CHAPTER 4

4. BALANCED DOMINATION NUMBER OF HELM AND PETERSEN GRAPH

4.1 Helm graph

The Helm graph $H_n$ is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the n-cycle.

Theorem 4.1.1

For Helm graph $H_n$ ($n \geq 3$), $\gamma_{bd}(H_n) = n$.

Proof:

Case I: Let $n$ be even.

Case i: Label the centre vertex with value 1.

In this case, Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n}{2}$ vertices of value 2 and $\frac{n}{2}$ vertices of value 3 and $n+1$ vertices of value 1.

Therefore, $\sum_{v \in V} f(v) = \frac{n}{2} (2) + \frac{n}{2} (3) + n+1$

$= n + \frac{3n}{2} + n + 1$

$= 2n + \frac{3n}{2} + 1$
Since $n$ is even, $7n+2$ is even and $\frac{7n+2}{2}$ is even or odd.

If $\frac{7n+2}{2}$ is odd, then there is no balanced dominating set.

If $\frac{7n+2}{2}$ is even, then $\sum_{v \in D} f(v) = \frac{7n+2}{4}$.

Also $\frac{7n+2}{2}$ is even only when $n=2m$ where $m$ is odd ($m=3,5,7,\ldots$).

That is, $n=6,10,14,18,\ldots$.

We prove by induction on $n$.

Let $n=6 (m=3)$.

The result is true for $H_6$ and $\gamma_{bd}(H_6) = 6$ as shown in Figure 40.

Figure 40: Helm graph $H_6$
In this Helm graph $H_6$, $f(V) = 22$

$D = \{ v_2, v_3, v_4, v_7, v_8, v_{12} \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3 + 3 + 2 + 1 + 1 + 1 = 11$$

$\gamma_{bd} = 6$

Assume that the result is true for $n-4$.

That is, $\gamma_{bd} (H_{n-4}) = n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{7(n-4) + 2}{2}$

$$= \frac{7n-28 + 2}{2}$$

$$= \frac{7n-26}{2}$$

And $\sum_{v \in D} f(v) = \frac{7n-26}{4}$

We have to prove for $H_n$, that is, $\gamma_{bd} (H_n) = n$.

From $H_n$ we get $\frac{7n+2}{2} - 14 = \frac{7n+2-28}{2}$

$$= \frac{7n-26}{2}.$$ 

$H_n$ has 8 vertices of value 14 more than $H_{n-4}$ in which four of the vertices are leaves.

To cover these four leaves, we have to take four vertices which may be the four leaves or vertices adjacent to the leaves.
Therefore, $\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4$

$= n-4 + 4$

$= n.$

Case ii: label the centre vertex with value 2.

In this case, Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n}{2}$ vertices of value 1 and $\frac{n}{2}$ vertices of value 3 and $n+1$ vertices of value 2.

Therefore, $\sum_{v \in V} f(v) = \frac{n}{2} (1) + \frac{n}{2} (3) + n+1(2)$

$= \frac{n}{2} + \frac{3n}{2} + 2n + 2$

$= \frac{8n+4}{2}$

$= 4n + 2$

Since $4n+2$ is even, $\sum_{v \in D} f(v) = \frac{4n+2}{2} = 2n+1$

Helm graph $H_n$ has $n$ leaves. To cover these $n$ leaves, we take these $n$ leaves or vertices adjacent to these leaves.

Taking $(n-1)$ vertices of value 2 and one vertex of value 3 gives $n$ vertices of value $2n+1$.

Therefore, $\gamma_{bd}(H_n) = n.$

Case iii: Label the centre vertex with value 3.
In this case, Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n}{2}$ vertices of value 1 and $\frac{n}{2}$ vertices of value 2 and $n+1$ vertices of value 3.

Therefore, $\sum_{v \in V} f(v) = \frac{n}{2} (1) + \frac{n}{2} (2) + n+1(3)$

$$= \frac{n}{2} + n + 3n+ 3$$

$$= 4n + \frac{n}{2} + 3$$

$$= \frac{8n+6+n}{2}$$

$$= \frac{9n+6}{2}$$

Since $n$ is even, $9n+6$ is even and $\frac{9n+6}{2}$ is even or odd.

If $\frac{9n+6}{2}$ is odd, then there is no balanced dominating set.

If $\frac{9n+6}{2}$ is even, then $\sum_{v \in D} f(v) = \frac{9n+6}{4}$.

Also $\frac{9n+6}{2}$ is even only when $n=2m$ where $m$ is odd ($m=3,5,7,\ldots$)

That is, $n= 6,10,14,18,\ldots$.

We prove by induction on $n$.

Let $n=6(m=3)$.

The result is true for $H_6$ and $\gamma_{bd}(H_6) = 6$ as shown in Figure 41
In this Helm graph $H_6$, $f(V) = 30$

$D = \{ v_1, v_6, v_9, v_{10}, v_{11}, v_{12} \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3 + 3 + 3 + 3 + 2 + 1 = 15$$

$\gamma_{bd} = 6$

Assume that the result is true for $n-4$.

That is, $\gamma_{bd}(H_{n-4}) = n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{9(n-4) + 6}{2}$
\[= \frac{9n-36+6}{2}\]
\[= \frac{9n-30}{2}\]

And \(\sum_{v \in D} f(v) = \frac{9n-30}{4}\)

We have to prove for \(H_n\), that is, \(\gamma_{bd}(H_n) = n\).

From \(H_n\) we get
\[\frac{9n+6}{2} - 18 = \frac{9n+6-36}{2}\]
\[= \frac{9n-30}{2}.\]

\(H_n\) has 8 vertices of value 18 more than \(H_{n-4}\) in which four of the vertices are leaves.

To cover these four leaves we have to take four vertices which may be the four leaves or vertices adjacent to the leaves.

Therefore, \(\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4\)
\[= n-4 + 4\]
\[= n.\]

Case II: Let \(n\) be odd.

Case i: Label the centre vertex with value 1.

In this case, Helm graph \(H_n\) has \(2n+1\) vertices in which \(\frac{n-1}{2}\) vertices of value 2 and \(\frac{n-1}{2}\) vertices of value 3 and \(n+1\) vertices of value 1 and one vertex of value 4.
Therefore, \( \sum_{v \in V} f(v) = \frac{n-1}{2} (2) + \frac{n-1}{2} (3) + n+1+4 \)

\[
= n-1 + \frac{3n-3}{2} + n + 5
\]

\[
= 2n + \frac{3n-3}{2} + 4
\]

\[
= \frac{4n+3n-3+8}{2}
\]

\[
= \frac{7n+5}{2}
\]

Since \( n \) is even, \( 7n+2 \) is even and \( \frac{7n+2}{2} \) is even or odd.

If \( \frac{7n+2}{2} \) is odd, then there is no balanced dominating set.

If \( \frac{7n+2}{2} \) is even, then \( \sum_{v \in D} f(v) = \frac{7n+2}{4} \).

Also \( \frac{7n+2}{2} \) is even only when \( n=2m-1 \) where \( m \) is odd (\( m=3,5,7,....... \))

That is, \( n= 5,9,13,17,........ \)

We prove by induction on \( n \).

Let \( n=5(m=3) \).

The result is true for \( H_3 \) and \( \gamma_{bd} (H_3) = 5 \) as shown in Figure 42
Figure 42: Helm graph $H_5$

In this Helm graph $H_5$, $f(V) = 20$

$D = \{ v_2, v_3, v_4, v_6, v_7 \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3 + 3 + 2 + 1 + 1 = 10$$

$\gamma_{bd} = 5$

Assume that the result is true for $n-4$.

That is, $\gamma_{bd}(H_{n-4}) = n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{7(n-4)+5}{2}$

$$= \frac{7n-28+5}{2}$$

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And \( \sum_{v \in D} f(v) = \frac{7n-23}{4} \)

We have to prove for \( H_n \), that is, \( \gamma_{bd}(H_n) = n \).

From \( H_n \) we get \( \frac{7n+5}{2} - 14 = \frac{7n+5-28}{2} = \frac{7n-23}{2} \).

\( H_n \) has 8 vertices of value 14 more than \( H_{n-4} \), in which four of the vertices are leaves.

To cover these four leaves we have to take four vertices which may be the four leaves or vertices adjacent to the leaves.

Therefore, \( \gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4 \)

\[ = n-4 + 4 \]

\[ = n. \]

Case ii: Label the centre vertex with value 2.

In this case, Helm graph \( H_n \) has \( 2n+1 \) vertices in which \( \frac{n-1}{2} \) vertices of value 3 and \( \frac{n-1}{2} \) vertices of value 1 and \( n+1 \) vertices of value 2 and one vertex of value 4.

Therefore, \( \sum_{v \in V} f(v) = \frac{n-1}{2} (3) + \frac{n-1}{2} (1) + n+1(2) + 4 \)

\[ = \frac{3n-3}{2} + \frac{n-1}{2} + 2n + 6 \]
Since $4n+4$ is even, $\sum_{v \in D} f(v) = \frac{4n+2}{2} = 2n+2$

Helm graph $H_n$ has $n$ leaves. To cover these $n$ leaves, we take these $n$ leaves or vertices adjacent to these leaves.

Taking $\frac{n-1}{2}$ vertices of value 3 and $\frac{n-1}{2}$ vertices of value 1 and one vertex of value 4 gives $n$ vertices of value $2n+2$.

Therefore, $\gamma_{bd}(H_n) = n$.

Case iii: Label the centre vertex with value 3.

In this case, Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n-1}{2}$ vertices of value 2 and $\frac{n-1}{2}$ vertices of value 1 and $n+1$ vertices of value 3 and one vertex of value 4.

Therefore, $\sum_{v \in V} f(v) = \frac{n-1}{2}(2) + \frac{n-1}{2}(1) + n+1(3) + 4$

$= \frac{n-1}{2} + n-1 + 3n + 3+4$

$= 4n + \frac{n-1}{2} + 6$

$= \frac{8n+12+n-1}{2}$

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Since n is odd, $9n+11$ is even and $\frac{9n+11}{2}$ is even or odd.

If $\frac{9n+11}{2}$ is odd, then there is no balanced dominating set.

If $\frac{9n+11}{2}$ is even, then $\sum_{v \in D} f(v) = \frac{9n+11}{4}$.

Also $\frac{9n+11}{2}$ is even only when $n=2m-1$ where $m$ is odd ($m=3,5,7,\ldots$)

That is, $n=5,9,13,17,\ldots$

We prove by induction on $n$.

Let $n=5(m=3)$.

The result is true for $H_5$ and $\gamma_{bd}(H_5) = 5$ as shown in Figure 43
In this Helm graph $H_5$, $f(V) = 28$

$D = \{ v_2, v_4, v_5, v_7, v_9 \}$ is a balanced dominating set.

$\sum_{v \in D} f(v) = 3 + 3 + 4 + 2 + 2 = 14$

$\gamma_{bd} = 5$

Assume that the result is true for $n-4$.

That is, $\gamma_{bd}(H_{n-4}) = n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{9(n-4)+11}{2}$

$= \frac{9n-36+11}{2}$

$= \frac{9n-25}{2}$
And $\sum_{v \in D} f(v) = \frac{9n-25}{4}$

We have to prove for $H_n$ that is, $\gamma_{bd}(H_n) = n$.

From $H_n$ we get

$$\frac{9n+11}{2} - 18 = \frac{9n+11-36}{2}$$

$$= \frac{9n-25}{2}.$$

$H_n$ has 8 vertices of value 18 more than $H_{n-4}$ in which four of the vertices are leaves.

To cover these four leaves we have to take four vertices which may be the four leaves or vertices adjacent to the leaves.

Therefore, $\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4$

$$= n-4 + 4$$

$$= n.$$

**Example 4.1.2**

For Helm graph $H_n$, $n$ is even
Figure 44: Helm graph $H_{14}$

In this graph, $f(V) = 50$

$D = \{v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{28}\}$ is a balanced dominating set.

$\sum_{v \in D} f(v) = 2 + 2 + 2 + 3 + 3 + 3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 25$

$\gamma_{bd} = 14$
Example 4.1.3

For Helm graph $H_n$, $n$ is odd

Figure 45: Helm graph $H_{11}$

In this graph, $f(V) = 48$

$D = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3 + 3 + 3 + 3 + 1 + 1 + 1 + 1 + 1 + 4 = 24$$

$\gamma_{bd} = 11$
4.2 Closed Helm graph

The Closed Helm graphs are formed by taking the Helm graph $H_n$ and placing an edge between each pendant vertex to form a second cycle in the graph.

It is denoted by $H_n$.

**Theorem 4.2.1**

For Closed Helm graph $H_n$ ($n \geq 3$), $\gamma_{bd}(H_n) \leq n$.

**Proof:**

Case I: Let $n$ be even.

Case i: Label the centre vertex with value 1.

Sub case i: All the vertices of the second cycle labeled as 1.

In this case, Closed Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n}{2}$ vertices of value 2 and $\frac{n}{2}$ vertices of value 3 and $n+1$ vertices of value 1.

Therefore, $\sum_{v \in V} f(v) = \frac{n}{2}(2) + \frac{n}{2}(3) + n+1$

$$= n + \frac{3n}{2} + n + 1$$

$$= 2n + \frac{3n}{2} + 1$$

$$= \frac{7n+2}{2}$$
Since $n$ is even, $7n+2$ is even and $\frac{7n+2}{2}$ is even or odd.

If $\frac{7n+2}{2}$ is odd, then there is no balanced dominating set.

If $\frac{7n+2}{2}$ is even, then $\sum_{v \in D} f(v) = \frac{7n+2}{4}$.

Also $\frac{7n+2}{2}$ is even only when $n=2m$ where $m$ is odd ($m=3,5,7,\ldots$)

That is, $n=6,10,14,18,\ldots$.

We prove by induction on $n$.

Let $n=6(m=3)$.

The result is true for $H_6$.

Assume that the result is true for $n-4$.

That is, $\gamma_{bd}(H_{n-4}) \leq n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{7(n-4)+2}{2}$

$= \frac{7n-28+2}{2}$

$= \frac{7n-26}{2}$

And $\sum_{v \in D} f(v) = \frac{7n-26}{4}$

We have to prove for $H_n$, that is, $\gamma_{bd}(H_n) \leq n$.

From $H_{n-4}$ we get $\frac{7n-26}{2} + 14 = \frac{7n-26+28}{2}$

$= \frac{7n+2}{2}$.
\( H_n \) has 8 vertices of value 14 more than \( H_{n-4} \) in which four of the vertices of value 1 and two vertices of value 2 and two vertices of value 3.

To cover these 8 vertices we have to take four vertices.

Therefore, \( \gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4 \)

\[ \leq n-4 + 4 \]

\[ \leq n. \]

Sub case ii: No vertices of the second cycle labeled as 1.

In this case, Closed Helm graph \( H_n \) has 2n+1 vertices in which n vertices of value 2 and n vertices of value 3 and 1 vertices of value 1.

Therefore, \( \sum_{v \in V} f(v) = n \times (2) + n \times (3) + 1 \)

\[ = 5n+1 \]

Since \( n \) is even, \( 5n+1 \) is odd.

Therefore, there is no balanced dominating set.

Case ii: Label the centre vertex with value 2.

Sub case i: All the vertices of the second cycle labeled as 2.

In this case, Closed Helm graph \( H_n \) has 2n+1 vertices in which \( \frac{n}{2} \) vertices of value 1 and \( \frac{n}{2} \) vertices of value 3 and n+1 vertices of value 2.

Therefore, \( \sum_{v \in V} f(v) = \frac{n}{2} \times (1) + \frac{n}{2} \times (3) + n+1 \times (2) \)
\[
\frac{n}{2} + \frac{3n}{2} + 2n + 2
= \frac{8n+4}{2}
= 4n + 2
\]

Since \(4n+2\) is even, \(\sum_{v \in D} f(v) = \frac{4n+2}{2} = 2n+1\).

We prove by induction on \(n\).

Let \(n=4\).

The result is true for \(H_4\).

Assume that the result is true for \(n-2\).

That is, \(\gamma_{bd}(H_{n-2}) \leq n-2\).

For \(H_{n-2}\), \(\sum_{v \in V} f(v) = 4(n-2) + 2\)

\[= 4n - 8 + 2\]

\[= 4n - 6.\]

We have to prove for \(H_n\), that is, \(\gamma_{bd}(H_n) \leq n\).

From \(H_{n-4}\) we get \(4n - 6 + 8 = 4n + 2\).

\(H_n\) has 4 vertices of value 8 more than \(H_{n-4}\) in which one vertex of value 1 and two vertices of value 2 and one vertex of value 3.

To cover these 4 vertices we have to take two vertices.

Therefore, \(\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-2}) + 2\).
\[ \leq n - 2 + 2 \]
\[ \leq n. \]

Sub case ii: No vertices of the second cycle labeled as 2.

In this case, Closed Helm graph \( \text{H}_n \) has 2n+1 vertices in which n vertices of value 1 and n vertices of value 3 and 1 vertex of value 2.

Therefore, \( \sum_{v \in V} f(v) = n \cdot 1 + n \cdot 3 + 2 \)
\[ = 4n + 2 \]

Since 4n+2 is even, \( \sum_{v \in D} f(v) = \frac{4n+2}{2} = 2n+1. \)

We prove by induction on n.

Let n=4.

The result is true for \( \text{H}_4 \).

Assume that the result is true for n-2.

That is, \( \gamma_{bd} (\text{H}_{n-2}) \leq n-2. \)

For \( \text{H}_{n-2}, \sum_{v \in V} f(v) = 4(n-2) + 2 \)
\[ = 4n - 8 + 2 \]
\[ = 4n - 6. \]

We have to prove for \( \text{H}_n \) that is, \( \gamma_{bd} (\text{H}_n) \leq n. \)

From \( \text{H}_{n-4} \) we get 4n -6 + 8 = 4n+2.
$H_n$ has 4 vertices of value 8 more than $H_{n-4}$ in which two vertices of value 1 and two vertices of value 3.

To cover these 4 vertices we have to take two vertices.

Therefore, $\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-2}) + 2$

\[
\leq n-2 + 2
\]

\[
\leq n.
\]

Case iii: Label the centre vertex with value 3.

Sub case i: All the vertices of the second cycle labeled as 3.

In this case, Closed Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n}{2}$ vertices of value 1 and $\frac{n}{2}$ vertices of value 2 and $n+1$ vertices of value 3.

Therefore, $\sum_{v \in V} f(v) = \frac{n}{2} (1) + \frac{n}{2} (2) + n+1(3)$

\[
= \frac{n}{2} + n + 3n + 3
\]

\[
= 4n + \frac{n}{2} + 3
\]

\[
= \frac{8n+6+n}{2}
\]

\[
= \frac{9n+6}{2}
\]

Since $n$ is even, $9n+6$ is even and $\frac{9n+6}{2}$ is even or odd.
If \( \frac{9n+6}{2} \) is odd, then there is no balanced dominating set.

If \( \frac{9n+6}{2} \) is even, then \( \sum_{v \in D} f(v) = \frac{9n+6}{4} \).

Also \( \frac{9n+6}{2} \) is even only when \( n = 2m \) where \( m \) is odd (\( m = 3, 5, 7, \ldots \))

That is, \( n = 6, 10, 14, 18, \ldots \).

We prove by induction on \( n \).

Let \( n = 6(m = 3) \).

The result is true for \( H_6 \).

Assume that the result is true for \( n-4 \).

That is, \( \gamma_{bd}(H_{n-4}) = n-4 \).

For \( H_{n-4}, \sum_{v \in V} f(v) = \frac{9(n-4)+6}{2} \)

\[ = \frac{9n-36+6}{2} \]

\[ = \frac{9n-30}{2} \]

And \( \sum_{v \in D} f(v) = \frac{9n-30}{4} \)

We have to prove for \( H_n \), that is, \( \gamma_{bd}(H_n) \leq n \).

From \( H_{n-4} \) we get \( \frac{9n-30}{2} - 18 = \frac{9n-30+36}{2} \)

\[ = \frac{9n+6}{2} \cdot \]
$H_n$ has 8 vertices of value 18 more than $H_{n-4}$ in which four vertices of value 3 and two vertices of value 1 and two vertices of value 2.

To cover these 8 vertices we have to take four vertices.

Therefore, $\gamma_{bd}(H_n) \leq \gamma_{bd}(H_{n-4}) + 4$

\[
\leq n-4 + 4
\]

\[
\leq n.
\]

Sub case ii: No vertices of the second cycle labeled as 3.

In this case, Closed Helm graph $H_n$ has 2n+1 vertices in which n vertices of value 2 and n vertices of value 1 and 1 vertices of value 3.

Therefore, $\sum_{v \in V} f(v) = n(2) + n(1) + 3$

\[
= 3n+3
\]

Since n is even, 3n+3 is odd.

Therefore, there is no balanced dominating set.

Case II: Let n be odd.

Case i: Label the centre vertex with value 1.

Sub case i: All the vertices of the second cycle labeled as 1.
In this case, Closed Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n-1}{2}$ vertices of value 2 and $\frac{n-1}{2}$ vertices of value 3 and $n+1$ vertices of value 1 and one vertex of value 4.

Therefore, $\sum_{v \in V} f(v) = \frac{n-1}{2} (2) + \frac{n-1}{2} (3) + n+1+4$

$$= n-1 + \frac{3n-3}{2} + n + 5$$

$$= 2n + \frac{3n-3}{2} + 4$$

$$= \frac{4n+3n-3+8}{2}$$

$$= \frac{7n+5}{2}$$

Since $n$ is even, $7n+5$ is even and $\frac{7n+5}{2}$ is even or odd.

If $\frac{7n+5}{2}$ is odd, then there is no balanced dominating set.

If $\frac{7n+5}{2}$ is even, then $\sum_{v \in D} f(v) = \frac{7n+5}{4}$.

Also $\frac{7n+5}{2}$ is even only when $n=2m+1$ where $m$ is even ($m=2,4,6,\ldots$).

That is, $n= 5,9,13,17,\ldots$.

We prove by induction on $n$.

Let $n=5$. 

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The result is true for $H_5$.

Assume that the result is true for $n-4$.

That is, $\gamma_{bd}(H_{n-4}) \leq n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{7(n-4)+5}{2}$

$= \frac{7n-28+5}{2}$

$= \frac{7n-23}{2}$

And $\sum_{v \in D} f(v) = \frac{7n-23}{4}$

We have to prove for $H_n$, that is, $\gamma_{bd}(H_n) \leq n$.

From $H_{n-4}$ we get $\frac{7n-23}{2} + 14 = \frac{7n-23+28}{2}$

$= \frac{7n+5}{2}$.

$H_n$ has 8 vertices of value 14 more than $H_{n-4}$ in which four vertices of value 1 and two vertices of value 2 and two vertices of value 3.

To cover these four leaves we have to take four vertices.

Therefore, $\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4$

$\leq n-4 + 4$

$\leq n$.

Sub case ii: No vertices of the second cycle labeled as 1.
In this case, Closed Helm graph $H_n$ has $2n+1$ vertices in which $n-1$ vertices of value 2 and $n-1$ vertices of value 1 and 1 vertex of value 1 and two vertices of value 4.

Therefore, $\sum_{v \in V} f(v) = 1+2(4) + (n-1)2 + (n-1)3$

$$= 1 + 8 + 2n - 2 + 3n - 3$$

$$= 5n + 4$$

Since $n$ is even, $5n+4$ is odd.

Therefore, there is no balanced dominating set.

Case ii: Label the centre vertex with value 2.

Sub case i: All the vertices of the second cycle labeled as 2.

In this case, Closed Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n-1}{2}$ vertices of value 3 and $\frac{n-1}{2}$ vertices of value 1 and $n+1$ vertices of value 2 and one vertex of value 4.

Therefore, $\sum_{v \in V} f(v) = \frac{n-1}{2} (3) + \frac{n-1}{2} (1) + n+1(2) + 4$

$$= \frac{3n-3}{2} + \frac{n-1}{2} + 2n + 6$$

$$= \frac{3n-3+n-1+4n+12}{2}$$
\[
\frac{8n+8}{2} = 4n + 4
\]

Since \(4n+4\) is even, \(\sum_{v \in D} f(v) = \frac{4n+2}{2} = 2n+2\)

We prove by induction on \(n\).

Let \(n=3\).

The result is true for \(H_3\).

Assume that the result is true for \(n-2\).

That is, \(\gamma_{bd}(H_{n-2}) \leq n-2\).

For \(H_{n-2}\), \(\sum_{v \in V} f(v) = 4(n-2) + 4\)

\[= 4n - 8 + 4\]

\[= 4n - 4.\]

We have to prove for \(H_n\) that is, \(\gamma_{bd}(H_n) \leq n\).

From \(H_{n-4}\) we get \(4n - 4 + 8 = 4n+2\).

\(H_n\) has 4 vertices of value 8 more than \(H_{n-4}\) in which two vertices of value 2 and one vertex of value 3 and one vertex of value 1.

To cover these 4 vertices we have to take two vertices.

Therefore, \(\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-2}) + 2\)

\[\leq n - 2 + 2\]
Sub case ii: No vertices of the second cycle labeled as 2.

In this case, Closed Helm graph \( H_n \) has \( 2n+1 \) vertices in which \( n-1 \) vertices of value 3 and \( n-1 \) vertices of value 1 and 1 vertices of value 2 and two vertices of value 4.

Therefore, \( \sum_{v \in V} f(v) = (n-1) \cdot 3 + (n-1) \cdot 1 + 1 \cdot 2 + 2 \cdot 4 \)

\[ = 3n-3 + n - 1 + 2 + 8 \]

\[ = 4n + 6 \]

Since \( 4n+6 \) is even, \( \sum_{v \in V} f(v) = \frac{4n+6}{2} = 2n+3 \)

We prove by induction on \( n \).

Let \( n=3 \).

The result is true for \( H_3 \).

Assume that the result is true for \( n-2 \).

That is, \( \gamma_{bd}(H_{n-2}) \leq n-2 \).

For \( H_{n-2} \), \( \sum_{v \in V} f(v) = 4(n-2) + 6 \)

\[ = 4n - 8 + 6 \]

\[ = 4n - 2. \]

We have to prove for \( H_n \) that is, \( \gamma_{bd}(H_n) \leq n \).
From $H_{n-2}$ we get $4n-2 + 8 = 4n+6$.

$H_n$ has 4 vertices of value 8 more than $H_{n-4}$ in which two vertices of value 1 and two vertices of value 3.

To cover these 4 vertices we have to take two vertices.

Therefore, $\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-2}) + 2$

$$\leq n-2 + 2$$

$$\leq n.$$

Case iii: Label the centre vertex with value 3.

Sub case i: All the vertices of the second cycle labeled as 3.

In this case, Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n-1}{2}$ vertices of value 2 and $\frac{n-1}{2}$ vertices of value 1 and $n+1$ vertices of value 3 and one vertex of value 4.

Therefore, $\sum_{v \in V} f(v) = \frac{n-1}{2} (2) + \frac{n-1}{2} (1) + n+1 (3) + 4$

$$= \frac{n-1}{2} + n+1 + 3n + 3+4$$

$$= 4n + \frac{n-1}{2} + 6$$

$$= \frac{8n+12+n-1}{2}$$

$$= \frac{9n+11}{2}$$

Since $n$ is odd, $9n+11$ is even and $\frac{9n+11}{2}$ is even or odd.
If $\frac{9n+11}{2}$ is odd, then there is no balanced dominating set.

If $\frac{9n+11}{2}$ is even, then $\sum_{v \in D} f(v) = \frac{9n+11}{4}$.

Also $\frac{9n+11}{2}$ is even only when $n=2m+1$ where $m$ is even ($m=2,4,6,\ldots$)

That is, $n= 5, 9, 13, 17, \ldots$

We prove by induction on $n$.

Let $n=5$.

The result is true for $H_5$.

Assume that the result is true for $n-4$.

That is, $\gamma_{bd}(H_{n-4}) \leq n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{9(n-4)+11}{2}$

$$= \frac{9n-36+11}{2}$$

$$= \frac{9n-25}{2}$$

And $\sum_{v \in D} f(v) = \frac{9n-25}{4}$

We have to prove for $H_n$ that is, $\gamma_{bd}(H_n) \leq n$.

From $H_{n-4}$ we get $\frac{9n-25}{2} + 18 = \frac{9n-25+36}{2}$

$$= \frac{9n+11}{2}.$$
\( H_n \) has 8 vertices of value 18 more than \( H_{n-4} \) in which four vertices of value 3, two vertices of value 1 and two vertices of value 2.

To cover these four leaves we have to take four vertices.

Therefore, \( \gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4 \)

\[ \leq n - 4 + 4 \]

\[ \leq n. \]

Sub case ii: No vertices of the second cycle labeled as 3.

In this case, Closed Helm graph \( H_n \) has 2n+1 vertices in which n-1 vertices of value 1 and n-1 vertices of value 2 and 1 vertices of value 3 and two vertices of value 4.

Therefore, \( \sum_{v \in V} f(v) = 1(3) + 2(4) + (n-1)1 + (n-1)2 \)

\[ = 3 + 8 + n - 1 + 2n - 2 \]

\[ = 3n + 8 \]

Since n is odd, 3n+8 is odd.

Therefore, there is no balanced dominating set.

Case ii: Label the centre vertex with value 4.

Sub case i: All the vertices of the second cycle labeled as 4.
In this case, Closed Helm graph $H_n$ has $2n+1$ vertices in which $\frac{n-1}{2}$ vertices of value 2 and $\frac{n-1}{2}$ vertices of value 1 and $n+1$ vertices of value 4 and one vertex of value 3.

Therefore, $\sum_{v \in V} f(v) = \frac{n-1}{2} (2) + \frac{n-1}{2} (1) + n+1(4) + 3$

$$= \frac{2n-2}{2} + \frac{n-1}{2} + 4n + 4 + 6$$

$$= \frac{10n+n-1+12}{2}$$

$$= \frac{11n+11}{2}$$

Since $n$ is odd, $11n+11$ is even and $\frac{11n+11}{2}$ is even or odd.

If $\frac{11n+11}{2}$ is odd, then there is no balanced dominating set.

If $\frac{11n+11}{2}$ is even, then $\sum_{v \in D} f(v) = \frac{11n+11}{4}$.

Also $\frac{11n+11}{2}$ is even only when $n=2m+1$ where $m$ is odd ($m=1,3,5,\ldots$)

That is, $n=3,7,11,15,\ldots$.

We prove by induction on $n$.

Let $n=3$.

The result is true for $H_3$.

Assume that the result is true for $n-4$. 
That is, $\gamma_{bd}(H_{n-4}) \leq n-4$.

For $H_{n-4}$, $\sum_{v \in V} f(v) = \frac{11(n-4) + 11}{2}$

$$= \frac{11n-44 + 11}{2}$$

$$= \frac{11n-33}{2}$$

And $\sum_{v \in D} f(v) = \frac{11n-33}{4}$

We have to prove for $H_n$ that is, $\gamma_{bd}(H_n) \leq n$.

From $H_{n-4}$ we get $\frac{11n-33}{2} + 22 = \frac{11n-33 + 44}{2}$

$$= \frac{11n+11}{2}.$$  

$H_n$ has 8 vertices of value 22 more than $H_{n-4}$ in which four vertices of value 4, two vertices of value 1 and two vertices of value 2.

To cover these four vertices we have to take four vertices.

Therefore, $\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-4}) + 4$

$$\leq n-4 + 4$$

$$\leq n.$$

Sub case ii: No vertices of the second cycle labeled as 4.

In this case, Closed Helm graph $H_n$ has $2n+1$ vertices in which $n-1$ vertices of value 2 and $n-1$ vertices of value 1 and 1 vertices of value 4 and two vertices of value 3.
Therefore, $\sum_{v \in V} f(v) = (n-1)(2) + (n-1)(1) + 1(4) + 2 \cdot 3$

$$= 2n-2 + n -1 + 4 + 6$$

$$= 3n+7$$

$3n + 7$ is even only when $n = 3, 5, 7, \ldots$

We prove by induction on $n$.

Let $n = 3$.

The result is true for $H_3$.

Assume that the result is true for $n-2$.

That is, $\gamma_{bd}(H_{n-2}) \leq n-2$.

For $H_{n-2}$, $\sum_{v \in V} f(v) = 3(n-2) + 7$

$$= 3n - 6 + 7$$

$$= 3n + 1.$$ 

We have to prove for $H_n$ that is, $\gamma_{bd}(H_n) \leq n$.

From $H_{n-2}$ we get $3n + 1 + 6 = 3n + 7$.

$H_n$ has 4 vertices of value 6 more than $H_{n-4}$ in which two vertices of value 1 and two vertices of value 2.

To cover these 4 vertices we have to take two vertices.

Therefore, $\gamma_{bd}(H_n) = \gamma_{bd}(H_{n-2}) + 2$
\[ \leq n - 2 + 2 \]

\[ \leq n. \]

**Example 4.2.2**

![Closed Helm graph \( H_{12} \)]

In this graph, \( f(V) = 50 \)

\[ D = \{ v_2, v_4, v_5, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{20}, v_{23} \} \] is a balanced dominating set.

\[ \sum_{v \in D} f(v) = 2 + 2 + 3 + 3 + 3 + 3 + 3 + 1 = 25 \]

\[ \gamma_{bd} = 10 \leq n. \]
Note 4.2.3:

The bound of the theorem 4.2.1 is sharp.

Example 4.2.4

![Diagram of a graph]

**Figure 47:** Closed Helm graph $H_4$

In this graph, $f(V) = 18$

$D = \{ v_1, v_2, v_4, v_7 \}$ is a balanced dominating set.

$\sum_{v \in D} f(v) = 3 + 3 + 2 + 1 = 9$

$\gamma_{bd} = 4 = n$
4.3. Petersen graph

The Petersen graph was introduced by Petersen as a counterexample. The Petersen graph is a 3-regular graph of order 10 and size 15.

The Petersen graph consists of two disjoint 5-cycles joined by a particular matching of cardinality 5. We denote two 5-cycles by $C: v_1, v_2, v_3, v_4, v_5, v_1$ and $C_1: u_1, u_2, u_3, u_4, u_5, u_1$. The length of the smallest cycle in this graph is 5.

**Theorem 4.3.1**

For Petersen graph $PG$, $\gamma_{bd}(PG) = 4$.

**Proof:**

The Petersen graph has two disjoint 5-cycles. The vertices of these two cycles can be labeled in three different types.

Type I: one of the cycle is labeled as $\{1,2,1,2,3\}$ and another cycle is labeled as $\{2,3,2,3,1\}$.

In this type, we have odd number of 1’s and odd number of 3’s and even number of 2’s.

Therefore $f(V)$ is even.
In this type $\gamma_{bd} = 4$ is shown in Figure 3.

![Graph](image)

**Figure 48:** Petersen graph

In this graph, $f(V) = 20$

$D = \{ v_1, u_2, u_4, v_5 \}$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 3+3+3+1 = 10$$

$\gamma_{bd} = 4$

Type II: one of the cycle is labeled as $\{ 1,3,1,3,2 \}$ and another cycle is labeled as $\{1,2,1,2,3\}$. 
In this type, we have odd number of 2’s and odd number of 3’s and even number of 1’s.

Therefore \( f(V) = \text{odd number of 2’s} + \text{odd number of 3’s} + \text{even number of 1’s} \)

\[ = \text{even} + \text{odd} + \text{even} \]

\[ = \text{odd} \]

Since \( f(V) \) is odd, there is no balanced dominating set.

Type III: one of the cycle is labeled as \( \{1,3,1,3,2\} \) and another cycle is labeled as \( \{2,3,2,3,1\} \).

In this type, we have odd number of 1’s and odd number of 2’s and even number of 3’s.

Therefore \( f(V) = \text{odd number of 1’s} + \text{odd number of 2’s} + \text{even number of 3’s} \)

\[ = \text{odd} + \text{even} + \text{even} \]

\[ = \text{odd} \]

Since \( f(V) \) is odd, there is no balanced dominating set.

Therefore, \( \gamma_{bd}(PG) = 4 \).
4.4 Union of Graphs

Lemma 4.4.1

Let $G$ be a graph obtained by attaching a pendant vertex to each of vertices of $C_{4n}$ then the number of vertices of value 1 is equal to the number of vertices of value 2.

That is, if $n_1$ is the number of vertices of value 1 and $n_2$ is the number of vertices of value 2 in the $\gamma_{bd}$ set of $G$ then $n_1 = n_2$. Also $n_1 = n_2 = 2n$.

Proof:

Let $G$ be a graph obtained by attaching a pendant vertex to each of vertices of $C_{4n}$.

The cycle $C_{4n}$ has $4n$ vertices.

Attaching a pendant vertex to each of vertices of $C_{4n}$, we get $8n$ vertices.

These $8n$ vertices divided into $4n$ vertices of value 1 and $4n$ vertices of value 2.

Therefore, $4n$ 1's + $4n$ 2's = $4n + 2(4n)$

= $4n + 8n$

= $12n$. 

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that is, \( f(V) = 12n \).

Hence \( \sum_{v \in D} f(v) = 6n \).

if \( n_1 \) is the number of vertices of value 1 and \( n_2 \) is the number of vertices of value 2 in the \( \gamma_{bd} \) set of G, then \( n_1 + 2n_2 = 6n \).

In this graph G, we have 4n pendant vertices in which 2n vertices of value 1 and 2n vertices of value 2.

To cover these 2n pendant vertices of value 2, we have to take either the pendant vertex of value 2 or vertex adjacent to that pendant vertex which is of value 1.

Therefore, if \( m_1 \) vertices of value 2 are from pendant vertices then we have to take \( 2n - m_1 \) vertices of value 1 other than pendant vertex.

Similarly, To cover these 2n pendant vertices of value 1, we have to take either the pendant vertex of value 1 or vertex adjacent to that pendant vertex which is of value 2.

Therefore, if \( m_2 \) vertices of value 1 are from pendant vertices then we have to take \( 2n - m_2 \) vertices of value 2 other than pendant vertex.

Hence \( n_1 = 2n - m_1 + m_2 \) and

\[
 n_2 = 2n - m_2 + m_1. 
\]
we have, \[ n_1 + 2n_2 = 6n \]

\[ 2n - m_1 + m_2 + 2(2n - m_2 + m_1) = 6n \]

\[ 2n - m_1 + m_2 + 4n - 2m_2 + 2m_1 = 6n \]

\[ 6n + m_1 - m_2 = 6n \]

\[ m_1 - m_2 = 0 \]

\[ m_1 = m_2. \]

since \( m_1 = m_2 \), we get \( n_1 = n_2 \) and \( n_1 = n_2 = 2n \).

**Theorem 4.4.2**

Let \( G \) be a graph obtained by attaching a pendant vertex to each of vertices of \( C_{4n} \). then \( \gamma_{bd} (G) = 4n \).

**Proof:**

Let \( G \) be a graph obtained by attaching a pendant vertex to each of vertices of \( C_{4n} \).

The cycle \( C_{4n} \) has \( 4n \) vertices.

Attaching a pendant vertex to each of vertices of \( C_{4n} \), we get \( 8n \) vertices.
These 8n vertices divided into 4n vertices of value 1 and 4n vertices of value 2.

Therefore, \( 4n \times 1's + 4n \times 2's = 4n + 2(4n) \)

\[ = 4n + 8n \]

\[ = 12n. \]

that is, \( f(V) = 12n. \)

Hence \( \sum_{v \in D} f(v) = 6n. \)

suppose \( n_1 + 2n_2 = 6n \) where \( n_1 \) is the number of vertices of value 1 and \( n_2 \) is the number of vertices of value 2.

then \( \gamma_{bd}(G) = n_1 + n_2. \)

we have to prove \( \gamma_{bd}(G) = n_1 + n_2 = 4n. \)

we prove this by induction on \( n. \)

Let \( n = 1. \)

we get \( n_1 + 2n_2 = 6 \)

since \( 2n_2 \) is even, \( n_1 \) must be even.

therefore, \( n_1 = 2 \) and \( n_2 = 2. \)
Hence $\gamma_{bd}(G) = n_1 + n_2 = 4 = 4n$.

Assume that the result is true for $n-1$.

Let $G'$ be the graph obtained by attaching a pendant vertex to each of vertices of $C_{4n-4}$.

then $G'$ has $8n-8$ vertices and $\sum_{v \in D} f(v) = 6n - 1 = 6n - 6$.

Let $m_1$ denote the number of vertices of value 1 and $m_2$ denote the number of vertices of value 2 of the graph $G'$.

then $\gamma_{bd}(G') = m_1 + m_2 = 4(n - 1) = 4n - 4$.

we have, $m_1 + 2m_2 = 6n - 6$

$m_1 + 2m_2 + 6 = 6n - 6 + 6$

$(m_1 + 2) + 2 (m_2 + 2) = 6n$ (by Lemma 4.4.1)

Therefore, $\gamma_{bd}(G) = m_1 + 2 + m_2 + 2$

$= m_1 + m_2 + 4$

$= 4n - 4 + 4$

$= 4n$. 

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Example 4.4.3:

Figure 49: C_{16} with pendant vertex

In this graph, \( f(V) = 48 \)

\[ \text{D}= \{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10},v_{11},v_{12},v_{13},v_{14},v_{15},v_{16}\} \] is a balanced dominating set.

\[ \sum_{v \in \text{D}} f(v)= 2+2+2+2+2+2+2+2+1+1+1+1+1+1+1+1= 24 \]

\( \gamma_{bd}=16 \)

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Lemma 4.4.4

Let G be a graph obtained by attaching a pendant edge to each vertices of Path $P_n$ then the number of vertices of value 1 is equal to the number of vertices of value 2 in the $\gamma_{bd}$ set of G.

that is, if $n_1$ is the number of vertices of value 1 and $n_2$ is the number of vertices of value 2 in the $\gamma_{bd}$ set of G then $n_1 = n_2$. Also $n_1 = n_2 = n/2$.

Proof:

Let G be a graph obtained by attaching a pendant edge to each vertices of $P_n$.

The Path $P_n$ has n vertices.

Attaching a pendant edge to each vertices of $P_n$, we get 2n vertices.

These 2n vertices divided into n vertices of value 1 and n vertices of value 2.

Therefore, $n \ 1's + n \ 2's = n + 2(n)$

$= 3n$.

that is, $f(V) = 3n$.

Hence $\sum_{v \in D} f(v) = 3n/2$. 

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if $n_1$ is the number of vertices of value 1 and $n_2$ is the number of vertices of value 2 in the $\gamma_{bd}$ set of $G$, then $n_1+2n_2 = \frac{3n}{2}$.

In this graph $G$, we have $n$ pendant vertices in which $n/2$ vertices of value 1 and $n/2$ vertices of value 2.

To cover these $n/2$ pendant vertices of value 2, we have to take either the pendant vertex of value 2 or vertex adjacent to that pendant vertex which is of value 1.

Therefore, if $m_1$ vertices of value 2 are from pendant vertices then we have to take $n/2-m_1$ vertices of value 1 other than pendant vertex.

Similarly, to cover these $n/2$ pendant vertices of value 1, we have to take either the pendant vertex of value 1 or vertex adjacent to that pendant vertex which is of value 2.

Therefore, if $m_2$ vertices of value 1 are from pendant vertices then we have to take $n/2-m_2$ vertices of value 2 other than pendant vertex.

Hence $n_1 = \frac{n}{2} - m_1 + m_2$ and

$$n_2 = \frac{n}{2} - m_2 + m_1.$$

we have, $n_1+2n_2 = \frac{3n}{2}$
\[
\frac{n}{2} - m_1 + m_2 + 2(\frac{n}{2} - m_2 + m_1) = \frac{3n}{2}
\]

\[
\frac{n}{2} - m_1 + m_2 + n - 2m_2 + 2m_1 = \frac{3n}{2}
\]

\[
\frac{3n}{2} + m_1 - m_2 = \frac{3n}{2}
\]

\[
m_1 - m_2 = 0
\]

\[
m_1 = m_2.
\]

since \( m_1 = m_2 \), we get \( n_1 = n_2 \) and \( n_1 = n_2 = n/2 \).

**Theorem 4.4.5**

Let \( G \) be a graph obtained by attaching a pendant edge to each vertices of Path \( P_n \). then \( \gamma_{bd}(G) = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \)

**Proof:**

Let \( G \) be a graph obtained by attaching a pendant edge to each vertices of Path \( P_n \).

The Path \( P_n \) has \( n \) vertices.

Attaching a pendant edge to each vertices of \( P_n \), we get \( 2n \) vertices.
These 2n vertices divided into n vertices of value 1 and n vertices of value 2.

Therefore, \( n \) 1's + n 2's = \( n + 2n \) = 3n.

that is, \( f(V) = 3n \).

\[ \text{Hence } \sum_{v \in D} f(v) = \frac{3n}{2}. \]

If \( n \) is odd, \( \frac{3n}{2} \) is odd.

Then \( \gamma_{bd}(G) = 0 \).

If \( n \) is even, suppose \( n_1 + 2n_2 = \frac{3n}{2} \) where \( n_1 \) is the number of vertices of value 1 and \( n_2 \) is the number of vertices of value 2.

then \( \gamma_{bd}(G) = n_1 + n_2 \).

we have to prove \( \gamma_{bd}(G) = n_1 + n_2 = n \).

we prove this by induction on \( n \).

Let \( n = 2 \).

we get \( n_1 + 2n_2 = 3 \)

therefore, \( n_1 = 1 \) and \( n_2 = 1 \).

Hence \( \gamma_{bd}(G) = n_1 + n_2 = 2 = n \).
Assume that the result is true for \( n-2 \).

Let \( G' \) be the graph obtained by attaching a pendant edge to each vertices of \( P_{n-2} \).

Then \( G' \) has \( n-4 \) vertices and \( \sum_{v \in D} f(v) = \frac{3(n-2)}{2} = \frac{3n-6}{2} \)

\[ = \frac{3n}{2} - 3 \]

Let \( m_1 \) denote the number of vertices of value 1 and \( m_2 \) denote the number of vertices of value 2 of the graph \( G' \).

Then \( \gamma_{bd}(G') = m_1 + m_2 = n-2 \).

We have, \( m_1 + 2m_2 = \frac{3n}{2} - 3 \)

\[ m_1 + 2m_2 + 3 = \frac{3n}{2} - 3 + 3 \]

\[ (m_1 + 1) + 2(m_2 + 1) = \frac{3n}{2} \text{ (by Lemma 4.4.4)} \]

Therefore, \( \gamma_{bd}(G) = m_1 + 1 + m_2 + 1 \)

\[ = m_1 + m_2 + 2 \]

\[ = n-2+2 \]

\[ = n. \]
Example 4.4.6:

Figure 50: $P_{10}$ with pendant edge at each vertex

In this graph $P_{10}$ (n is even) with pendant edge at each vertex, $f(V) = 30$

$$D = \{ v_2, v_5, v_8, v_9, v_{11}, v_{14}, v_{15}, v_{17}, v_{18}, v_{20} \}$$ is a balanced dominating set.

$$\sum_{v \in D} f(v) = 2+2+2+2+1+1+1+1+1+1 = 15$$

$\gamma_{bd}=10$

Figure 51: $P_7$ with pendant edge at each vertex

In this graph $P_7$ (n is odd) with pendant edge at each vertex, $f(V) = 21$, $\gamma_{bd}=0$