Chapter 5
Chapter 5

Mathematical modeling and analysis for control of infectious diseases with time delay

5.1 Introduction

To study the spread of infectious diseases and its control has been the stated goal of Academicians. Various SIS and SIRS mathematical models have been proposed and analyzed for the spread of infectious diseases [6, 9]. In these models it is assumed that disease spreads through the direct contact between susceptibles and infectives. In these studies it is shown that if the basic reproduction number $R_0 < 1$, then the disease dies out and if $R_0 > 1$, then disease persists. To prevent the epidemic some vaccination mathematical models have also been proposed and analyzed [2, 35, 57, 75, 99, 117]. These studies show that vaccination of susceptibles help in preventing the epidemics. The critical vaccination rate has been derived above which the disease may be eradicated.
However, there are some diseases like typhoid fever and other enteric diseases, which spread in human population by direct contact between susceptibles and infectives and through carriers (flies, ticks, mites, etc) [1, 16, 22, 25, 34, 44, 45, 46, 62, 99, 118, 119]. The developing countries are most affected by such diseases due to lack of education, wide occurrence of carriers. Hethcote [62] studied a non-linear mathematical model by assuming that the carrier population is constant. But, in general the carrier population is not constant and it grows in the environment depending on the natural conditions of the environment as well as on various human related activities. This aspect has been considered in the models proposed by Singh et. al [118, 119] and Ghosh et. al [44, 45, 46]. In this model it is also assumed that the carrier population follow the logistic growth in its environment. Ghosh et. al [45] studied the spread of carrier dependent infectious diseases with environmental effects using variable carrier population. The carriers present in the environment enhances the chance of carrying more bacteria from environment to the susceptibles in the population leading to fast spread of carrier dependent infectious diseases.

The outbreak of infectious diseases cause mortality of millions of people as well as expenditure of enormous amount of money in health care and disease control. It is, therefore, essential that adequate attention must be paid to stop spreading of such diseases by taking control measures. In particular, Shulgin et. al [117] studied a simple SIR epidemic model with pulse vaccination and showed that pulse
vaccination leads to epidemics eradication if certain conditions regarding the magnitude of vaccination proportion and on the period of pulses are satisfied. Kribs-Zaleta and Velasco-Hernandez [75] presented a simple two dimensional SIS model with vaccination exhibiting backward bifurcation. Farrington [35] analyzed the impact of vaccination program on the transmission potential of the infection in large populations and derived relation between vaccine efficacy against transmission, vaccine coverage and reproduction numbers. Gumel and Moghadas [57] proposed a model for the dynamics of an infectious disease in the presence of a preventive vaccine considering non-linear incidence rate and found the optimal vaccine coverage threshold needed for disease control and eradication. Naresh et.al [99] proposed and analyzed a mathematical model for the control of spread of carrier dependent infectious diseases by incorporating vaccination of susceptibles. In this model the control of carrier population through chemical disinfectant has not been considered, which may be a more effective tool to control the spread of carrier dependent infectious diseases.

Therefore, in this chapter, we consider that the chemical disinfectant is used by the government to control the growth rate of carrier population. The measured data for the density of carrier population is usually few days old, thus the rate of discharge of this disinfectant is assumed to be proportional to the density of this few days old carrier population, which leads for incorporating delay in the discharge of chemical disinfectant. The natural depletion rate of chemical disinf-
fectant is assumed to be proportional to its concentration and its depletion rate due to the uptake by carriers is assumed to be proportional to both the density of carriers as well as the concentration of chemical disinfectant. The decay rate of carrier population is also assumed to be proportional to the density of carriers as well as the concentration of chemical disinfectant.

5.2 Mathematical Model

In the modeling process it is assumed that infection transmit via direct contact between susceptibles and infectives as well as via carriers present in the environment (indirect contact). Further, the growth of carriers is assumed to be logistic. It is further assumed that the carriers are controlled by chemical disinfectant, whose growth rate is assumed to be proportional to the density of carriers, $\tau$ time before, as the data available to the policy makers may not be current.

Let $X(t), Y(t)$ and $N(t)$ be the susceptible, infective and total human population at any time $t$ respectively. $C(t), C_h(t)$ be the density of carrier population and concentration of chemical disinfectant used for controlling carriers at time $t$. It is assumed that the rate of immigration of susceptibles is $A$, a constant. In the modeling process, we have assumed simple mass action interaction between susceptibles and infectives. It is also assumed that the growth rate of chemical disinfectant is proportional to $C(t - \tau)$ and its natural depletion rate is proportional to
its concentration. The depletion rate of chemical disinfectant due to the uptake by carriers is assumed to be proportional to both the density of carriers as well as the concentration of chemical disinfectant. The decay rate of carrier population is also assumed to be proportional to the density of carriers as well as the concentration of chemical disinfectant.

In view of the above considerations, the model dynamics is governed by the following system of nonlinear ordinary differential equations:

\[
\begin{align*}
\frac{dX(t)}{dt} &= A - \beta X(t)Y(t) - \lambda X(t)C(t) - dX(t) + \nu Y(t), \\
\frac{dY(t)}{dt} &= \beta X(t)Y(t) + \lambda X(t)C(t) - (\nu + \alpha + d)Y(t), \\
\frac{dC(t)}{dt} &= rC(t) \left(1 - \frac{C(t)}{K}\right) - \theta_2 C(t)Ch(t), \\
\frac{dCh(t)}{dt} &= \theta C(t - \tau) - \theta_0 Ch(t) - \theta_1 C(t)Ch(t),
\end{align*}
\] (5.2.1)

where \(X(t) + Y(t) = N(t)\) represents the total human population at any time \(t\) and the initial conditions for model system (5.2.1) are as follows:

\[X(0) = X_0 > 0, \; Y(0) = Y_0 \geq 0, \; C(\theta) = C_0 \geq 0 \text{ for } \theta \in [-\tau, 0], \; Ch(0) = Ch_0 \geq 0\]

and \(X(0) + Y(0) = N(0) = N_0 > 0\).

Using the fact that \(X(t) + Y(t) = N(t)\), the above system reduces to the following system:

\[
\frac{dY(t)}{dt} = \beta(N(t) - Y(t))Y(t) + \lambda(N(t) - Y(t))C(t) - (\nu + \alpha + d)Y(t),
\]
\[
\frac{dN(t)}{dt} = A - dN(t) - \alpha Y(t),
\]
\[
\frac{dC(t)}{dt} = rC(t) \left(1 - \frac{C(t)}{K}\right) - \theta_2 C(t) C_h(t),
\]
\[
\frac{dC_h(t)}{dt} = \theta C(t - \tau) - \theta_0 C_h(t) - \theta_1 C(t) C_h(t).
\]

In the above model (5.2.1), the constants \(\beta\) and \(\lambda\) represent the contact rate between susceptibles and infectives and the growth rate of infectives due to the presence of carriers in the environment respectively. The constants \(d\), \(\alpha\) and \(\nu\) represents the natural death rate, diseased induced death rate and recovery rate of human population. \(r\) represents the intrinsic growth rate of carrier population whereas \(K\) is the carrying capacity of carrier population. The constant \(\theta_2\) represent decay rate due to chemical disinfectant respectively. The constant \(\theta\) represents the growth rate of chemical disinfectant, \(\theta_0\) and \(\theta_1\) represent the natural decay rate and decay rate due to uptake of carriers of this disinfectant respectively.

Now, we analyze the model system (5.2.2) using stability theory of differential equations in detail. For this, we prove the following lemma, which gives the region of attraction \([40, 118]\) and shows the boundedness of the solutions of model system (5.2.2).

**Lemma 5.2.1.** The set:

\[
\Omega = \left\{ (Y, N, C, C_h) : 0 \leq Y \leq N \leq \frac{A}{d}, 0 \leq C \leq K, 0 \leq C_h \leq \frac{\theta}{\theta_0} K \right\},
\]

is a region of attraction and it attracts all solutions initiating in the interior of the
Proof: Consider the second equation of model (5.2.2), we get

\[
\frac{dN(t)}{dt} \leq A - dN(t)
\]  

(5.2.4)

Now, using comparison theorem [59], we get

\[
\limsup_{t \to +\infty} N(t) \leq \frac{A}{d}
\]

By using the fact that \(X(t) + Y(t) = N(t)\) and \(X(t) \geq 0\), we have

\[
Y(t) \leq N(t) \leq \frac{A}{d}
\]

For the third equation of model (5.2.2), we get

\[
\frac{dC(t)}{dt} \leq rC(t) \left(1 - \frac{C(t)}{K}\right)
\]  

(5.2.5)

Again by making use of comparison theorem [59], we have

\[
\limsup_{t \to +\infty} C(t) \leq K
\]

Using the last equation of model (5.2.2), we have

\[
\frac{dC_h(t)}{dt} \leq \theta K - \theta_0 C_h(t)
\]  

(5.2.6)
Using comparison theorem, we get

\[ \limsup_{t \to +\infty} C_h(t) \leq \frac{\theta}{\theta_0} K \]

Hence lemma is proved.

### 5.3 Equilibrium Analysis

The model (5.2.2) has three non-negative equilibria, which are as follows:

(i) Disease free equilibrium \( E_0 \left( 0, \frac{A}{d}, 0, 0 \right) \) exists without any condition.

(ii) Carrier free equilibrium \( E_1 \left( \frac{\beta A - d(\nu + \alpha + d)}{\beta(\alpha + d)}, \frac{\beta A + \alpha(\nu + \alpha + d)}{\beta(\alpha + d)}, 0, 0 \right) \) exists, provided:

\[ R_0 = \frac{\beta A}{d(\nu + \alpha + d)} > 1 \quad (5.3.1) \]

It should be noted here that \( R_0 \) is known as basic reproduction number.

(iii) Endemic equilibrium \( E_2(Y^*, N^*, C^*, C_h^*) \) exists without any condition.

The existence of \( E_0 \) and \( E_1 \) is obvious, hence omitted. In the following, we show the existence of interior equilibrium \( E_2 \) by solving the following set of algebraic equations:

\[ \beta(N - Y)Y + \lambda(N - Y)C - (\nu + \alpha + d)Y = 0, \quad (5.3.2) \]

\[ A - dN - \alpha Y = 0, \quad (5.3.3) \]
From equation (5.3.2) and (5.3.3), we get

\[ r \left(1 - \frac{C}{K}\right) - \theta_2 C_h = 0, \quad (5.3.4) \]

\[ \theta C - \theta_0 C_h - \theta_1 CC_h = 0, \quad (5.3.5) \]

Again using equation (5.3.3) and equation (5.3.5) in equation (5.3.4), we get an equation in \( C \) as follows:

\[ (A - (\alpha + d)Y)(\beta Y + \lambda C) - d(\nu + \alpha + d)Y = 0. \quad (5.3.6) \]

This can be written as,

\[ r \left(1 - \frac{C}{K}\right) - \frac{\theta \theta_2 C}{\theta_0 + \theta_1 C} = 0 \quad (5.3.7) \]

Now it is apparent that a unique positive value of \( C \) (say \( C^* \)) can be obtained from equation (5.3.8). Finally using this values of \( C^* \) in equations (5.3.6), (5.3.3) and (5.3.5), we get the positive values of \( Y^* \), \( N^* \) and \( C_h^* \) respectively. Thus the equilibrium \( E_2(Y^*, N^*, C^*, C_h^*) \) exists without any condition.

### 5.4 Stability Analysis

In this section the local stability behavior of equilibria \( E_0 \), \( E_1 \) and \( E_2 \) is studied with and without time delay. We also derive the conditions for the occurrence
5.4.1 Local stability analysis without delay (i.e. $\tau = 0$)

The local stability behavior of equilibria $E_0$, $E_1$ and $E_2$, can be studied using the eigenvalues of the corresponding variational matrices obtained for model system (5.2.2).

The general variational matrix obtained for model system (5.2.2) is follows:

$$
M = \begin{pmatrix}
-a_1 & \beta Y + \lambda C & \lambda (N - Y) & 0 \\
-\alpha & -d & 0 & 0 \\
0 & 0 & -a_2 & -\theta_2 C \\
0 & 0 & \theta - \theta_1 C_h & -a_3
\end{pmatrix},
$$

where $a_1 = (\beta Y + \lambda C - \beta(N - Y) + (\nu + \alpha + d)$, $a_2 = -r \left(1 - \frac{2C}{K}\right) + \theta_2 C_h$ and $a_3 = \theta_0 + \theta_1 C$.

Let $M_i$ be the variational matrix $M$ evaluated at the equilibrium $E_i(i = 0, 1, 2)$.

**Local stability of equilibria** $E_0 \left(0, \frac{A}{d}, 0, 0\right)$:

The variational matrix $M_0$ is given as follows:
It is apparent from the above matrix $M_0$ that its eigenvalues are $\frac{\beta A}{d} - (\nu + \alpha + d)$, $-d$, $r$ and $-\theta_0$. Therefore, two eigenvalues of $M_0$ are negative and one eigenvalue is $r$, which is always positive whereas one eigenvalue is positive or negative according to $R_0 > 1$ or $R_0 < 1$. Hence, we can say that the equilibrium $E_0$ always has stable manifold locally in $N - C_h$ - plane and unstable manifold locally in $C$-direction.

It has stable or unstable manifold locally in $Y$ - direction depending on whether $R_0 < 1$ or $R_0 > 1$. Here, it is clear that the existence of equilibrium $E_1$ leads to the instability of $E_0$ in $Y$ - direction, as $E_1$ exists for $R_0 > 1$ (see condition (5.3.1)).

**Local stability of equilibria** $E_1 \left( \frac{\beta A - d(\nu + \alpha + d)}{\beta(\alpha + d)}, \frac{\beta A + \alpha(\nu + \alpha + d)}{\beta(\alpha + d)}, 0, 0 \right)$:

The variational matrix $M_1$ can be written as:

$$M_1 = \begin{pmatrix}
-\beta Y_1^* & \beta Y_1^* & \lambda(N_1^* - Y_1^*) & 0 \\
-\alpha & -d & 0 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & \theta & -\theta_0
\end{pmatrix}.$$
where $Y_1^* = \frac{\beta A - d(\nu + \alpha + d)}{\beta(\alpha + d)}$ and $N_1^* = \frac{\beta A + \alpha(\nu + \alpha + d)}{\beta(\alpha + d)}$.

Here, it can be seen that two eigenvalues of matrix $M_1$ are $r$ and $-\theta_0$. The remaining two eigenvalues can be obtained by solving the following quadratic equation

$$\xi^2 + (\beta Y_1^* + d)\xi + \beta Y_1^*(\alpha + d) = 0. \quad (5.4.1)$$

It should be noted here that both the roots of above equation (5.4.1) are either negative or with negative real part as $Y_1^*$ must be positive for the existence of $E_1$ (see condition (5.3.1)).

Now one eigenvalue of matrix $M_1$ is $r$, which is positive whereas the other three eigenvalues are either negative or with negative real part. Thus the equilibrium $E_1$ has stable manifold locally in $Y - N - C_h$ - space and unstable manifold locally in $C$ - direction.

Now, we study the local stability behavior of interior equilibrium $E_2$ by using Routh-Hurwitz criterion.

**Local stability of equilibria $E_2(Y^*, N^*, C^*, C_h^*)$:**

The characteristic equation for the matrix $M_2$ can be written as

$$\Phi^4 + p_1 \Phi^3 + p_2 \Phi^2 + p_3 \Phi + p_4 = 0 \quad (5.4.2)$$

where $p_1 = a_1^* + d + a_2^* + a_3^*$,

$$p_2 = (a_1^* + d)(a_2^* + a_3^*) + a_2^* d + a_2^* a_3^* + \alpha(\beta Y^* + \lambda C^*) + \theta_0 \theta_2 C_h^*,$$

$$p_3 = \alpha(\beta Y^* + \lambda C^*)(a_2^* + a_3^*) + a_2^* d(a_2^* + a_3^*) + (a_1^* + d)(a_2^* a_3^* + \theta_0 \theta_2 C_h^*)$$

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and $p_4 = (a_1^*d + \alpha(\beta Y^* + \lambda C^*)) (a_2^*a_3^* + \theta_0\theta_2C_h^*)$

While writing the values of $p_i$'s, we have used the fact that $\theta - \theta_1C_h^* = \frac{\theta_0C_h^*}{C^*}$.

Here, it should be noted that

$a_1^* = (\beta Y^* + \lambda C^*) - \beta(N^* - Y^*) + (\nu + \alpha + d) = (\beta Y^* + \lambda C^*) + \frac{\lambda(N^* - Y^*)C^*}{Y^*} > 0,$

$a_2^* = \frac{rC^*}{K} > 0$ and $a_3^* = \theta_0 + \theta_1C^* > 0$.

It is easy to see that all $p_i$'s are positive. Now for the local stability of interior equilibrium $E_2(Y^*, N^*, C^*, C_h^*)$ of model system (5.2.2) without time delay, the main findings are stated in next theorem.

**Theorem 5.4.1.** The equilibrium $E_2(Y^*, N^*, C^*, C_h^*)$ is locally asymptotically stable for $\tau = 0$.

### 5.4.2 Local stability analysis with delay (i.e. $\tau \neq 0$)

Here, we analyze model system (5.2.2) with time delay, i.e. we assume that $\tau \neq 0$.

In this section we derive the stability conditions for the equilibrium $E_2$ as well as the conditions for occurrence of Hopf-bifurcation.

Now linearizing the system (5.2.2) about $E_2(Y^*, N^*, C^*, C_h^*)$ by using the following transformations:

$Y = Y^* + y$, $N = N^* + n$, $C = C^* + c$, $C_h = C_h^* + c_h$, where $y$, $n$, $c$ and $c_h$ are small perturbations.

The linearized system around the equilibrium $E_2(Y^*, N^*, C^*, C_h^*)$ is given by:
\[ \frac{du}{dt} = Au(t) + Bu(t - \tau), \] (5.4.3)

where \( u(t) = [y, n, c, ch]^T \), \( A = (a_{ij})_{4 \times 4} \), and \( B = (b_{ij})_{4 \times 4} \).

The values of \( a_{ij} \) and \( b_{ij} \) are as follows:

\[ a_{11} = -\beta Y^* - \lambda C^* - \beta(N^* - Y^*) + \nu + \alpha + d, \quad a_{12} = \beta Y^* + \lambda C^*, \quad a_{13} = \lambda(N^* - Y^*), \]
\[ a_{14} = 0, \quad a_{21} = -\alpha, \quad a_{22} = -d, \quad a_{23} = 0, \quad a_{24} = 0, \quad a_{31} = 0, \quad a_{32} = 0, \quad a_{33} = -\frac{rC^*}{K}, \]
\[ a_{34} = -\theta_2 C^*, \quad a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = -\theta_1 C_h^*, \quad a_{44} = -(\theta_0 + \theta_1 C^*), \quad b_{43} = \theta \text{ and all other } b_{ij} = 0. \]

The characteristic equation for the linearized system (5.2.2) is given by the following equation

\[ P(\psi) + Q(\psi)e^{-\psi\tau} = 0, \] (5.4.4)

\[ P(\psi) = \psi^4 + A_1\psi^3 + A_2\psi^2 + A_3\psi + A_4 \text{ and } Q(\psi) = B_1\psi^2 + B_2\psi + B_3. \]

In the above expressions of \( P(\psi) \) and \( Q(\psi) \), value of \( A_i \)'s and \( B_i \)'s are given as follows:

\[ A_1 = a_1^* + d + a_2^* + a_3^*, \]
\[ A_2 = (a_1^* + d)(a_2^* + a_3^*) + a_1^*d + a_2^*a_3^* - \theta_1\theta_2 C^* C_h^* + \alpha(\beta Y^* + \lambda C^*), \]
\[ A_3 = \alpha(\beta Y^* + \lambda C^*)(a_2^* + a_3^*) + a_1^*d(a_2^* + a_3^*) + (a_1^* + d)(a_2^*a_3^* - \theta_1\theta_2 C^* C_h^*), \]
\[ A_4 = (a_1^*d + \alpha(\beta Y^* + \lambda C^*))(a_2^*a_3^* - \theta_1\theta_2 C^* C_h^*), \]
\[ B_1 = \theta\theta_2 C^*, \]
\[ B_2 = \theta\theta_2 C^*(a_1^* + d), \]
\[ B_3 = \theta\theta_2 C^*(a_1^*d + \alpha(\beta Y^* + \lambda C^*)). \]
To show the Hopf-bifurcation analysis, we must have a pair of purely imaginary roots of characteristic equation (5.4.4). By substituting $\psi = i\omega (\omega > 0)$ into equation (5.4.4) and separating real and imaginary parts, we get the following transcendental equations:

$$\omega^4 - A_2\omega^2 + A_4 = -[(B_3 - B_1\omega^2)\cos\omega\tau + B_2\omega\sin\omega\tau]. \quad (5.4.5)$$

and

$$-A_3\omega + A_1\omega^3 = B_2\omega\cos\omega\tau - (B_3 - B_1\omega^2)\sin\omega\tau. \quad (5.4.6)$$

Now squaring and adding equations (5.4.5) and (5.4.6) we get following equation in $\omega$:

$$\left(\omega^4 - A_2\omega^2 + A_4\right)^2 + \left(-A_3\omega + A_1\omega^3\right)^2 = (B_3 - B_1\omega^2)^2 + (B_2\omega)^2 \quad (5.4.7)$$

Substituting $\omega^2 = \eta$ in above equation (5.4.7), we get the following equation in $\eta$:

$$\left(\eta^2 - A_2\eta + A_4\right)^2 + \eta(A_1\eta - A_3)^2 = (B_3 - B_1\eta)^2 + B_2^2\eta \quad (5.4.8)$$

This can be rewritten as:

$$h(\eta) = \eta^4 + D_1\eta^3 + D_2\eta^2 + D_3\eta + D_4 = 0 \quad (5.4.9)$$

where $D_1 = A_1^2 - 2A_2$, $D_2 = A_2^2 + 2A_4 - 2A_1A_3 - B_1^2$, $D_3 = A_3^2 - 2A_2A_4 + 2B_1B_2 - B_2^2$ and $D_4 = A_4^2 - B_3^2$. 

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Now if the coefficients in $h(\eta)$ are satisfying the conditions of Routh-Hurwitz criterion, then equation (5.4.9) will not have any positive real root, i.e. we may not get any positive value of $\omega$, which satisfy the transcendental equations (5.4.5) and (5.4.6). In this case the result may be written in the form of following theorem:

**Theorem 5.4.2.** If the coefficients in $h(\eta)$ (i.e. $D_i$s) satisfy the conditions of Routh-Hurwitz criterion, then the interior equilibrium $E_2$ of model (5.2.2) is asymptotically stable for all delay $\tau > 0$ provided conditions stated in Theorem 5.4.1 are satisfied.

On the contrary, if the values of $D_i$'s, $(i = 1, 2, 3, 4)$ in (5.4.9) do not satisfy the Routh-Hurwitz criterion, then equation (5.4.9) may have a positive root provided $D_4 < 0$

$$A_4 - B_3 < 0$$  \hspace{1cm} (5.4.10)

Now if (5.4.10) holds, then Equation (5.4.9) has a positive root $\eta_0$, and equation (5.4.6) has a pair of purely imaginary roots of the form $\pm i\omega_0$. From transcendental equations (5.4.5) and (5.4.6), we may obtain

$$\tan \omega_0 \tau = \frac{B_2 \omega_0 (\omega_0^4 - A_2 \omega_0^2 + A_4) + \omega_0 (A_1 \omega_0^2 - A_3)(B_3 - B_1 \omega_0^2)}{(B_3 - B_1 \omega_0^2)(\omega_0^4 - A_2 \omega_0^2 + A_4) - B_2 \omega_0^2 (A_1 \omega_0^2 - A_3)}.$$  \hspace{1cm} (5.4.11)

Now $\tau_k$ corresponding to this positive value of $\omega_0$ is given as follows

$$\tau_k = \frac{k \pi}{\omega_0} + \frac{1}{\omega_0} \arctan \frac{B_2 \omega_0 (\omega_0^4 - A_2 \omega_0^2 + A_4) + \omega_0 (A_1 \omega_0^2 - A_3)(B_3 - B_1 \omega_0^2)}{(B_3 - B_1 \omega_0^2)(\omega_0^4 - A_2 \omega_0^2 + A_4) - B_2 \omega_0^2 (A_1 \omega_0^2 - A_3)},$$  \hspace{1cm} (5.4.12)
where \( k = 0, 1, 2, 3, \ldots \)

By using Butler's lemma[39], we can say that the interior equilibrium of model (5.2.2) remains stable for \( \tau < \tau_0 \).

Now we investigate the presence of Hopf-bifurcation as \( \tau \) increases through \( \tau_0 \).

For this we need the following lemma.

**Lemma 5.4.3.** The following transversality condition is satisfied:

\[
\text{sgn} \left[ \frac{d(\text{Re}(\psi))}{d\tau} \right]_{\tau=\tau_0} > 0. \tag{5.4.13}
\]

**Proof:** Differentiating equation (5.4.4) with respect to \( \tau \), we get

\[
\left( \frac{d\psi}{d\tau} \right)^{-1} = \frac{(4\psi^3 + 3A_1\psi^2 + 2A_2\psi + A_3) + (2B_1\psi + B_2)e^{-\psi\tau}}{\psi(B_1\psi^2 + B_2\psi + B_3)e^{-\psi\tau}} - \frac{\tau}{\psi} \\
= \frac{4\psi^3 + 3A_1\psi^2 + 2A_2\psi + A_3}{\psi(B_1\psi^2 + B_2\psi + B_3)e^{-\psi\tau}} + \frac{2B_1\psi + B_2}{\psi(B_1\psi^2 + B_2\psi + B_3)} - \frac{\tau}{\psi^2} \tag{5.4.14}
\]

Now we know that

\[
\left[ \frac{d(\text{Re}(\psi))}{d\tau} \right]^{-1}_{\psi=i\omega_0} = \left[ \text{Re} \left( \frac{d\psi}{d\tau} \right)^{-1} \right]_{\psi=i\omega_0} = \text{Re} \left[ G_1 \right]_{\psi=i\omega_0} + \text{Re} \left[ G_2 \right]_{\psi=i\omega_0}, \tag{5.4.15}
\]

where, \( G_1 = \frac{4\psi^3 + 3A_1\psi^2 + 2A_2\psi + A_3}{\psi(B_1\psi^2 + B_2\psi + B_3)e^{-\psi\tau}} \) and \( G_2 = \frac{2B_1\psi + B_2}{\psi(B_1\psi^2 + B_2\psi + B_3)} \).
Now

\[ Re[G_1]_{\psi = i\omega_0} = Re \left[ \frac{4\psi^3 + 3A_1\psi^2 + 2A_2\psi + A_3}{\psi(B_1\psi^2 + B_2\psi + B_3)e^{-\psi\tau}} \right]_{\psi = i\omega_0} \]

\[ = \frac{1}{\omega_0} Re \left[ \frac{(A_3 - 3A_1\omega_0^2) + i2\omega_0(A_2 - 2\omega_0^2)}{(A_3 - A_1\omega_0^2)\omega_0 - i(\omega_0^4 - A_2\omega_0^2 + A_4)} \right] \]

\[ = \frac{1}{\Lambda_1} [(A_3 - 3A_1\omega_0^2)(A_3 - A_1\omega_0^2) - 2(A_2 - 2\omega_0^2)(\omega_0^4 - A_2\omega_0^2 + A_4)] \]

\[ = \frac{1}{\Lambda_1} [4\omega_0^6 + 3(A_1^2 - 2A_2)\omega_0^4 + 2(A_2^2 - 2A_1A_3 + 2A_4)\omega_0^2 + A_3^2 - 2A_2A_4] \]

where \( \Lambda_1 = (A_3 - A_1\omega_0^2)\omega_0^2 + (\omega_0^4 - A_2\omega_0^2 + A_4)^2 \). Here we have used equations (5.4.5) and (5.4.6).

Similarly

\[ Re[G_2]_{\psi = i\omega_0} = Re \left[ \frac{2B_1\psi + B_2}{\psi(B_1\psi^2 + B_2\psi + B_3)} \right]_{\psi = i\omega_0} \]

\[ = \frac{1}{\omega_0} Re \left[ \frac{B_2 + i2B_1\omega_0}{-B_2\omega_0 + i(B_3 - B_1\omega_0^2)} \right] \]

\[ = \frac{1}{\Lambda_2} [-2B_1\omega_0^2 + 2B_1B_3 - B_2^2] \]

where \( \Lambda_2 = B_2\omega_0^2 + (B_3 - B_1\omega_0^2)^2 \).

From equation (5.4.7), it is easy to see that \( \Lambda_1 = \Lambda_2 = \Lambda \) (say).
Using expressions of $Re\{G_1\}_{\psi=i\omega_0}$ and $Re\{G_2\}_{\psi=i\omega_0}$ in equation (5.4.15), we get

$$
\left[ \frac{d(Re(\psi))}{d\tau} \right]^{-1}_{\psi=i\omega_0} = \frac{h'(\eta_0)}{\Lambda}
$$

(5.4.16)

where $\Lambda = B_2\omega_0^2 + (B_3 - B_1\omega_0^2)^2 = (A_3 - A_1\omega_0^2)\omega_0^2 + (\omega_0^4 - A_2\omega_0^2 + A_4)^2$

It should be noted that $h'(\eta_0) > 0$ if the condition (5.4.10) is satisfied. This proves the lemma. The principal findings of the Hopf-bifurcation analysis are stated in the next theorem.

**Theorem 5.4.4.** If the condition (5.4.10) is satisfied then the interior equilibrium $E_2$ of model system (5.2.2) is locally asymptotically stable for $\tau < \tau_0$ and becomes unstable for $\tau > \tau_0$. The condition for Hopf-bifurcation is also satisfied yielding the required periodic solution from the interior equilibrium $E_2$ as $\tau$ passes through $\tau_0$, i.e. Hopf-bifurcation occurs at $\tau = \tau_0$ [49].

### 5.5 Numerical Simulation

In the previous sections of chapter 5, we made qualitative analysis of the system and obtained some results about the dynamics of the system. In this section, we perform a numerical simulation of the model system (5.2.2) based on the previous results using MATLAB 7.0.1. We choose the following set of parameter values in model system (5.2.2).

$$A = 100, \beta = 0.00003, \lambda = 0.00002, \nu = .02, \alpha = 0.2, d = .01, r = 0.065,$$
\[ K = 2000, \theta = 0.04, \theta_0 = 0.02, \theta_1 = 0.00002, \theta_2 = 0.0002. \] (5.5.1)

We have chosen these parameters to illustrate the theoretical results for Hopf bifurcation rather than as an application to a specific disease.

For the above set of parameters it is found that the conditions of existence of carrier free equilibria (CFE) \( E_1 \) (i.e. \( R_0 > 1 \)) is also satisfied. The equilibrium values in the interior equilibrium \( E_2 \) for this data are obtained as:

\[ Y^* = 232.66, \quad N^* = 5346.78, \quad C^* = 174.18, \quad C_h^* = 296.69. \]

For \( \tau = 0 \), the eigenvalues of the variational matrix corresponding to the equilibria \( E_2 \) for the model system (5.2.2) are:

\(-0.0145 + 0.0332 i \quad -0.0145 - 0.0332 i \]
\(-0.0485 + 0.0246 i \quad -0.0485 - 0.0246 i \). We note that two eigenvalues of variational matrix \( P_2 \) are negative whereas the remaining two eigenvalues are with negative real part. Hence, we can say that the interior equilibria \( E_2 \) is locally asymptotically stable in absence of delay (i.e. \( \tau = 0 \)).

For non-delay model, the variation of infective population \( Y(t) \), total population \( N(t) \), density of carrier population \( C(t) \) and concentration of chemical disinfectant with respect to time \( t \) is shown in Fig.5.6.1. From this figure, it is apparent that all the variables are approaching to their equilibrium values as time increases, which shows the stability of interior equilibrium \( E_2 \) of model system (5.2.2).
Further for the above set of parameter values, the condition (5.4.10) for existence of a pair of purely imaginary roots of characteristic equation (5.4.4) is also satisfied. By using equation (5.4.12), we have computed numerical value of $\tau_0$, which comes out to be 22.80.

By making use of Theorem 5.4.4, it is easy to note that the interior equilibrium $E_2$ of model system (5.2.2) is stable for $\tau < 22.80$ days and unstable for $\tau > 22.80$ days. This result has been shown in Fig.5.6.2 and Fig.5.6.3, for $\tau = 20$ days and $\tau = 24$ days respectively. From Fig.5.6.2, it may be noted that all the variables are approaching to their equilibrium states, showing the stability of interior equilibrium $E_2$ for $\tau < \tau_0$. From Fig.5.6.3, it can be seen that all the variables are showing oscillatory behavior, thus the interior equilibrium $E_2$ is unstable for $\tau > \tau_0$.

We have drawn the solution trajectories of model system (5.2.2) for the above set of parameter values in $Y - C_h$-plane and $C - C_h$-plane for $\tau = 20$ days and $\tau = 24$ days in Fig.5.6.4 and Fig.5.6.5 respectively. From Fig.5.6.4, it is clear that trajectory is approaching to their equilibrium values, which shows that the endemic equilibria $E_2$ is nonlinearly stable in $Y - C_h$-plane and $C - C_h$-plane for $\tau = 20$ days. However, in Fig.5.6.5, it can be seen that the solution trajectory is not approaching to the equilibrium values and forming a limit cycle. This implies that the endemic equilibrium is unstable for $\tau = 24$ days.

Biologically speaking, if the measured data for the density of carriers is older
than 22.80 days then the number of infective will oscillate and in this case it is
difficult to predict the size and severity of epidemic. Hence, if one wants to predict
the size of epidemic accurately and want to make use of chemical disinfectants
efficiently the measured data should not be older than 22.80 days.

5.6 Conclusion

In this chapter, a non-linear delay mathematical model for the control of carrier
dependent infectious diseases, using chemical disinfectant to control the growth
of carriers, is proposed and analyzed. It is found that disease free equilibria (DFE)
and interior equilibria always exists, however carrier free equilibria (CFE) exists
only when basic reproduction number $R_0 > 1$. It is shown that in absence of delay
(i.e. $\tau = 0$), the DFE and CFE are always unstable, however the interior equilibria
is locally asymptotically stable under certain conditions as stated in Theorem
5.4.1.

Further in Section 5.4, we have derived the conditions for the bifurcation of the
interior equilibrium. It is found that stable interior equilibrium $E_2$ remains stable
for all $\tau > 0$ under certain conditions (see Theorem 5.4.2) It is also shown that
interior equilibrium enters a Hopf - bifurcation as $\tau$ crosses some critical value $\tau_0$.
This critical value of $\tau_0$ has been obtained analytically and is given by equation
(5.4.12).

Biologically the model analysis shows that to predict the size of epidemic one
should use the chemical disinfectant with the rate proportional to the density of carrier population, which should not be older than $\tau_0$ time.
Figure 5.6.1: Variation of $Y$, $N$, $C$ and $C_h$ with respect to time $t$ in absence of delay.
Figure 5.6.2: Variation of $Y$, $N$, $C$ and $C_h$ with respect to time $t$ for $\tau = 20$ days.
Figure 5.6.3: Variation of $Y$, $N$, $C$ and $C_h$ with respect to time $t$ for $\tau = 24$ days.
Figure 5.6.4: Nonlinear Stability in $Y - C_h$-plane and $C - C_h$-plane for $\tau = 20$ days.

Figure 5.6.5: Limit Cycle in $Y - C_h$-plane and $C - C_h$-plane for $\tau = 24$ days.