CHAPTER 5

5. BALANCED AND INDEPENDENT BALANCED DOMINATION

5.1 Balanced domination for web graph

To form a web graph, take a closed helm graph remove the central vertex and attach a pendant edge to each vertex of the n-cycle. We denote generalized web graph as $W(t,n)$ where $t$ is the number of n-cycles.

**Theorem 5.1.1:**

Let $G$ be a web graph $W(2,n)$ and $n$ be even. Let $n=2m$ then $\gamma_{bd}(G) \geq n+1$ if $m$ is even and $\gamma_{bd} = 0$ if $m$ is odd.

**Proof:**

Let $G$ be a web graph $W(2,n)$ and $n$ be even. the web graph $W(2,n)$ has $3n$ vertices.

$$f(v) = \frac{3n}{2} (1) + \frac{3n}{2} (2)$$

$$= \frac{9n}{2}$$

Since $n=2m$, $f(V) = 9m$
If m is odd, 9m is odd. Therefore, $\gamma_{bd} = 0$

If m is even, 9m is even.

Let n=4. W(2,4) has 12 vertices and $f(V) = \frac{9n}{2} = 18$.

![Figure 52: Web graph W(2,4)](image)

Here $\gamma_{bd} = 5 = n+1$.

Assume that the theorem is true for n-4.
That is, for $W(2, n-4)$, $\gamma_{bd} \geq n-3$.

$W(2, n)$ has 12 vertices more than $W(2, n-4)$. Out of these 12 vertices, 4 are pendant vertices.

Therefore, $\gamma_{bd}(W(2, n)) \geq \gamma_{bd}(W(2, n-4)) + 4$

$\geq n-3 + 4$

$\geq n + 1$.

**Theorem 5.1.2:**

For the web graph $W(2, n)$ where $n$ is odd then $\gamma_{bd} \geq n+1$.

**Proof:**

The web graph $W(2, n)$ has $3n$ vertices.

The web graph $W(2, n)$ has two cycles each of $n$ vertices and $n$ pendant vertices.

The total value of $2n$ vertices of two cycles is $3(n+1)$.

The $n$ pendant vertices can be labeled in different ways.

Case 1: among the $n$ pendant vertices, label $\frac{n-1}{2}$ vertices of value 1 and $\frac{n-1}{2}$ vertices of value 2.
Subcase (i): the remaining one pendant vertex can be labeled as 1.

\[
f(V) = 3(n+1) + \frac{n-1}{2} (2) + \frac{n-1}{2} (1) + 1 \\
= \frac{9n+5}{2}.
\]

If \(\frac{9n+5}{2}\) is odd then \(\gamma_{bd}=0\)

If \(\frac{9n+5}{2}\) is even, then \(\sum_{v \in D} f(v) = \frac{9n+5}{4}\).

For the balanced dominating set, if we take all the vertices in the second cycle we cover all the vertices but sum of values is not equal to \(\frac{9n+5}{4}\). Therefore, we need to take more vertices in the balanced dominating set.

If we take some vertices in the pendant vertices and some vertices in the second cycle, these vertices can cover only some vertices in first cycle. In order to cover other vertices in first cycle we need more than \(n\) vertices.

Therefore \(\gamma_{bd} \geq n+1\).

Subcase (ii): the remaining one pendant vertex can be labeled as 2.

\[
f(V) = 3(n+1) + \frac{n-1}{2} (2) + \frac{n-1}{2} (1) + 2
\]
\[ f(V) = 3(n+1) + \frac{n-1}{2} (2) + \frac{n-1}{2} (1) + 3 = \frac{9n+9}{2}. \]

If \( \frac{9n+9}{2} \) is odd then \( \gamma_{bd}=0 \)

If \( \frac{9n+9}{2} \) is even, then \( \sum_{v \in D} f(v) = \frac{9n+9}{4} \).

For the balanced dominating set, if we take all the vertices in the second cycle we cover all the vertices but sum of values is not equal to \( \frac{9n+9}{4} \). Therefore, we need to take more vertices in the balanced dominating set.

If we take some vertices in the pendant vertices and some vertices in the second cycle, these vertices can cover only some vertices in first cycle. In order to cover other vertices in first cycle we need more than \( n \) vertices.

Therefore \( \gamma_{bd} \geq n+1 \).
Case 2: Among the n pendant vertices, label \( \frac{n-1}{2} \) vertices of value 1 and \( \frac{n-1}{2} \) vertices of value 3.

Subcase (i): the remaining one pendant vertex can be labeled as 1.

\[
f(V) = 3(n+1) + \frac{n-1}{2}(3) + \frac{n-1}{2}(1) + 1
\]

\[
= 5n+2 \text{ which is odd.}
\]

Subcase (ii): the remaining one pendant vertex can be labeled as 2.

\[
f(V) = 3(n+1) + \frac{n-1}{2}(3) + \frac{n-1}{2}(1) + 2
\]

\[
= 5n+3
\]

Same as subcase (i) of case 1. Therefore \( \gamma_{bd} \geq n+1 \).

Subcase (iii): the remaining one pendant vertex can be labeled as 3.

\[
f(V) = 3(n+1) + \frac{n-1}{2}(3) + \frac{n-1}{2}(1) + 3
\]

\[
= 5n+4 \text{ which is odd.}
\]

Case 3: among the n pendant vertices, label \( \frac{n-1}{2} \) vertices of value 3 and \( \frac{n-1}{2} \) vertices of value 2.
Subcase (i): the remaining one pendant vertex can be labeled as 1.

\[ f(V) = 3(n+1) + \frac{n-1}{2} \cdot (2) + \frac{n-1}{2} \cdot (3) + 1 \]

\[ = \frac{11n+3}{2}. \]

Hence \( \sum_{v \in D} f(v) = \frac{11n+3}{4}. \)

Same as subcase (i) of case 1. Therefore \( \gamma_{bd} \geq n+1. \)

Subcase (ii): the remaining one pendant vertex can be labeled as 2.

\[ f(V) = 3(n+1) + \frac{n-1}{2} \cdot (2) + \frac{n-1}{2} \cdot (3) + 2 \]

\[ = \frac{11n+5}{2}. \]

Same as subcase (i) of case 1.

Subcase (iii): the remaining one pendant vertex can be labeled as 3.

\[ f(V) = 3(n+1) + \frac{n-1}{2} \cdot (2) + \frac{n-1}{2} \cdot (1) + 3 \]

\[ = \frac{11n+7}{2}. \]

Hence \( \sum_{v \in D} f(v) = \frac{11n+7}{4}. \) Same as subcase (i) of case 1. Therefore \( \gamma_{bd} \geq n+1. \)
Example 4.2.4

Figure 53: Web graph $W(2,7)$

In this Web graph $W(2,7)$, $f(V) = 38$

$D = \{ v_4, v_7, v_8, v_9, v_{10}, v_{13}, v_{18}, v_{19}, v_{22} \}$ is a balanced dominating set.

$$\Sigma_{v \in D} f(v) = 3 + 3 + 3 + 2 + 2 + 1 + 1 + 1 = 19$$

$\gamma_{bd} = 9$. 

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**5.2 Independent Balanced Domination**

A set $S$ of vertices in a graph $G$ is an independent balanced dominating set if $S$ is a balanced dominating set and the set of vertices $S$ is independent.

The independent balanced domination number $\gamma_{ibd}(G)$ is the minimum cardinality of the independent balanced dominating set.

**Theorem 5.2.1:**

Let $G$ be a complete bipartite graph $K_{m,n}$ $(m>n)$, then $G$ has two independent balanced dominating sets if $m = 2n$.

**Proof:**

Let $G$ be a complete bipartite graph. $G$ can be partitioned into 2 sets $S_1$ and $S_2$ with $|S_1|=m$, $|S_2|=n$ and each set of vertices have labeling 1 and 2.

Also $S_1$ and $S_2$ are independent.

If $m=2n$, give the labeling 1 to each of vertices of $S_2$ and 2 to each of vertices of $S_1$.

Therefore, we get $\sum_{u \in S_1} f(u) = \sum_{v \in S_2} f(v)$ and both the set $S_1$ and $S_2$ are independent.

Therefore $G$ has two independent balanced dominating sets.
**Theorem 5.2.2:**

Let $G$ be a complete bipartite graph $K_{m,n}$ and if $m = 2n$ then $\gamma_{ibd}(G) = n$.

**Proof:**

Let $m = 2n$.

By theorem 5.2.1, $G$ has 2 independent balanced dominating set $S_1$ and $S_2$.

And $|S_1|=m$, $|S_2|=n$.

Since $\gamma_{ibd}(G)$ is the minimum cardinality of the independent balanced dominating set,

$$\gamma_{ibd}(G) = \min \{m, n\}.$$ 

Since $m > n$, $\gamma_{ibd}(G) = n$. 

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5.3 Balanced dominating function

For a graph \( G = (V, E) \), let \( f: V \rightarrow \{1, 2\} \) and let \((V_1, V_2)\) be the ordered partition of \( V \) induced by \( f \), where \( V_i = \{v \in V / f(v) = i\} \) and \( |v_i| = n_i \) for \( i = 1, 2 \).

A function \( f = (v_1, v_2) \) is balanced domination function if \( n_1 \) is even. The weight of \( f \) is \( f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 \).

The Balanced domination function number is denoted by \( BD (G) \) equals the minimum weight of an balanced dominating function of \( G \) and we say that a function \( f = (V_1, V_2) \) is a \( BD \)-function if it is an balanced domination function and \( f(V) = BD (G) \).

**Theorem 5.3.1:**

For any connected graph \( G \), \( \gamma (G) \leq BD (G) \)

**Proof:**

Let \( f = (V_1, V_2) \) be a \( BD \) – function and let \( S \) be a \( \gamma \) set of \( G \).

Then \( V_1 \) is a dominating set of \( G \) or \( V_2 \) is a dominating set of \( G \)

\[ \gamma (G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = BD (G) \]. Therefore, \( \gamma (G) \leq BD (G) \)
**Theorem 5.3.2:**

If $G$ is a connected graph of order $n$ and if $BD\,(G) = \gamma\,(G) + 1$ then there is a vertex $v \in V$ of degree $n - \gamma\,(G)$.

**Proof:**

Assume that $BD\,(G) = \gamma\,(G) + 1$.

For a balanced dominating function $f = (V_1, V_2)$ to have weight $\gamma\,(G) + 1$.

Either $|V_1| = \gamma\,(G) + 1$ and $|V_2| = 0$ or $|V_1| = \gamma\,(G)-1$ and $|V_2| = 1$.

Any other arrangement of weight $\gamma\,(G) + 1$ would have $|V_1| + |V_2| < \gamma\,(G)$

Case (1): $|V_1| = \gamma\,(G) + 1$ and $|V_2| = 0$

$G$ is disconnected

Case (1): $|V_1| = \gamma\,(G)-1$ and $|V_2| = 1$

$|V_2| = 1$

Let $V_2 = \{v\}$

$\text{Deg } v = n - 1 = n - \gamma\,(G)$
Therefore, there is a vertex \( v \in V \) of degree \( n - \gamma(G) \)

**Theorem 5.3.3:**

For any graph \( G \) with maximum degree \( \Delta \geq 1 \), \( BD(G) \geq 3n/\Delta + 1 \)

**Proof:**

Let \( f = (V_1, V_2) \) be an BD- function of \( G \). Since each vertex \( v \in V \), is adjacent to a vertex of \( V_2 \),

\[ |V_1| \leq \Delta|V_2| \]

Similarly, \( |V_2| \leq \Delta|V_1| \)

\[(\Delta+1) \ BD \ G = (\Delta+1)|V_1| + (\Delta+1)2|V_2| \]

\[ \geq (\Delta+1)|V_1| + 2|V_1| + 2|V_2| \]

\[ \geq |V_2| + |V_1| + 2|V_1| + 2|V_2| \]

\[ \geq 3|V_1| + 3|V_2| \]

\[ \geq 3n \]

\( BD(G) \geq 3n/\Delta + 1 \).
Theorem 5.3.4:

For path $P_n$, if $n$ is even then $BD(G) = \frac{3n}{2}$.

Proof:

Let $n_1$ and $n_2$ be the cardinality of $V_1$ and $V_2$ where $V_1 = \{v \in V / f(v) = 1\}$ and

$V_2 = \{v \in V / f(v) = 2\}$.

Clearly $n_1 + n_2 = n$.

If $n$ is even, we have $n_1 = n/2$ and $n_2 = n/2$.

Then $BD(G) = f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = \frac{3n}{2}$.

Hence $BD(G) = \frac{3n}{2}$.

Result 5.3.5:

For path $P_n$, $BD(G) = \begin{cases} \left\lfloor \frac{3n}{2} \right\rfloor & \text{if } n_1 > n_2 \\ \left\lceil \frac{3n}{2} \right\rceil & \text{if } n_1 < n_2 \end{cases}$. 
Theorem 5.3.6:

For complete bipartite graph $K_{m,n}$, $BD(G) =$ \begin{cases} 2m + n & \text{if } n \text{ is even} \\ 2n + m & \text{if } m \text{ is even} \end{cases}

Proof:

If $n$ is even, take $n_1 = n$ and $n_2 = m$.

Therefore, we get $n$ vertices that assign 1 and $m$ vertices that assign 2.

Then $BD(G) = f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1$

\[= 2m + n.\]

If $m$ is even, take $n_1 = m$ and $n_2 = n$.

Therefore, we get $m$ vertices that assign 1 and $n$ vertices that assign 2.

Then $BD(G) = f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1$

\[= 2n + m.\]

Hence $BD(G) =$ \begin{cases} 2m + n & \text{if } n \text{ is even} \\ 2n + m & \text{if } m \text{ is even} \end{cases}
Theorem 5.3.7:

For cycle $C_n$ ($n > 3$), if $n$ is even then $BD(G) = \frac{3n}{2}$.

Proof:

Let $n_1$ and $n_2$ be the cardinality of $V_1$ and $V_2$ where $V_1 = \{v \in V / f(v) = 1\}$ and $V_2 = \{v \in V / f(v) = 2\}$.

Clearly $n_1 + n_2 = n$.

If $n$ is even, we have $n_1 = n/2$ and $n_2 = n/2$.

Then $BD(G) = f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = 3n/2$.

Theorem 5.3.8:

For the $2 \times m$ grid graph, if $m$ is even then $BD(G) = 3m$.

Proof:

Let $m$ be even. For $2 \times m$ grid graph, we have $2m$ vertices. We can split the graph into two sets each having $m$ vertices. That is, one set of $m$ vertices assign 1 and another set of $m$ vertices assign 2. Therefore, we get $n_1 = m$ and $n_2 = m$.

Then $BD(G) = f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = 3m$. 