v_2, v_3, v_5\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 2 + 2 + 1 = 5$$

Therefore, $\gamma_{bd}=3$
7) The **Goldner-Harary graph** is a simple undirected graph with 11 vertices and 27 edges as shown in Figure 25. It is named after A. Goldner and Frank Harary, who proved in 1975 that it was the smallest non Hamiltonian maximal planar graph.

![Goldner-Harary graph](image)

**Figure 25: Goldner-Harary graph**

For the Goldner-Harary graph, $\gamma_{bd}=3$

Here $f(V) = 24$ and $D = \{v_4, v_8, v_{10}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 5 + 4 + 3 = 12$$

Therefore, $\gamma_{bd}=3$. 

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8) The **Grotzch graph** is a triangle-free graph with 11 vertices, 20 edges, chromatic number 4 and crossing number 5 as shown in Figure 26. It is named after German mathematician Herbert Grotzsch.

![Grotzch graph diagram](image)

**Figure 26: Grotzch graph**

For the Grotzch graph, $\gamma_{bd}=5$

Here $f(V)=24$ and $D=\{v_3,v_4,v_5,v_{10},v_{11}\}$ is a balanced dominating set.

$$\sum_{v\in D} f(v) = 4+3+2+2+1 = 12$$

Therefore, $\gamma_{bd}=5$. 

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9) The **Hoffman graph** is a 4-regular graph with 16 vertices and 32 edges as shown in Figure 27 discovered by Alan Hoffman.

![Figure 27: Hoffman graph](image)

For the Hoffman graph, $\gamma_{bd}=7$

Here $f(V) = 24$ and $D= \{v_4, v_6, v_8, v_{11}, v_{12}, v_{14}, v_{15}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 2+2+2+2+2+1+1 = 12$$

Therefore, $\gamma_{bd}=7$. 

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10) The **Mobius-Kantor graph** is a symmetric bipartite cubic graph with 16 vertices and 24 edges as shown in Figure 28, named after August Ferdinand Mobius and Seligmann Kantor.

![Mobius-Kantor graph](image)

**Figure 28: Mobius-Kantor graph**

For the Mobius-Kantor graph, $\gamma_{bd}=7$

Here $f(V) = 24$ and $D = \{v_2, v_3, v_8, v_{10}, v_{11}, v_{13}, v_{15}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 2+2+2+2+2+1+1 = 12$$

Therefore, $\gamma_{bd}=7$. 
11) The **Truncated Tetrahedron** is an Archimedean solid. It has 4 regular faces, 4 regular triangular faces, 12 vertices and 18 edges as shown in Figure 29. Archimedean solid means one of 13 possible solids whose faces are all regular polygons whose polyhedral angles are all equal.

![Figure 29: Truncated Tetrahedron](image)

For the Truncated Tetrahedron graph, \( \gamma_{bd}=4 \)

Here \( f(V) = 24 \) and \( D= \{ v_2, v_5, v_8, v_{11} \} \) is a balanced dominating set.

\[
\sum_{v \in D} f(v) = 3+3+3+3 = 12
\]

Therefore, \( \gamma_{bd}=4 \).
12) The **Desargues graph** is a distance-transitive graph with 20 vertices and 30 edges as shown in Figure 30. It is named after Gerard Desargues.

![Desargues Graph](image)

**Figure 30: Desargues graph**

For the Desargues graph, $\gamma_{bd}=8$

Here $f(V) = 30$ and $D = \{ v_2, v_4, v_6, v_8, v_{11}, v_{12}, v_{16}, v_{20} \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 2+2+2+2+2+2+1 = 15$$

Therefore, $\gamma_{bd}=8$. 72
13) The **chvatal graph** is an undirected graph with 12 vertices and 24 edges.

For the chvatal graph, $\gamma_{bd}=4$

Here $f(V) = 30$ and $D = \{v_4, v_8, v_{10}, v_{12}\}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 4 + 4 + 4 + 3 = 15$$

Therefore, $\gamma_{bd}=4$. 

**Figure 31: chvatal graph**
14) The Dürer graph is an undirected cubic graph with 12 vertices and 18 edges.

![Dürer graph diagram]

**Figure 32: Dürer graph**

For the Dürer graph, $\gamma_{bd}=4$

Here $f(V) = 28$ and $D = \{ v_4, v_6, v_8, v_9 \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 4 + 4 + 4 + 2 = 14$$

Therefore, $\gamma_{bd}=4$.  

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# 3.2. Grid graphs

**Theorem 3.2.1**

For Grid graph $P_2 \times P_n$ then $\gamma_{bd}(P_2 \times P_n) \leq n$ if $n$ is even.

**Proof:**

For Grid graph $P_2 \times P_n$ we have $2n$ vertices.

These $2n$ vertices divided into $n$ vertices of value 1 and $n$ vertices of value 2.

Therefore, $n$ 1's + $n$ 2's = $n + 2n = 3n$.

that is, $f(V) = 3n$.

Hence sum value of vertices belonging to $\gamma_{bd}$ - set must be $3n/2$

that is, $\sum_{v \in D} f(v) = 3n/2$.

Since $n$ is even, let $n = 2k$.

Therefore $\sum_{v \in D} f(v) = 3k$.

Suppose $3k = n_1$ 1's + $n_2$ 2's where $n_1$ is the number of vertices having value 1 and $n_2$ is the number of vertices having value 2.

That is, $3k = n_1 + 2n_2$
then $\gamma_{bd} = n_1 + n_2$.

We have to prove $\gamma_{bd} (P_2 \times P_n) \leq n$.

We prove this by induction on $k$.

Let $k=1$. We get $n_1 + 2n_2 = 3$.

Then $n_1 = 1$ and $n_2 = 1$.

Therefore $\gamma_{bd} = n_1 + n_2 = 2 = 2k = n$.

Assume that this is true for $n-2$.

For $P_2 \times P_{n-2}$ grid graph, there are $2n-4$ vertices.

These $2n-4$ vertices divided into $n-2$ vertices of value 1 and $n-2$ vertices of value 2.

Therefore $(n-2)$ 1's + $(n-2)$ 2's

$$= n-2 + 2(n-2)$$

$$= n-2 + 2n-4$$

$$= 3n-6.$$ 

Hence $3k = \frac{3n-6}{2}$.
Let $m_1$ denote the number of vertices of value 1 and $m_2$ denote the number of vertices of value 2 for the Grid graph $P_2 \times P_{n-2}$.

Therefore, $m_1$ 1's + $m_2$ 2's = $\frac{3n-6}{2}$. and

\[ \gamma_{bd} (P_2 \times P_{n-2}) = m_1 + m_2 \leq n - 2. \] (Since the result is true for n-2)

We have, \[ m_1 \text{ 1's} + m_2 \text{ 2's} = \frac{3n-6}{2}. \]

\[ m_1 \text{ 1's} + m_2 \text{ 2's} + 3 = \frac{3n-6}{2} + 3 \]

\[ (m_1+1) \text{ 1's} + (m_2+1) \text{ 2's} = 3n/2. \]

Therefore, \[ \gamma_{bd} (P_2 \times P_n) = m_1 + 1 + m_2 + 2 \]

\[ = m_1 + m_2 + 2 \]

\[ \leq n - 2 + 2 \]

\[ \leq n. \]

Therefore, \[ \gamma_{bd} (P_2 \times P_n) \leq n \text{ if n is even.} \]
Example 3.2.2:

Figure 33: $P_2XP_{10}$ graph

In this $P_2XP_{10}$ graph, $f(V)= 30$

$D= \{v_2, v_6, v_8, v_{10}, v_{12}, v_{16}, v_{17}, v_{20}\}$ is a balanced dominating set.

$\sum_{v \in D} f(v) = 2+2+2+2+2+2+1= 15$

$\gamma_{bd}=8$

Note 3.2.3:

The bound of theorem 3.2.1 is sharp.
Example 3.2.4:

![Figure 34: P₂X₄ graph](image)

In this P₂X₄ graph, f (V) = 12

\[ D = \{ v_2, v_5, v_7, v_8 \} \]

\[ \sum_{v \in D} f(v) = 2 + 2 + 1 + 1 = 6 \]

\[ \gamma_{bd} = 4 = n. \]

**Theorem 3.2.5**

For Grid graph P₂Xₙ then \( \gamma_{bd} (P₂Xₙ) = 0 \) if \( n \) is odd.

**Proof:**

The Grid graph P₂Xₙ has 2n vertices.

These 2n vertices divided into n vertices of value 1 and n vertices of value 2.

Therefore, n 1's + n 2's = n + 2n = 3n.
That is, \( f(V) = 3n \).

Since \( n \) is odd, \( n \) number of 1's gives an odd number and \( 2n \) is even.

Therefore, \( n+2n \) must be odd.

That is, \( f(V) \) is odd.

By theorem 2.1.3, \( \gamma_{bd}(P_2 X P_n) = 0 \).

**Example 3.2.6:**

![Figure 35: \( P_2 X P_9 \) graph](image)

In this \( P_2 X P_9 \) graph, \( f(V) = 27 \)

Therefore, \( \gamma_{bd} = 0 \).

**Theorem 3.2.7**

For Grid graph \( P_4 X P_n \) then \( \gamma_{bd}(P_4 X P_n) \leq 2n \).
Proof:

The Grid graph $P_4 \times P_n$ has $4n$ vertices.

These $4n$ vertices divided into two $2n$ vertices that is, $2n$ vertices of value 1 and $2n$ vertices of value 2.

Therefore, $2n$ 1's + $2n$ 2's = $2n + 2(2n)$

= $2n + 4n$

= $6n$.

that is, $f(V) = 6n$.

Hence sum value of vertices belonging to $\gamma_{bd}$ - set must be $6n/2$.

that is, $\sum_{v \in D} f(v) = 3n$.

Suppose $3n = n_1$ 1's + $n_2$ 2's where $n_1$ is the number of vertices having value 1 and $n_2$ is the number of vertices having value 2.

that is, $3n = n_1 + 2n_2$

then $\gamma_{bd} = n_1 + n_2$.

we have to prove $\gamma_{bd}(P_4 \times P_n) \leq 2n$. 
We prove this by induction on \( n \).

Let \( n=1 \). we get \( n_1 + 2n_2 = 3 \).

Then \( n_1 = 1 \) and \( n_2 = 1 \).

Therefore \( \gamma_{bd} = n_1 + n_2 = 2 = 2n \).

Assume that this is true for \( n-1 \).

that is, the result is true for \( P_4 \times P_{n-1} \).

that is, \( \gamma_{bd}(P_4 \times P_{n-1}) \leq 2(n-1) \leq 2n-2 \).

For \( P_4 \times P_{n-1} \) grid graph, there are \( 4n-4 \) vertices.

These \( 4n-4 \) vertices divided into \( 2n-2 \) vertices of value 1 and \( 2n-2 \) vertices of value 2.

Therefore \( (2n-2) \) 1's + \( (2n-2) \) 2's = \( 2n-2 + 2(2n-2) \)

\[ = 2n-2 + 4n-4 \]

\[ = 6n-6. \]

Hence \( \sum_{v \in D} f(v) = \frac{6n-6}{2} = 3n-3. \)
Let $m_1$ denote the number of vertices of value 1 and $m_2$ denote the number of vertices of value 2 for the Grid graph $P_4 \times P_{n-1}$.

Therefore, $m_1$ 1's + $m_2$ 2's = $3n-3$ and

$$\gamma_{bd} (P_4 \times P_{n-1}) = m_1 + m_2 \leq 2n-2.$$ (Since the result is true for $n-1$)

we have

$$m_1 \text{ 1's + } m_2 \text{ 2's } = 3n-3$$

$$m_1 \text{ 1's + } m_2 \text{ 2's + 3 } = 3n-3 + 3$$

$$(m_1+1) \text{ 1's + (m}_2+1) \text{ 2's } = 3n.$$ 

Therefore, $\gamma_{bd} = m_1 + 1 + m_2 + 1$

$$= m_1 + m_2 + 2$$

$$\leq 2n-2+2$$

$$\leq 2n.$$ 

Therefore, $\gamma_{bd} (P_4 \times P_n) \leq 2n$
Example 3.2.8:

Figure 36: \( P_4 \times P_6 \) graph

In this \( P_4 \times P_6 \) graph, \( f(V) = 36 \)

\[ D = \{v_2, v_4, v_6, v_9, v_{12}, v_{14}, v_{18}, v_{20}, v_{22}, v_{23}\} \] is a balanced dominating set.

\[ \sum_{v \in D} f(v) = 2+2+2+2+2+2+2+2+1+1 = 18 \]

\[ \gamma_{bd} = 10 \]

Note 3.2.9:

The bound of theorem 3.2.7 is sharp.
Example 3.2.10:

In this $P_4 \times P_2$ graph, $f(V) = 12$

$D = \{v_2, v_5, v_7, v_8\}$

$\sum_{v \in D} f(v) = 2 + 2 + 1 + 1 = 6$

$\gamma_{bd} = 4 = 2n.$

3.3. Friendship graph

A Friendship graph $F_n$ is a graph which consists of $n$ triangles with a common vertex.
Theorem 3.3.1

For friendship graph $F_n$, $\gamma_{bd} \leq n$.

Proof:

We prove the theorem by induction on $n$.

Let $n=1$. Then we have $F_1$ is a triangle.

Therefore, $\gamma_{bd} = 1 = n$.

We assume that the result is true for friendship graph $F_{n-1}$.

That is, $\gamma_{bd} (F_{n-1}) \leq n - 1$.

We have to prove for friendship graph $F_n$.

$F_n$ is a graph which consists of $n$ triangles with a common vertex.

Case i: The common vertex having value 1.

$$f(V) = \sum_{v \in V} f(v) = 6n - 1(n-1)$$

$$= 6n - n + 1$$

$$= 5n + 1$$

$5n+1$ is even only if $n$ is odd.
Let \( n \) be odd and \( \sum_{v \in D} f(v) = \frac{5n+1}{2} \).

For a dominating set, it must have the common vertex or one vertex from each \( n \) triangle other than the common vertex.

If one vertex from each \( n \) triangle other than the common vertex in the balanced dominating set then \( \gamma_{bd} = n \).

Otherwise, Taking the common vertex in the balanced dominating set we get \( \gamma_{bd} \) greater than \( n \).

By choosing the minimum cardinality, we get \( \gamma_{bd} = n \).

Case ii: The common vertex having value 2.

\[
f(V) = \sum_{v \in V} f(v) = 6n - 2(n-1) \\
= 6n - 2n + 2 \\
= 4n + 2
\]

\( 4n + 2 \) is always even.

Hence \( \sum_{v \in D} f(v) = 4n + \frac{2}{2} = 2n + 1 \).
For a dominating set, it must have the common vertex or one vertex from each n triangle other than the common vertex.

If one vertex from each n triangle other than the common vertex in the balanced dominating set then \( \gamma_{bd} = n \).

Otherwise, Taking the common vertex \( v_1 \) which is of value 2 in the balanced dominating set \( D \).

\[
\sum_{v \in D} f(v) = f(v_1) + \sum_{v \in D - \{v_1\}} f(v)
\]

\[
\sum_{v \in D - \{v_1\}} f(v) = \sum_{v \in D} f(v) - f(v_1)
\]

\[
= 2n+1-2 = 2n-1.
\]

By our assumption, \( \gamma_{bd}(F_{n-1}) \leq n - 1 \).

For \( F_{n-1} \), \( f(V) = \sum_{v \in V} f(v) = 6(n-1) - 2(n-2) \)

\[
= 6n-6-2n+4
\]

\[
= 4n-2
\]

Hence \( \sum_{v \in D} f(v) = 4n - 2 \) \( \leq 2n-1 \).

For 2n-1 value, \( \gamma_{bd} \leq n - 1 \).
Therefore, $\gamma_{bd}(F_n) = 1 + \gamma_{bd}(F_{n-1})$

$\leq 1 + n - 1$

$\gamma_{bd}(F_n) \leq n.$

Case iii: The common vertex having value 3.

$f(V) = \sum_{v \in V} f(v) = 6n - 3(n - 1)$

$= 6n - 3n + 3$

$= 3n + 3$

$3n + 3$ is even only if $n$ is odd.

Hence $\sum_{v \in D} f(v) = \frac{3n + 3}{2}$.

For a dominating set, it must have the common vertex or one vertex from each $n$ triangle other than the common vertex.

If one vertex from each $n$ triangle other than the common vertex in the balanced dominating set then $\gamma_{bd} = n$.

Otherwise, Taking the common vertex $v_1$ which is of value 3 in the balanced dominating set $D$. 
\[ \sum_{v \in D} f(v) = f(v_1) + \sum_{v \in D - \{v_1\}} f(v) \]

\[ \sum_{v \in D - \{v_1\}} f(v) = \sum_{v \in D} f(v) - f(v_1) \]

\[ = \frac{3n+3}{2} - 3 = \frac{3n-3}{2}. \]

By our assumption, the result is true for \( F_{n-2} \) (since \( n \) is odd).

For \( F_{n-2} \), \( f(V) = \sum_{v \in V} f(v) = 6(n-2) - 3(n-3) \)

\[ = 6n-12-3n+9 \]

\[ = 3n-3 \]

Hence \( \sum_{v \in D} f(v) = \frac{3n-3}{2} \).

For \( \frac{3n-3}{2} \). value, \( \gamma_{bd} (F_{n-2}) \leq n - 2 \).

Therefore, \( \gamma_{bd} \leq n - 2 + 1 \leq n - 1 \leq n. \)
Example 3.3.2:

Figure 38: Friendship graph $F_7$

In this friendship graph $F_7$, $f(V) = 24$

$D = \{v_1, v_3, v_5, v_7, v_9, v_{10}\}$ is a balanced dominating set.

$\sum_{v \in D} f(v) = 3 + 2 + 2 + 2 + 2 + 1 = 12$

$\gamma_{bd} = 6 \leq n$
Note 3.3.3:

The bound of theorem 3.3.1 is sharp.

Example 3.3.4:

![Friendship graph $F_3$](image)

**Figure 39: Friendship graph $F_3$**

In this friendship graph $F_3$, $f(V) = 16$

$D = \{ v_3, v_4, v_7 \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3 + 3 + 2 = 8$$

$$\gamma_{bd} = 3 = n$$