CHAPTER VI
Chapter – VI

To Obtain the Apparent Viscosity of Various Non-Newtonian Fluids and its Variation with Respect to other Rheological Parameters

Introduction:

Steady flow of non-Newtonian viscous fluids is very interesting from the standpoint of blood flow in narrow vessels. Viscous non-Newtonian fluids where the shear stress is non-linearly proportional to the $n^{th}$ power of the velocity gradient may be considered to be the appropriate model for discussion under consideration. Bugliarello and Sevilla [1], Charm et al. [2], Chaturani and Samy [3 and 4], Cho and Kensey [5], Kiani and Hudetz [6], Oka [7], Pries et al. [8] and Reinke et al. [9] have considered non-Newtonian visco-elastic fluid models of blood flow in narrow vessels. Van Dam et al. [12] studied the linear visco-elastic behavior of aortic stenosed thrombus. Boyd et al. [11] analysed the Casson and Carreau – Yasuda flow model of blood for steady and oscillatory flows. They have shown that for steady flows, the difference between corresponding Newtonian and non-Newtonian profiles increases for decreasing Reynolds number. The parameters used are in agreement with the parameter taken in my results, Tables – 1, 2, 3 and 4.

Scott – Blair [10] reported that the blood obeys the Casson equation only in the limited range except at very high and
very low shear rates and that there is no difference between the Casson plots and Herschel – Bulkley plots of experimental data over the range where the Casson plot is valid. It is also suggested that the assumptions included in the Casson equation are unsuitable for the Cow’s blood and that the Herschel – Bulkley equation represents fairly closely what is occurring in the blood. Since the Herschel – Bulkley equation contains one more parameter than the Casson equation does, so it would be expected that more detailed information about blood properties can be obtained by the use of Herschel – Bulkley equation. Further more, the Herschel – Bulkley equation is reduced to the mathematical models which describes the behaviour of Newtonian fluid, Bingham fluid and Power law fluid by taking appropriate values of the parameters. Here, in this chapter we use the Herschel – Bulkley equation for the analysis of steady blood flow and investigate in detail the effects of yield stress, the parameter which expresses the degree of shear thinning, apparent viscosity and its variation with respect to other rheological parameters. Flow rates and apparent viscosity for Casson, Bingham Plastic, Power law and Newtonian fluids have been determined as the particular cases. Discussion has been supplemented with the help of Tables and Graphs.

Assumptions:

Before we obtain the results, we make following assumptions in order to simplify the problem.

(1) The fluid is incompressible, and the vessel is long rigid circular tube whose radius is constant.
(2) Motion of the fluid is fully-developed and has an axial symmetry.

(3) Motion is steady.

(4) Flowing fluid in the vessel obey the Herschel–Bulkley equation. The shear stress $\tau$ and the shear rate $\dot{\gamma}$ relationship is given by

$$
\dot{\gamma} = \frac{1}{k} (\tau - f_H)^n \quad \text{if} \quad \tau \geq f_H
$$

$$
= 0 \quad \text{if} \quad |\tau| \leq f_H
$$

(1)

where $k$ and $n$ represents non-Newtonian effects, $f_H$ is the yield stress.

The equation is reduced to that for Bingham fluid when $n = 1$, to that for a power law fluid when $f_H = 0$, to that for Newtonian fluid when $f_H = 0$, $n = 1$, to that for Casson fluid when $n = 2$, $\tau$ and $f_H$ are taken with under root sign. We shall take cylindrical co-ordinate system whose origin is located on the axis of the vessel. Momentum and continuity equation for fully developed laminar viscous incompressible fluid are

$$
0 = -\frac{\partial p}{\partial r}
$$

(2)
\[
0 = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{rz} \right)
\]  \hspace{1cm} (3)

and
\[
\frac{\partial u}{\partial z} = 0
\]  \hspace{1cm} (4)

where \( p \) is pressure, \( \tau_{rz} \) is the shear stress normal to \( r \) along \( z \)-direction, \( u \) is the velocity field along \( z \)-direction and which is the function of \( r \). Body force is neglected in equations (1) and (2).

From equation (2) we have
\[
r \tau_{rz} = \frac{1}{2} \frac{\partial p}{\partial z} r^2 + C
\]

when the stress is assumed to be finite at the axis of the vessel, then \( C = 0 \), and so the above equation becomes
\[
\tau_{rz} = \frac{r}{2} \frac{\partial p}{\partial z}
\]  \hspace{1cm} (5)

If \( \tau_w \) is the shear stress at the wall, that is \( r = R; \tau_{rz} = \tau_w \)

Then
\[
\tau_{rz} = \frac{\tau_w}{R}, \quad r = \tau
\]  \hspace{1cm} (6)

We now assume that there exists a constitutive relation relating the stress with the strain rate in the form
\[
\dot{\gamma}(\tau) = \left| \frac{du}{dr} \right| = f(\tau)
\]  \hspace{1cm} (7)
where $\dot{\gamma}$ is the strain rate, $u$ is the velocity and $f(\tau)$ is a function which describes the dependence of strain rate on the shear stress $\tau$.

Integrate (6) and get

$$u(r) = \int f(\tau) \cdot d\tau$$

or

$$u(r) = \frac{R}{\tau_n} \int^r f(\tau) \cdot d\tau$$

(8)

**Velocity Field**

From equation (7), we have

$$-\frac{du}{dr} = f(\tau)$$

$$u = -\int f(\tau) \cdot d\tau + C$$

For Herschel – Bulkley fluids, axial velocity $u_H$ is given by

$$u_H = -\int \frac{1}{k} (\tau - f_H)^n \cdot d\tau + C$$

$$= -\frac{1}{k} \int \left( \frac{\tau_w}{R} \cdot r - f_H \right)^n \cdot d\tau + C \text{ from (6)}$$

$$= -\frac{1}{k(n+1)} \left( \frac{\tau_w}{R} \cdot r - f_H \right)^{n+1} \cdot \frac{R}{\tau_w} + C,$$

arbitrary constant $C$ is obtained from the boundary condition, $w = 0$, when $r = R$. 

119
So,

$$u_B = \frac{R \tau_w}{2k} \left[ 1 - \left( \frac{r}{R} \right)^{n+1} \right]$$

(9)

For Bingham fluids, $n = 1$, the axial velocity $u_B$ is given by

$$u_B = \frac{R \tau_w}{2k} \left[ 1 - \left( \frac{r}{R} \right)^2 - 2\frac{f_H}{\tau_w} \left( 1 - \frac{r}{R} \right) \right]$$

(10)

For Power law fluids, axial velocity $u_p$ is

$$u_p = \frac{R \tau_w}{2k} \left[ 1 - \left( \frac{r}{R} \right)^{n+1} \right], \quad n = 1, 2, 3,$$

(11)

For Newtonian fluids, $n = 1$, $f_H = 0$ and velocity $u_N$ is

$$u_N = \frac{R \tau_w}{2k} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] = \frac{R^2}{4} \frac{dp}{dz} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]$$

(12)

Casson fluids, whose stress–strain relation is given by

$$\dot{\gamma} = \frac{1}{k} \left[ \tau^{1/2} - f_H^{1/2} \right] = f(\tau) ,$$

axial velocity is

$$u_c = \frac{R \tau_w}{2k} \left[ 1 - \left( \frac{r}{R} \right)^2 + 2\frac{f_H}{\tau_w} \left( 1 - \frac{r}{R} \right) - \frac{8}{3} \left( \frac{f_H}{\tau_w} \right)^{1/2} \left[ 1 - \left( \frac{r}{R} \right)^{3/2} \right] \right]$$

(13)
Flow – Rate and Apparent Viscosity –

Single phase flow

The volumetric flow rate \( Q \) across a cross-section of radius \( R \) is given by

\[
Q = \int_0^R 2 \pi r u \cdot dr
\]

or

\[
Q = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau^2 f(\tau) \, d\tau
\]  

(14)

For Herschel – Bulkley fluid given by equation (1), the flow rate \( Q_H \) is calculated as

\[
Q_H = \frac{\pi R^3}{\tau_w^3} \int_0^{\tau_w} \tau^2 \cdot \frac{1}{k} \left( \tau - f_H \right)^n \cdot d\tau
\]

\[
= \pi R^3 \left[ \frac{\tau_w^2}{k} \cdot \left( \frac{\tau_w - f_H}{\tau_w} \right)^{n+1} \right]
- \frac{2 \tau_w \left( \tau_w - f_H \right)^{n+2}}{n+1} \left( n + 1 \right) \left( n + 2 \right)
+ \frac{2 \left( \tau_w - f_H \right)^{n+3}}{n+1} \left( n + 1 \right) \left( n + 2 \right) \left( n + 3 \right)
\]

or

\[
Q_H = \frac{\pi R^3}{k} \frac{\tau_w^n}{n+1} \left( 1 - \frac{f_H}{\tau_w} \right)^{n+1}
\]

\[
\left[ \frac{1}{n+3} + \frac{2}{(n+2)(n+3)} \left( \frac{f_H}{\tau_w} \right) + \frac{2}{(n+1)(n+2)(n+3)} \left( \frac{f_H}{\tau_w} \right)^2 \right]
\]  

(15)
The apparent viscosity $\eta_{ap}$ can be obtained as

$$\frac{1}{\eta_{ap}} = \frac{Q}{\pi R^3 \tau_w}$$

or

$$\frac{1}{\eta_{ap}} = \phi(a_H) = \frac{\tau_w^{n-1}}{k(n+1)(n+2)(n+3)}$$

$$\left[ (n+1)(n+2) + 2(n+1) \left( \frac{f_H}{\tau_w} \right) + 2 \left( \frac{f_H}{\tau_w} \right)^2 \right]$$

(16)

where $\phi(a_H)$ is the apparent fluidity.

**Bingham fluid**

Constitutive relation is obtained by putting $n = 1$ in equation (1). Thus the stress – strain relation becomes

$$\tau = f_H + k \dot{\gamma}$$

(17)

And so the flow rate $Q_B$ and apparent viscosity $\eta_{ap}$ are obtained as

$$Q_B = \frac{\pi R^3}{4k} \tau_w \left[ 1 - \frac{4}{3} \left( \frac{f_H}{\tau_w} \right) + \frac{1}{3} \left( \frac{f_H}{\tau_w} \right)^4 \right]$$

(18)

and

$$\frac{1}{\eta_{ap}} = \phi(a_H) = \frac{1}{4k} \left[ 1 - \frac{4}{3} \left( \frac{f_H}{\tau_w} \right) + \frac{1}{3} \left( \frac{f_H}{\tau_w} \right)^4 \right]$$

(19)
Casson fluids –

For Casson fluids obeying stress – strain relation of the form

\[ \tau^{1/2} = f_{H}^{1/2} + (k \dot{\gamma})^{1/2} \]

we calculate flow rate \( Q_{c} \) and apparent viscosity \( \eta_{a_{c}} \) in the forms

\[
Q_{c} = \frac{\pi R^{3} \tau_{w}}{4k} \left[ 1 - \frac{16}{7} \left( \frac{f_{H}}{\tau_{w}} \right)^{1/2} + \frac{4}{3} \left( \frac{f_{H}}{\tau_{w}} \right) - \frac{1}{21} \left( \frac{f_{H}}{\tau_{w}} \right)^{4} \right]
\]  

(20)

and

\[
\frac{1}{\eta_{a_{c}}} = \phi(a_{c}) = \frac{1}{4k} \left[ 1 - \frac{16}{7} \left( \frac{f_{H}}{\tau_{w}} \right)^{1/2} + \frac{4}{3} \left( \frac{f_{H}}{\tau_{w}} \right) - \frac{1}{21} \left( \frac{f_{H}}{\tau_{w}} \right)^{4} \right]
\]

(21)

Power law fluids –

For Power law fluids, the stress-strain relation can be obtained by putting \( f_{H} = 0 \) in equation (1), that is

\[ \dot{\gamma} = \frac{1}{k} \tau^{n} = f(\tau) \]

The volume flow rate \( Q_{p} \) and apparent viscosity \( \eta_{a_{p}} \) are determined as

\[
Q_{p} = \frac{\pi R^{3} \tau_{w}^{n}}{k (n + 3)}
\]  

(22)

and

\[
\frac{1}{\eta_{a_{p}}} = \phi(a_{p}) = \frac{1}{k (n + 3)}
\]

(23)
Newtonian fluids –

For Newtonian fluids, the stress-strain relation is obtained by putting \( f_{ii} = 0, \ n = 1 \) in equation (1), that is

\[ \tau = k \dot{\gamma} \]

Thus, the volume flow rate \( Q_N \) and apparent viscosity \( \eta_{an} \) are determined as

\[
Q_N = \frac{\pi R^4 \tau_w}{4k} = \frac{\pi R^4}{8k} \frac{dp}{dz} \tag{24}
\]

and

\[ \eta_{an} = k \tag{25} \]

Equation (22) is a well known Poiseuille formula for determining the viscosity of water in a fine glass tube.

Discussions:

For Herschel – Bulkley, Bingham, Power law and Casson fluids, the variation of apparent viscosity \( \eta_{ai} \) (\( i = H, B, P, C \)) with respect to other rheological parameters \( K \) and \( n \) are shown in the Tables : (1 – 4) and Graphs : (1 – 4). From the graphs of Herschel – Bulkley and Power law fluids it is observed that apparent viscosity increases rapidly with \( K \) at large \( n \) (that is when the non-Newtonian behaviour of the fluid is more pronounced).
Validation of Result:

We have carried out the analysis for non-Newtonian fluid given by different models (Herschel – Bulkley, Casson and Power law fluids). The values of parameters given in Tables 1 – 4 can be compared with the values considered in Graphs – 7 of Boyd et al. [11] paper wherein Newtonian and non-Newtonian velocities, shear rate and viscosities profiles comparative study is given. The predicted non-Newtonian flows are smaller than corresponding Newtonian counterpart showing the validity of Graphs 1 – 4.

Table – 1

Variation of $\eta_{\text{hu}}$ for different values of $K$ and $n$

<table>
<thead>
<tr>
<th></th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.00345</td>
<td>0.01267</td>
<td>0.0259</td>
</tr>
<tr>
<td>0.03</td>
<td>0.004672</td>
<td>0.01901</td>
<td>0.03890</td>
</tr>
<tr>
<td>0.04</td>
<td>0.006320</td>
<td>0.02595</td>
<td>0.05180</td>
</tr>
<tr>
<td>0.05</td>
<td>0.007788</td>
<td>0.03165</td>
<td>0.06480</td>
</tr>
</tbody>
</table>
Table – 2

Variation of $\eta_{na}$ for different values of $K$ and $f_{H}/\tau_{w}$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$f_{H}/\tau_{w} = 0.1$</th>
<th>$f_{H}/\tau_{w} = 0.2$</th>
<th>$f_{H}/\tau_{w} = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.0230</td>
<td>0.0272</td>
<td>0.0331</td>
</tr>
<tr>
<td>0.03</td>
<td>0.0346</td>
<td>0.0408</td>
<td>0.0497</td>
</tr>
<tr>
<td>0.04</td>
<td>0.0461</td>
<td>0.0545</td>
<td>0.0663</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0576</td>
<td>0.0681</td>
<td>0.0829</td>
</tr>
</tbody>
</table>

Table – 3

Variation of $\eta_{n}$ with $K$ and $n$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$n = 1/2$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.07</td>
<td>0.08</td>
<td>0.10</td>
</tr>
<tr>
<td>0.03</td>
<td>0.10</td>
<td>0.12</td>
<td>0.15</td>
</tr>
<tr>
<td>0.04</td>
<td>0.14</td>
<td>0.16</td>
<td>0.20</td>
</tr>
<tr>
<td>0.06</td>
<td>0.21</td>
<td>0.24</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table – 4

Variation of $\eta_{nc}$ with $K$ and $f_{H}/\tau_{w}$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$f_{H}/\tau_{w} = 0.1$</th>
<th>$f_{H}/\tau_{w} = 0.2$</th>
<th>$f_{H}/\tau_{w} = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.053</td>
<td>0.077</td>
<td>0.133</td>
</tr>
<tr>
<td>0.03</td>
<td>0.080</td>
<td>0.115</td>
<td>0.200</td>
</tr>
<tr>
<td>0.04</td>
<td>0.107</td>
<td>0.154</td>
<td>0.267</td>
</tr>
<tr>
<td>0.06</td>
<td>0.160</td>
<td>0.231</td>
<td>0.401</td>
</tr>
</tbody>
</table>
Graph - 1: Variation of $\eta_{ab}$ for different values of $K$ and $n$
Graph – 2: Variation of $\eta_{\nu}$ for different values of $K$ and $f_H/\tau_w$.
Graph - 3: Variation of $\eta_a$ with $K$ and $n$. 

Power law Fluid
Graph - 4: Variation of $\eta_\infty$ with K and $f_H/\tau_w$