Chapter 3
CHAPTER-3

EFFECT OF ELECTRIC LOAD PARAMETER ON UNSTEADY MHD CONVECTIVE FLOW OF VISCIOUS IMMISCIBLE LIQUIDS IN A HORIZONTAL CHANNEL: TWO FLUID MODEL


ABSTRACT

An analytical study of the two-fluid hydromagnetic unsteady mixed convective flow of viscous, incompressible, electrically conducting, immiscible liquids in a horizontal channel is investigated. Both the liquids are considered to have different densities, viscosities, thermal and electrical conductivities and occupy equal heights. The transport properties of both the liquids are taken to be constant and the bounding channel walls are maintained at constant and equal temperature. Regular perturbation technique is applied to obtain the solutions of velocity field and temperature distribution of both the liquids. Expressions for skin-frictions and heat transfer rates at the plates are also derived. The effects of
The various parameters entered into the equations of momentum and energy are evaluated numerically and plotted graphically, while numerical values of variations in skin-fricutions and heat transfer rates at the plates are presented in tabular form. The results of the study are discussed.

**NOMENCLATURE**

- \( B_0 \) uniform magnetic field
- \( C_p \) specific heat at constant pressure
- \( C = K_{T1}/K_{T2} \) ratio of thermal conductivities of the liquids
- \( D = \rho_1/\rho_2 \) ratio of densities of the liquids
- \( E = \sigma_1/\sigma_2 \) ratio of electrical conductivities of the liquids
- \( E_y \) electric field intensity
- \( Ec \) Eckert number
- \( h \) half width of parallel walls
- \( K_{T1}, K_{T2} \) thermal conductivities
- \( M_1, M_2 \) magnetic parameters
- \( n \) positive constant
- \( Nu_1, Nu_2 \) rate of heat transfer at \( y = 1 \) and \( y = -1 \)
- \( Pr \) Prandtl number
- \( Q \) flux of flow
- \( Re \) electric load parameter
- \( t \) time
- \( T \) temperature
- \( u \) velocity component along \( x \)-axis
- \( U_0 \) free stream velocity
\[ V = \frac{\mu_1}{\mu_2} \]  
\[ (x, y) \]  

**Cartesian coordinate**

**Greek Symbols**

\( \mu_1, \mu_2 \)  
viscosities of the liquids

\( \rho_1, \rho_2 \)  
densities of the liquids

\( \sigma_1, \sigma_2 \)  
electrical conductivities of the liquids

\( \tau_1, \tau_2 \)  
skin-frictions at \( y = 1 \) and \( y = -1 \)

\( \theta_1, \theta_2 \)  
kinematic viscosities of the liquids

**1. INTRODUCTION**

The study of flow of two or more immiscible liquids is of immense importance due to their abundance use in flow sciences. Such flows occur in soil mechanics, ground water hydrology, purification of crude oil in petroleum industry, oil recovery through ocean wells, flow of water containing oil in packed rocks, industrial filtration, ceramic engineering, etc., where immiscible liquids move in two or more phases. Besides the study of flow and heat transfer for an electrically conducting fluid under the influence of magnetic field has their application in many engineering problems such as MHD generators, nuclear reactors, plasma studies, geothermal energy extractions and the boundary layer control in the field of aerodynamics (Soundalgekar and Takhar, [1977], Kim [2001]).

Kapur and Shukla [1963] studied unsteady flow of two viscous incompressible fluids between two parallel plates. These authors (Kapur and Shukla [1964]) also studied the same problem considering pressure gradient
under the same boundary conditions. Pathak [1966] studied this problem for two immiscible viscous fluid flows, but under different boundary conditions. Gupta and Goyal [1973] extended the problem for $n$-immiscible viscous incompressible fluids under pulsatile fluid flow between two coaxial rotating circular cylinders with injection at the inner cylinder.

Rudraiah [1985] presented a theoretical analysis for coupled flows in a parallel plate channel and a bounding porous channel of finite thickness. An important step towards the understanding of heat transfer and fluid mechanics in the interface region of two immiscible fluid flows in the parallel plate channel was made by Vafai and Thiagaraja [1987]. Bhargava and Sachati [1989] extended this study for heat transfer in generalized Couette flow of two immiscible viscous incompressible fluids considering porous walls of the channel and employing Darcy’s law to model the flow in porous medium. Following Beavers and Joseph [1967], fully developed steady flow was studied by Vafai and Kim [1990], where a fluid layer was sandwiched between a semi-infinite porous body and an external impermeable porous boundary. The main concern of this study was to introduce an analysis for exact solution of the fluid flow by the use of matching the velocity and tangential stress at the interface. Aldoss et. al. [1993] reported non-similarity solutions using the Darcy’s law for the problem of mixed convection along the horizontal surfaces in a porous medium by dividing the entire mixed convection flow regime into two regions: one covers the forced convection dominated regime and other covers the free-convection dominated regime.
The theoretical studies concerning coupled flows in the presence of electromagnetic field have received considerable attention in last four decades or so. It, however, appears that relatively less attention has been paid involving applications on hydromagnetic convective flows, where immiscible liquids flow through channels. The gradual developments of magnetohydrodynamics have been exhibited in the work of Cowling [1957], Ferraro and Plumpton [1961]. Several authors, including Singh and coworkers [1994, 96, 2000, 2004] have investigated flow problems on immiscible liquids. Recently, Singh and Takhar [2005] have presented an analytical study on free convection in fully developed, laminar, free convection flow of two viscous, immiscible, incompressible liquids bounded above and below by two naturally permeable beds of high porosity and finite thickness. More recently, Singh et. al. [2008, 2008] have studied hydromagnetic convective flow of viscous immiscible liquids in a horizontal parallel wall channel considering two liquids of different viscosities, thermal and electrical conductivities occupying different heights. In addition, Lahiri and Ganguli [1986], Sengupta and Ray [1991], Maity et. al. [1996], Malashetty et. al. [2005] and Umavati et. al. [2006] have studied steady and unsteady two fluid models under different physical situations and relevant boundary conditions.

In the above stated studies, the electric load parameter is not given due importance, although it has important applications in aerospace technologies, design of MHD generators (Ferraro and Plumpton [1961]), cross-fired accelerators (Perlmutter and Seigel [1961]), coal-fired MHD generators (Postlathwaite and Sluyter [1978]), etc. However, Singha and Deka [2005] have
analytically studied two phase flow problem in a parallel plate channel under uniform magnetic field for two different values of electric load parameter. The aim of the present study is to extend the work of Singha and Deka [2005] considering that the upper bounding plate of the channel moves in its own plane with a velocity exponentially decreasing function of time. The present study is expected to be useful in understanding the presence of slag layers on the heat transfer characteristics of the coal-fired MHD generators and will enhance the applications of the study of Singha and Deka [2005].

2. BASIC EQUATIONS AND DESCRIPTION OF THE PROBLEM

In two-dimensional Cartesian coordinate system \((x, y)\), the flow of two immiscible, incompressible, electrically conducting, viscous liquids is considered. The geometry of the model (see Fig. Schematic diagram) consists of two infinite parallel walls (width \(2h\)) extending in the direction of \(x\)-axis. Both the liquids flow in the direction of \(x\)-axis under the influence of uniform magnetic field \(B_0\) applied normal to the flow region. The flow region \(0 \leq y \leq h\) is occupied by the liquid of density \(\rho_1\), viscosity \(\mu_1\), electrical conductivity \(\sigma_1\) and thermal conductivity \(K_{T1}\), while the region \(-h \leq y \leq 0\) is occupied by the fluid of density \(\rho_2\), viscosity \(\mu_2\), electrical conductivity \(\sigma_2\) and thermal conductivity \(K_{T2}\). In addition, the analysis of the present model is based on the following assumptions:
1. The flow of both the immiscible, incompressible liquids is unsteady, laminar and fully developed.

2. The transport properties of both the liquids are constant.

3. Both the liquids are immiscible such that viscosity \( \mu_1 < \mu_2 \), density \( \rho_1 < \rho_2 \), electrical conductivity \( \sigma_1 < \sigma_2 \) and thermal conductivity \( K_{T_1} < K_{T_2} \) (see Bhargava et al. [2001]).

4. Initially, at \( t = 0 \), the bounding walls are at the constant temperature \( T_1 = T_0 \) and \( T_2 = T_0 \), also when \( t > 0 \), the temperature of the upper bounding wall is suddenly raised to \( T_1 = T_0 + (T_w - T_0)(1 + \epsilon e^{-nt}) \) and thereafter maintained constant.

5. The fluid velocity and magnetic field distributions are

\[
\bar{V} = [u(y), 0, 0] \text{ and } \bar{B} = [0, B_0, 0],
\]

respectively.
6. The dissipation function $\phi$ is $(\partial u/\partial y)^2$ and current density $\vec{J}$ due to the magnetic field is defined by $\vec{J} = \sigma \left[ \vec{E} + \vec{V} \times \vec{B} \right]$, where $\vec{E}$ is the electric field intensity, i.e., $\vec{E} = (E_y, 0, 0)$.

7. The term involving $\frac{\partial T}{\partial x}$ in the energy equation is neglected since both the walls are maintained at constant and equal temperature.

8. The upper wall moves in its own plane with exponentially decreasing function of time, i.e., $u = U_0 (1 + e^{-at})$, while the lower wall is fixed.

9. The walls are long enough, so that all the physical variables are functions of $y$ and $t$ only.

10. The magnetic Reynolds number is small, so that the induced magnetic field is neglected in comparison to the applied magnetic field.

11. The temperature of the upper wall is high enough, so that viscous heating effects and load parameter effects are considered significantly.

12. The free stream velocities temperatures of both the fluids follow an exponentially decreasing small perturbation law, so that the system remains stable under assumption 4 and 8. The condition of continuity of velocity and shear stress along with continuity of temperature and heat flux at the interface will be satisfied under present configuration (Kim [2008]).
The general equations, governing the MHD flow in vector notations
(Cowling [1957]; Cramer and Pai [1973]), are:

\[ \vec{V} \cdot \vec{V} = 0 \] \hspace{1cm} (1)

\[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{V}) \vec{V} = \left( \frac{1}{\rho} \right) \nabla p + \left( \frac{\mu}{\rho} \right) \nabla^2 \vec{V} + \left( \frac{1}{\rho} \right) (J \times B) \] \hspace{1cm} (2)

\[ \rho C_p \left( \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) = K_f \nabla^2 T + \rho \phi + \frac{J^2}{\sigma} \] \hspace{1cm} (3)

Therefore, under the above stated assumptions, the equations governing
the motion and energy (2)-(3) in Region-I and Region-II are transformed to:

Region-I \((0 \leq y \leq h)\)

\[ \frac{\partial u_1}{\partial t} = g_1 \frac{\partial^2 u_1}{\partial y^2} - \frac{\sigma_1}{\rho_1} \left( E_y + u_1 B_0 \right) B_0 \] \hspace{1cm} (4)

\[ \frac{\partial T_1}{\partial t} = \frac{K_{T_1}}{\rho_1 C_p} \frac{\partial^2 T_1}{\partial y^2} + \frac{g_1}{C_p} \left( \frac{\partial u_1}{\partial y} \right)^2 + \frac{\sigma_1}{\rho_1 C_p} \left( E_y + u_1 B_0 \right)^2 \] \hspace{1cm} (5)

Region-II \((-h \leq y \leq 0)\)

\[ \frac{\partial u_2}{\partial t} = g_2 \frac{\partial^2 u_2}{\partial y^2} - \frac{\sigma_2}{\rho_2} \left( E_y + u_2 B_0 \right) B_0 \] \hspace{1cm} (6)

\[ \frac{\partial T_2}{\partial t} = \frac{K_{T_2}}{\rho_2 C_p} \frac{\partial^2 T_2}{\partial y^2} + \frac{g_2}{C_p} \left( \frac{\partial u_2}{\partial y} \right)^2 + \frac{\sigma_2}{\rho_2 C_p} \left( E_y + u_2 B_0 \right)^2 \] \hspace{1cm} (7)

The no slip conditions at the lower wall, the continuity of velocity and
shear stress at the interface along with continuity of temperature and heat flux at
the interface, relevant to the problem (Kim [2001]) are:
\[ u_1 = U_0 \left(1 + e^{-nt}\right), \quad T_1 = T_0 + \left(T_w - T_0\right) \left(1 + e^{-nt}\right) \quad \text{at} \quad y = h \]

\[ u_1 = u_2, \quad \mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y} \quad \text{at} \quad y = 0 \]

\[ T_1 = T_2, \quad K_{T_1} \frac{\partial T_1}{\partial y} = K_{T_2} \frac{\partial T_2}{\partial y} \quad \text{at} \quad y = 0 \]

\[ u_2 = 0, \quad T_2 = T_0 \quad \text{at} \quad y = -h \quad (8) \]

We introduce the following non-dimensional variables and parameters:

\[ u_1^* \frac{u_1}{U_0}, \quad u_2^* = \frac{u_2}{U_0}, \quad y^* = \frac{y}{h}, \quad T_1^* = \frac{T_1 - T_0}{T_w - T_0}, \]

\[ T_2^* = \frac{T_2 - T_0}{T_w - T_0}, \quad t^* = \frac{\partial t}{h^2}, \quad n^* = \frac{n h^2}{\partial t}, \]

\[ E = \frac{\sigma_1}{\sigma_2} \quad \text{(ratio of electrical conductivities of the liquids)}, \]

\[ E_c = \frac{U_0^2}{(T_w - T_0) C_p} \quad \text{(Eckert number)}, \]

\[ V = \frac{\mu_1}{\mu_2} \quad \text{(ratio of viscosities of the liquids)}, \]

\[ R_e = \frac{E_y}{U_0 B_0} \quad \text{(Electric load parameter)}, \quad P_r = \frac{\mu_1 C_p}{K_{T_1}} \quad \text{(Prandtl number)}, \]

\[ D = \frac{\rho_1}{\rho_2} \quad \text{(ratio of densities of the liquids)}, \]

\[ M_2 = \frac{M_1 V}{E} \quad \text{(magnetic parameter in Region-II)}, \]
\[ M_1 = B_0 h \sqrt{\frac{\sigma_1}{\mu_1}} \] (magnetic parameter in Region-I),

and

\[ C = \frac{K_{T1}}{K_{T2}} \] (ratio of thermal conductivities of the liquids).

Here the subscripts 1 and 2 refer to the Region-I and Region-II respectively. The measure of the applied magnetic filed is the Hartman number \( M_1 \) in Region-I and Hartman number \( M_2 \) in Region-II, where \( M_1 \) and \( M_2 \) is related by \( M_2 = \frac{E}{V} M_1 \).

Introducing the above stated non-dimensional variables and parameters, after ignoring the stars over them, the equations (4)-(7) are transformed to:

**Region-I** \((0 \leq y \leq 1)\)

\[
\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial y^2} - M_1^2 \left( R_e + u_1 \right) \quad (9)
\]

\[
P_r \frac{\partial T_1}{\partial t} = \frac{\partial^2 T_1}{\partial y^2} + E_c P_r \left( \frac{\partial u_1}{\partial y} \right)^2 + M_1^2 E_c P_r \left( R_e + u_1 \right)^2 \quad (10)
\]

**Region-II** \((-1 \leq y \leq 0)\)

\[
V \frac{\partial u_2}{\partial t} = D \frac{\partial^2 u_2}{\partial y^2} - M_2^2 D \left( R_e + u_2 \right) \quad (11)
\]

\[
V C_{p_r} \frac{\partial T_2}{\partial t} = V D \frac{\partial^2 T_2}{\partial y^2} + C_D E_c P_r \left( \frac{\partial u_2}{\partial y} \right)^2 + M_2^2 C_D E_c P_r \left( R_e + u_2 \right)^2 \quad (12)
\]

The boundary conditions (8) transform to:

...
3. Method of Solution

To obtain the velocity and temperature field of the coupled equations for both the phases, we use regular perturbation technique, choosing \( \varepsilon \) as perturbation parameter. For the purpose we assume \( \varepsilon << 1 \):

\[
\begin{align*}
    u_1(y,t) &= U_1(y) + \varepsilon U_2(y) e^{-nt} \\
    u_2(y,t) &= V_1(y) + \varepsilon V_2(y) e^{-nt} \\
    T_1(y,t) &= \theta_1(y) + \varepsilon \theta_2(y) e^{-nt} \\
    T_2(y,t) &= \phi_1(y) + \varepsilon \phi_2(y) e^{-nt}
\end{align*}
\]

Introducing (14) in Eqs. (9)-(12), we obtain:

Region -I \((0 \leq y \leq 1)\)

\[
\begin{align*}
    U''_1(y) - M_1^2 U_1(y) &= M_1^2 R_e \\[15]
    U''_2(y) - M_3^2 U_2(y) &= 0 \\[16]
    \theta''_1(y) + P_1 &= -P_2 U_1(y) - P_3 U_1^2(y) - P_4 \left[ U'_1(y) \right]^2 \\[17]
\end{align*}
\]
\[
\theta_2''(y) + M_2^2 \theta_2(y) = -P_2 U_2(y) - 2P_3 U_1(y) U_2(y) - 2P_4 U_1'(y) U_2'(y) \tag{18}
\]

Region -II \((-1 \leq y \leq 0)\)

\[
V_1''(y) - M_2^2 V_1(y) = M_2^2 \rho_e \tag{19}
\]

\[
V_2''(y) - M_4^2 V_2(y) = 0 \tag{20}
\]

\[
\phi_1''(y) + Q_1 = -Q_2 V_1(y) - Q_3 V_2^2(y) - Q_4 [V_1'(y)]^2 \tag{21}
\]

\[
\phi_2''(y) + M_6^2 \phi_2(y) = -Q_2 V_2(y) - 2Q_3 V_1(y) V_2(y) - 2Q_4 V_1'(y) V_2'(y) \tag{22}
\]

The boundary conditions (13) are transformed to:

\[
U_1 = 1, \quad U_2 = 1, \quad \theta_1 = 1, \quad \theta_2 = 1 \quad \text{at} \quad y = 1
\]

\[
U_1 = V_1, \quad U_2 = V_2, \quad \frac{\partial U_1}{\partial y} = \frac{1}{V} \frac{\partial V_1}{\partial y}, \quad \frac{\partial U_2}{\partial y} = \frac{1}{V} \frac{\partial V_2}{\partial y} \quad \text{at} \quad y = 0
\]

\[
\theta_1 = \phi_1, \quad \theta_2 = \phi_2, \quad \frac{\partial \theta_1}{\partial y} = \frac{1}{C} \frac{\partial \phi_1}{\partial y}, \quad \frac{\partial \theta_2}{\partial y} = \frac{1}{C} \frac{\partial \phi_2}{\partial y} \quad \text{at} \quad y = 0
\]

\[
V_1 = 0, \quad V_2 = 0, \quad \phi_1 = 0, \quad \phi_2 = 0 \quad \text{at} \quad y = -1 \tag{23}
\]

The solution of equations (15)-(22) satisfying the boundary conditions (23) are obtained as follows:

Region -I \((0 \leq y \leq 1)\)

\[
\psi_1(y,t) = C_1 e^{M_1 y} + C_2 e^{-M_1 y} - \rho_e + \epsilon \left[ C_3 e^{M_3 y} + C_4 e^{-M_3 y} \right] e^{-\alpha t} \tag{24}
\]

\[
T_1(y,t) = C_{10} + C_9 y + \frac{P_5}{2} y^2 + \frac{P_6}{M_1^2} e^{M_1 y} + \frac{P_7}{M_1^2} e^{-M_1 y}
\]
Region II \((-1 \leq y \leq 0)\)

\begin{align}
    u_2(y,t) &= C_5 \varepsilon^2 \varepsilon^y + C_6 \varepsilon^{-M_2 y} - R_c + \varepsilon \left[ C_7 \varepsilon^M_4 y + C_8 \varepsilon^{-M_4 y} \right] e^{-nt} \\
    T_2(y,t) &= C_{14} + C_{13} y + \frac{Q_5}{2} y^2 + \frac{Q_6}{M_2^2} \varepsilon^2 \varepsilon^y + \frac{Q_7}{M_2^2} \varepsilon^{-M_2 y} \\
        &- \frac{Q_8}{4M_2^2} \varepsilon^2 \varepsilon^y - \frac{Q_9}{4M_2^2} \varepsilon^{-2M_2 y} \\
        &+ \varepsilon \left[ C_{15} \cos M_6 y + C_{16} \sin M_6 y + Q_{10} \varepsilon^M_4 y + Q_{11} \varepsilon^{-M_4 y} - Q_{12} \varepsilon^{(M_2 + M_4)} y \\
        &- Q_{13} \varepsilon^{-(M_2 + M_4)} y + Q_{14} \varepsilon^{(M_2 - M_4)} y + Q_{15} \varepsilon^{-M_2 - M_4} y \right] e^{-nt}
\end{align}

The constants are given in the appendix.

4. SKIN-FRICTION AND HEAT TRANSFER RATE

The skin-friction \((\tau_1)\) at the upper channel wall \((y=1)\) and skin-friction \((\tau_2)\) at the lower channel wall \((y=-1)\) is given by

\begin{align}
    \tau_1 &= \frac{dU_1}{dy} \bigg|_{y=1} + \varepsilon \frac{dU_2}{dy} \bigg|_{y=-1} e^{-nt} = G_1 + \varepsilon G_2 e^{-nt} \\
    \tau_2 &= \frac{dV_1}{dy} \bigg|_{y=1} + \varepsilon \frac{dV_2}{dy} \bigg|_{y=-1} e^{-nt} = G_3 + \varepsilon G_4 e^{-nt}
\end{align}
The rate of heat transfer \( (Nu_1) \) at the upper channel wall \((y = 1)\) and rate of heat transfer \( (Nu_2) \) at the lower channel wall \((y = -1)\) is given by

\[
Nu_1 = \left. \frac{d\theta_1}{dy} \right|_{y=1} + \varepsilon \left. \frac{d\theta_2}{dy} \right|_{y=-1} e^{-nt} = G_5 + \varepsilon G_6 e^{-nt} \tag{30}
\]

\[
Nu_2 = \left. \frac{d\phi_1}{dy} \right|_{y=1} + \varepsilon \left. \frac{d\phi_2}{dy} \right|_{y=-1} e^{-nt} = G_7 + \varepsilon G_8 e^{-nt} \tag{31}
\]

The constants are given in the appendix.

5. PARTICULAR CASES

Case I : If the problem is reduced to steady case such that the upper wall moves with velocity \( U_1 \) and the lower wall moves with velocity \( U_2 \) then the analytical results obtained are similar to those obtained by Singh et. al. [2008].

Case II : If the problem is reduced to steady case such that the upper wall moves with velocity \( U_1 \) and the lower wall is fixed then the analytical results obtained are for Couette flow.

Case III : If the problem is reduced to steady case and both the walls are set fixed, the analytical results obtained are similar to those obtained by Singha and Deka [2005].

6. RESULTS AND DISCUSSION

The solution of Eqs. (15)-(22) satisfying boundary conditions (23) are shown in (24)-(27). The skin-friction at the upper and lower wall is given in (28)-(29). Also the heat transfer rate at the upper and lower wall is expressed in (30)-
In order to establish the effects of various parameters on the flow and heat transfer processes in Region-I and Region-II, and also to get physical insight into the problem, numerical calculations have been performed. In the analysis, the value of Prandtl number \((P_r)\) is chosen to be 0.71, 1.0 and 7.0, which corresponds to air, electrolyte solution and water at 20\(^\circ\)C and one atmospheric pressure – three important fluids generally used in energy and aerospace technologies, design of MHD generators, cross-field accelerators (Perlumutter and Siegal [1961], Romig [1961]) and the coal–fired MHD generators channel (Postlethwaite and Sluyter [1978]). The values of the remaining parameters entered into the problem are chosen arbitrarily but do retain physical significance in real energy systems applications (Blums [1987]). Observations regarding variations in velocity and temperature distributions in both the regions are made with the aid of a number of graphical figures, while skin-friction and rate of heat transfer at the channel walls are recorded in tabular form. These variations simulate the effects of Prandtl number \((P_r)\), Eckert number \((Ec)\), magnetic parameter \((M_1)\) in region-I, magnetic parameter \((M_2)\) in Region-II, electric load parameter \((Re)\), ratio of the viscosities \((V)\) of the liquids, ratio of the densities \((D)\) of the liquids, ratio of the thermal conductivities \((C)\) of the liquids on the temperature distribution and/or velocity field. The values of the Eckert number \((Ec)\) and the perturbation parameter \((\varepsilon)\) are chosen non-zero; so in all cases a dissipative (viscous heating) flow regime is studied in both the flow regions in the channel. The convergence of this series solution has been checked and verified for some simple cases on the computer.
Fig. 1 Effect of Prandtl number on temperature distribution
$M_1 = 1.2$, $Ee = 0.5$, $C = 0.6$, $D = 0.5$, $V = 0.4$

Fig. 1 illustrates the variations in temperature (termed as $T_1$ in Region-I and $T_2$ in Region-II) versus non-dimensional $y$ for three values of Prandtl number ($Pr$) viz. $Pr = 0.71$, $Pr = 1.0$ and $Pr = 7.0$ and two values of electric load parameter, $Re$ (doted curves pertaining to electric load parameter $Re = 0.5$ and solid curves pertaining to $Re = 1.0$) choosing $M_1 = 1.2$, $Ee = 0.5$, $C = 0.8$, $D = 0.6$, $V = 0.5$ and $e = 0.02$. It is also observed that all profiles are positive rising from $y = -1.0$ to $y = 0$ (i.e., from Region-II to the interface) and again from $y = 0$ to $y = 0.5$ (i.e., from interface to approximately middle of Region-I) and then decrease smoothly towards the $y$-axis and ultimately become constant at $y = 1.0$. Clearly maximum magnitude of temperature $T_1$ in Region-I is recorded at the interface, while maximum magnitude of temperature $T_2$ in Region-II is recorded at the middle of Region-I. Besides, an increase in electric load parameter ($Re$) increases the temperature significantly, indicating that it has
dominant effect in increasing the temperature. In fact, an increase in the electric load parameter enhances the resistance, which in turn produces the energy development in both the flow regions, thereby enhancing temperature, consistent with Singha and Deka [2005], Singh et. al [2008]. In addition, it is observed that in both the regions, the temperature increases more rapidly for water ($P_r = 7.0$) higher-Prandtl number fluid in comparison to electrolyte solution ($P_r = 1.0$) and air ($P_r = 0.71$). Physically, this increase in Prandtl number is due to the fact that Prandtl number mathematically defines the ratio of the momentum diffusivity to the thermal diffusivity. As such, higher $P_r$-fluids transfer heat less effectively than do lower $P_r$-fluids; consequently, lower temperatures are observed in profiles for $P_r = 0.71$ (air) and $P_r = 1.0$ (electrolyte solution) compared with profile $P_r = 7.0$ (water).

![Fig. 2 Effect of $M_1$ on temperature distribution](image)

$E_e = 0.2$, $R_e = 0.5$, $C = 0.6$, $D = 0.5$, $V = 0.4$
Fig. 2 demonstrates the variations in temperature ($T_1$ in Region-I and also temperature $T_2$ in Region-II) versus non-dimensional transverse coordinate $y$ for different values of magnetic parameter ($M_1 = 1.2, 1.6, 2.0$) and Prandtl number ($Pr = 1.0, 7.0$) with fixed values of $Ec = 0.3$, $Re = 0.5$, $C = 0.6$, $D = 0.5$, $V = 0.4$ and $\varepsilon = 0.02$. Since the viscosities and electrical conductivities of the liquids of both the region are different, the magnetic parameters $M_1$ and $M_2$ are associated with the relations $M_1 = M_2 V/E$. Physically, this relation implies the need of the inequality of viscosities and electrical conductivities of the fluid described in assumption 3. The solid curves are associated with electrolyte solution ($Pr = 1.0$) and dotted curves are associated to water ($Pr = 7.0$). It is observed that the effect of increase in magnetic parameter is to increase the temperature. This is due to the fact that variation of magnetic parameter leads to the variation in Lorentz force. The Lorentz force produces more resistance to the transport phenomena, which ultimately increases the temperature. Also, higher temperatures are noted in temperature profiles $T_1$ (for Region-I) and $T_2$ (for Region-II) for $Pr = 7.0$ in comparison to $Pr = 1.0$. This leads to the conclusion that the electrolyte solution is more effective in diminishing the temperature in comparison with water. Besides, it is noted that the temperature $T_2$ ultimately attains steady state in Region-II at $y = 1.0$. These effects are in good arrangement with Singh et. al. [2008] and Blums [1987].
Fig. 3a. Effect of $R_e$ on temperature distribution
$M_1=1.2$, $E_c=0.2$, $C=0.6$, $D=0.5$, $V=0.4$

Fig.-3a and Fig.-3b represent the variations in temperature in Region-I, Region-II versus $y$ due to change in the electric load parameter ($R_e$) choosing $M_1=1.2$, $E_c=0.2$, $C=0.6$, $D=0.5$, $V=0.4$ and $\varepsilon=0.02$ in the range $(0 \leq R_e \leq 1.5)$ and $(0 \geq R_e \geq -1.5)$ respectively. Fig.-3a clearly indicates that the case of open circuit problem for positive values of electric load parameter ($R_e > 0$), is to accelerate the temperature and to decrease the temperature in short circuit case ($R_e = 0$). Also, it is observed that the temperature increases smoothly in the Region-II and shows increasing trend up to the interface of two regions. Thereafter, it also increases up to nearly middle part of the Region-I and then decreases, ultimately becomes constant at $y=1.0$. Besides, an increase in Prandtl number increases the temperature in the entire flow regime. However, for negative values of electric load parameter (see Fig.-3b), the temperature increases more rapidly in region-II in comparison to Region-I. Physically, in the short
circuit case \((R_e = 0)\) a disturbance is created by Lorentz force, which change as the initial behaviour of open circuit case \((R_e < 0)\). Hence, more resistance is created in the Region-I, due to velocity of wall associated with Region-I. As such, the temperature in Region-II increases more rapidly than Region-I. Again, higher temperatures are seen to be associated with water \((P_r = 7.0)\) compared with electrolyte solution \((P_r = 1.0)\).

![Fig. 3b. Effect of \(R_e\) on temperature distribution](image)

\(M_1 = 1.2, Ec = 0.2, C = 0.6, D = 0.5, V = 0.4\)

Fig. 4 represents variations in the temperature \(T_1\) and \(T_2\) versus \(y\) for different values of viscosity ratio \((V)\) with fixed values of \(M_1 = 1.2, Ec = 0.2,\)
\(C = 0.6, D = 0.5, R_e = 0.5\) and \(\varepsilon = 0.02\) in the range \(0.4 \leq V \leq 1.0\) and \(1.0 \leq P_r \leq 7.0\). Obviously, an increase in \(V\) leads to an increase in temperatures \(T_1, T_2\) in Region-I and Region-II. Physically, an increase in viscosity ratio \((V = \mu_1 / \mu_2)\) implies more friction in layers of the flowing fluids. Hence, Lorentz force produces more resistance which in turn increases the temperature in
Region-I as well as Region-II. This conclusion is in excellent agreement with those of Bhaargava and Sacheti [1989].

Fig. 4 Effect of $V$ on temperature distribution
$M_1 = 1.2$, $Ec = 0.2$, $D = 0.5$, $C = 0.6$, $Re = 0.5$

Fig. 5 Effect of $D$ on temperature distribution
$M_1 = 1.2$, $Ec = 0.2$, $V = 0.4$, $C = 0.6$, $Re = 0.5$
Fig. 5 illustrates variations in the temperature distribution versus $y$ for different values of viscosity ratio ($D$) with fixed values of $M_1 = 1.2$, $Ec = 0.2$, $C = 0.6$, $V = 0.4$, $Re = 0.5$ and $\epsilon = 0.02$ in the range $0.5 \leq D \leq 1.0$ and $1.0 \leq Pr \leq 7.0$. As expected, an increase in density ratio decreases the temperature in Region-I and Region-II. Again, in Region-II, the maximum temperature is recorded at the interface ($y = 0$), while maximum magnitudes of temperatures are noted in the middle ($y = 0.5$) of the Region-I, which tends to steady state at $y = 1.0$. Physically, an increase in density ratio implies that the density, $\rho_1$ of the liquid of Region-I increases more rapidly in comparison to density, $\rho_2$ of the liquid of Region-II. This reduces the speed of the convective flow, so that the temperature decreases. Also, the temperature decreases more rapidly with water in comparison to air.

![Fig. 6 Effect of C on temperature distribution](image)

$M_1 = 1.2$, $Ec = 0.2$, $V = 0.4$, $D = 0.5$, $Re = 0.5$
Fig. 7 Effect of $M_1$ on velocity field at $V = 0.4$ and $D = 0.5$

Fig.-6 demonstrates effects of thermal conductivity ratio ($C$) on variations in temperature $T_1$ in Region-I and $T_2$ in Region-II for fixed values of $M_1 = 1.2$, $Ec = 0.2$, $D = 0.5$, $V = 0.4$, $Re = 0.5$ and $\varepsilon = 0.02$ in the range $0.6 \leq C \leq 1.0$ and $1.0 \leq Pr \leq 7.0$. As expected, an increase in thermal conductivity ratio ($C$) boosts the temperature in Region-I and Region-II.

Fig.-7 depicts variations in velocity field ($u_1$) in Region-I and velocity field ($u_2$) in Region-II versus $y$ for the effects of $M_1$ with viscosity ratio $V = 0.4$ and $D = 0.5$. The effects are shown pertaining to the range $1.2 \leq M_1 \leq 2.2$ and $-1.5 \leq Re \leq 1.5$. It is observed that an increase in $M_1$ decreases the velocity for $Re = 1.5$ (solid curves), while a decrease in $Re$ increases the velocity field.
Fig. 8 Effect of $V$ on velocity field at $M_1 = 1.2$ and $D = 0.5$

Fig. 8 represents the effects of viscosity ratio and electric load parameter on velocity field ($u_1$) in Region-I and velocity field ($u_2$) in Region-II versus $y$ with $M = 1.2$ and $D = 0.5$. The effects are shown pertaining to the range $0.4 \leq V \leq 1.0$ and $-1.5 \leq R_e \leq 1.5$. It is observed that an increase in $V$ increases the velocity for $R_e = -1.5$ (dotted curves), while an increase in $R_e$ decreases the velocity field.

Fig. 9 shows variations in velocities $u_1(y)$ and $u_2(y)$ due to change in density ratio, and also in electric load parameter with $D = 1.4$ and $M_1 = 1.2$. The effects are shown pertaining to the range $-1.5 \leq R_e \leq 1.5$ including short circuit case. We note that as $R_e$ is increased from $-1.5$ to $1.5$ via short circuit case ($R_e = 0$), the velocity decreases. The decrease is more prominent in Region-II, where the channel wall is fixed. As expected, an increase in density $D$ leads to a decrease in velocity in both the regions.
Fig. 9 Effect of $D$ on velocity field at $V = 0.4$ and $M_1 = 1.2$

Fig. 10 Effect of $Re$ on velocity field at $M_1 = 1.2$, $V = 0.4$ and $D = 0.5$

Fig. 10 demonstrates variations in the velocity field for different values of electric load parameter in the range $-1.5 \leq Re \leq 1.5$ with fixed values $M_1 = 1.2$,
$V = 0.4$ and $D = 0.5$. It is noted that an increase in electric load parameter decreases the velocity. The physics behind this phenomenon is that increase in electric load parameter increases magnetic field via Lorentz force, thereby reduced velocity.

Fig.-11 shows velocity profiles of the present case and the particular cases shown in section-4. The variations in the velocity are shown for the present case ($M_1 = 1.2$, $R_e = 0.0$, $D = 0.5$ and $V = 0.4$); Singh et. al. [2008] ($M_1 = 1.5$, $R_e = -0.8$, $G = 1.5$ and $\beta = 0.5$, $\alpha = 0.3$, $U_1 = 0.2$ and $U_2 = 1.0$), and also Singha and Deka [2005] ($M_1 = 1.5$, $\alpha = 0.3$, $\beta = 1.0$, $R_e = -1.0$, $G = 1.0$. The velocity profiles shown in figure are in good agreement to the present study. This confirms and verifies that the present study is in improvement on the above mentioned studies.

![Fig.-11. Comparison of present problem and particular cases](image-url)
Table-1 represents variations in the skin-frictions ($\tau_1$, $\tau_2$) at the channel walls and Table-2 represents heat transfer rate ($Nu_1$, $Nu_2$) at the walls due to change in the values of the parameters entered into the equations of the problem. The tables are self explanatory and any discussion about them seems to be redundant.

7. CONCLUSIONS

The effect of electric load parameter on unsteady MHD convective flow of viscous, incompressible immiscible liquids in a horizontal channel is examined for two fluids model. The flow field is subjected to the presence of uniform transverse magnetic field of low Reynolds number, so that induced magnetic field is neglected. Both the liquid are assumed to be electrically conducting and transport properties of the two liquids are taken to be constant. Also, the density, viscosity, electrically conductivity and thermal conductivity of the two liquids are different. The method of solution is applied for small perturbation approximation. Numerical results are presented to illustrate the details of the flow and heat transfer characteristics and their dependence on the material parameters. Important observations of the study are:

❖ An increase in Prandtl number or magnetic parameter decreases the temperature, while increases in electric load parameter or viscosity ratio or density ratio or thermal conductivity ratio increase it.
The velocity decreases with increase in magnetic parameter or electric load parameter, while increases with increase in viscosity ratio or density ratio.

Increased magnetic field or viscosity ratio or electric load parameter decreases skin-friction at both the channel walls.

The rates of heat transfer $Nu_1$ and $Nu_2$ decrease with increase in magnetic field, Prandtl number, electric load parameter or viscosity ratio; but increase with conductivity ratio of density ratio.

It is recognized that there are many other methods that could be considered in order to describe some reasonable solutions for this particular type of problem. Besides for the better understanding of the thermal behaviour of this work, however it may be necessary to perform the experimental works. In the near future, we shall compare these theoretical results with those obtained in another problem where the bounding walls execute periodic motion.

### Table-1

**Variations in skin-friction ($\tau_1$, $\tau_2$) at the channel walls**

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<tr>
<th>$M_1$</th>
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<th>$Re$</th>
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<th>$\tau_2$</th>
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Table-2
Variations in rate of heat transfer \((\text{Nu}_1, \text{Nu}_2)\) at the channel walls

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<tr>
<th>(M_1)</th>
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<th>(R_e)</th>
<th>(P_r)</th>
<th>(D)</th>
<th>(C)</th>
<th>(\text{Nu}_1)</th>
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APPENDIX

\[
M_3 = \sqrt{M_1^2 - n}, \quad M_4 = \sqrt{M_2^2 - \frac{nV}{D}}, \quad M_5 = \sqrt{nP_r},
\]

\[
M_6 = \frac{C}{D} nP_r, \quad P_1 = M_1^2 R_e^2 E_c P_r, \quad P_2 = 2M_1^2 R_e E_c P_r, \quad P_3 = M_1^2 E_c P_r, \quad P_4 = E_c P_r,
\]

\[
P_5 = 2M_1^2 P_4 C_1 C_2 - P_3 \left( R_e^2 + 2C_1 C_2 \right) + P_2 R_e - P_1, \quad P_6 = C_1 \left( 2P_3 R_e - P_2 \right),
\]
\[ P_7 = C_2 \left( 2P_3 R_e - P_2 \right), \]
\[ P_9 = C_2^2 \left( P_4 M_1^2 + P_3 \right), \]
\[ P_{11} = \frac{C_4 \left( 2P_3 R_e - P_2 \right)}{M_3^2 + M_5^2}, \]
\[ P_{13} = \frac{C_2 C_4 \left( 2P_4 M_1 M_3 + 2P_3 \right)}{(M_1 + M_3)^2 + M_5^2}, \]
\[ P_{15} = \frac{C_2 C_3 \left( 2P_4 M_1 M_3 - 2P_3 \right)}{(M_1 - M_3)^2 + M_5^2}, \]
\[ Q_2 = \frac{C}{V} 2M_2^2 R_e E_e P_r, \]
\[ Q_3 = \frac{C}{V} M_2^2 E_e P_r, \]
\[ Q_4 = \frac{C}{V} E_e P_r, \]
\[ Q_5 = 2M_2^2 Q_4 C_5 C_6 - Q_3 \left( R_e^2 + 2C_5 C_6 \right) + Q_2 R_e - Q_1, \]
\[ Q_6 = C_5 \left( 2Q_3 R_e - Q_2 \right), \]
\[ Q_7 = C_6 \left( 2Q_3 R_e - Q_2 \right), \]
\[ Q_8 = C_5^2 \left( Q_4 M_2^2 + Q_3 \right), \]
\[ Q_{11} = \frac{C_8 \left( 2Q_3 R_e - Q_2 \right)}{M_4^2 + M_6^2}, \]
\[ Q_{13} = \frac{C_6 C_8 \left( 2Q_4 M_2 M_4 + 2Q_3 \right)}{(M_2 + M_4)^2 + M_6^2}, \]
\[ Q_{15} = \frac{C_6 C_7 \left( 2Q_4 M_2 M_4 - 2Q_3 \right)}{(M_2 - M_4)^2 + M_6^2}, \]
\[ K_1 = 1 - e^{2M_1}, \]
\[ P_8 = C_1^2 \left( P_4 M_1^2 + P_3 \right), \]
\[ P_{10} = \frac{C_3 \left( 2P_3 R_e - P_2 \right)}{M_3^2 + M_5^2}, \]
\[ P_{12} = \frac{C_1 C_3 \left( 2P_4 M_1 M_3 + 2P_3 \right)}{(M_1 + M_3)^2 + M_5^2}, \]
\[ P_{14} = \frac{C_1 C_4 \left( 2P_4 M_1 M_3 - 2P_3 \right)}{(M_1 - M_3)^2 + M_5^2}, \]
\[ Q_1 = \frac{C}{V} M_2^2 R_e E_e P_r, \]
\begin{align*}
K_2 &= 1 - e^{-2M_2}, \\
K_4 &= 1 - e^{2M_3}, \\
K_6 &= -e^{M_3}, \\
K_8 &= \frac{M_2}{M_1 V} \left(1 + e^{-2M_2}\right), \\
K_{10} &= 1 + e^{2M_3}, \\
K_{12} &= e^{M_3}, \\
L_1 &= 1 - \frac{P_5}{2} - \frac{P_6}{M_1^2} e^{M_1} - \frac{P_7}{M_1^2} e^{-M_1} + \frac{P_8}{4M_1^2} e^{2M_1} + \frac{P_9}{4M_1^2} e^{-2M_1}, \\
L_2 &= 1 - P_{10} e^{M_3} - P_{11} e^{-M_3} + P_{12} e^{(M_1 + M_3)} + P_{13} e^{-(M_1 + M_3)}, \\
L_3 &= \frac{Q_5}{2} + \frac{Q_6}{M_2^2} e^{-M_2} + \frac{Q_7}{M_2^2} e^{M_2} - \frac{Q_8}{4M_2^2} e^{-2M_2} - \frac{Q_9}{4M_2^2} e^{2M_2}, \\
L_4 &= Q_{10} e^{-M_4} + Q_{11} e^{M_4} - Q_{12} e^{-(M_2 + M_4)} - Q_{13} e^{(M_2 + M_4)} + Q_{14} e^{-(M_2 - M_4)} + Q_{15} e^{(M_2 - M_4)}, \\
L_5 &= \frac{P_6}{M_1^2} + \frac{P_7}{M_1^2} - \frac{P_8}{4M_1^2} - \frac{P_9}{4M_1^2} - \frac{1}{C} \left( \frac{Q_6}{M_2^2} - \frac{Q_7}{M_2^2} - \frac{Q_8}{4M_2^2} + \frac{Q_9}{4M_2^2} \right), \\
L_6 &= P_{10} + P_{11} - P_{13} + P_{14} + P_{15} - Q_{10} - Q_{11} - Q_{12} + Q_{13} - Q_{14} - Q_{15}, \\
L_7 &= \frac{P_6}{M_1} - \frac{P_7}{M_1} - \frac{P_8}{2M_1} + \frac{P_9}{2M_1} - \frac{Q_6}{M_2} + \frac{Q_7}{M_2} + \frac{Q_8}{2M_2} - \frac{Q_9}{2M_2}, \\
L_8 &= M_3 P_{10} - M_3 P_{11} - (M_1 + M_3)(P_{12} - P_{13}) + (M_1 - M_3)(P_{14} - P_{15}).
\end{align*}
\[ + \frac{1}{C} \left[ -M_4 Q_{10} + M_4 Q_{11} + (M_2 + M_4)(Q_{12} - Q_{13}) - (M_2 - M_4)(Q_{14} - Q_{15}) \right], \]

\[ L_9 = L_1 + L_3 + L_5, \quad L_{10} = L_8 + \frac{M_5 L_2}{\sinh M_5} - \frac{M_6 L_4}{\sinh M_6}, \]

\[ C_1 = \frac{K_3 K_8 - K_2 K_9}{K_1 K_8 - K_2 K_7}, \quad C_2 = (1 + R_e) e^{M_1} - C_1 e^{2M_1}, \]

\[ C_3 = \frac{K_5 K_{12} - K_6 K_{11}}{K_5 K_{10} - K_4 K_{11}}, \quad C_4 = e^{M_3} - C_3 e^{2M_3}, \]

\[ C_5 = \frac{K_3 K_7 - K_1 K_9}{K_1 K_8 - K_2 K_7}, \quad C_6 = R e^{-M_2} - C_5 e^{-2M_2}, \]

\[ C_7 = \frac{(K_4 K_{12} - K_6 K_{10})}{(K_5 K_{10} - K_{11} K_4)}, \quad C_8 = -C_7 e^{-2M_4}, \]

\[ C_9 = \frac{L_9 - CL_7}{1 + C}, \quad C_{10} = L_1 - C_9, \]

\[ C_{11} = \frac{L_{10} - C^{-1} L_6 M_6 \cot M_6}{M_5 \cot M_5 + C^{-1} M_6 \cot M_6}, \quad C_{12} = \frac{L_2 - C_{11} \cos M_5}{\sin M_5}, \]

\[ C_{13} = \frac{C (L_7 + L_9)}{1 + C}, \quad C_{14} = C_{13} - L_3, \]

\[ C_{15} = C_{11} + L_6, \quad C_{16} = \frac{L_4 + C_{15} \cos M_6}{\sin M_6}, \]

\[ G_1 = M_1 \left( C_1 e^{M_1} - C_2 e^{-M_1} \right), \quad G_2 = M_3 \left( C_3 e^{-M_3} - C_4 e^{M_3} \right), \]

\[ G_3 = M_2 \left( C_5 e^{M_2} - C_6 e^{-M_2} \right), \quad G_4 = M_4 \left( C_7 e^{-M_4} - C_8 e^{M_4} \right), \]

\[ G_5 = C_9 + P_3 + \frac{P_6}{M_1} e^{M_1} - \frac{P_7}{M_1} e^{-M_1} - \frac{P_8}{2M_1} e^{2M_1} + \frac{P_9}{2M_1} e^{-2M_1}, \]
\[ G_6 = M_5 c_1 \sin M_5 + M_5 c_{12} \cos M_5 + M_3 r_{10} e^{-M_3} - M_3 r_{11} e^{M_3} - (M_1 + M_3) r_{12} e^{-(M_1 + M_3)} \]
\[ + (M_1 + M_3) h_{13} e^{(M_1 + M_3)} + (M_1 - M_3) r_{14} e^{-(M_1 - M_3)} - (M_1 - M_3) r_{15} e^{(M_1 - M_3)} \]

\[ G_7 = c_{13} + Q_5 + \frac{Q_6}{M_2} e^{M_2} - \frac{Q_7}{M_2} e^{-M_2} - \frac{Q_8}{2M_2} e^{2M_2} + \frac{Q_9}{2M_2} e^{-2M_2} , \]

\[ G_8 = M_6 c_{15} \sin M_6 + M_6 c_{16} \cos M_6 + M_4 q_{10} e^{-M_4} - M_4 q_{11} e^{M_4} - (M_2 + M_4) q_{12} e^{-(M_2 + M_4)} \]
\[ + (M_2 + M_4) q_{13} e^{(M_2 + M_4)} + (M_2 - M_4) q_{14} e^{-(M_2 - M_4)} - (M_2 - M_4) q_{15} e^{(M_2 - M_4)} \]

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