CHAPTER IV

Structure Connection in a Lorentzian Paracontact Manifold
4.1. Preliminaries: Matsumoto and Mihai [1] introduced the idea of Lorentzian paracontact structure in a differentiable manifold. Prasad and Ojha [3] studied its several properties. The structure connection in an almost contact metric manifold has been studied by Sinha and Yadav [4]. In the present chapter we have studied the structure connection in a Lorentzian paracontact Riemannian manifold and established various results.

Let $\mathbb{M}^n$ be an $n$-dimensional differentiable manifold of class $C^0$ equipped with a $(1,1)$ tensor field $\mathcal{F}$ of class $C^0$ and of rank $n-1$, a vector field $\mathcal{U}$ and a 1-form $\mathcal{U}$ satisfying

$$\mathcal{F}^2 = \mathcal{I} + \mathcal{U} \otimes \mathcal{U}, \quad \mathcal{U} = 0,$$

where $\mathcal{X} \overset{\text{def}}{=} \mathcal{F}(\mathcal{X})$, for arbitrary vector field $\mathcal{X}$. Then $\mathbb{M}^n$ is called a Lorentzian paracontact manifold [1,3]. On a Lorentzian paracontact manifold $\mathbb{M}^n$, we have

$$(4.1.2) \quad \begin{align*}
(a) & \quad \mathcal{U}(\mathcal{U}) = -1, \\
(b) & \quad \mathcal{U} \circ \mathcal{F} = 0.
\end{align*}$$

Let $\mathbb{M}^n$ be also endowed with a non-zero metric tensor $g(\mathcal{X}, \mathcal{Y})$, defined by

$$(4.1.3) \quad \begin{align*}
(a) & \quad \mathcal{F}(\mathcal{X}, \mathcal{Y}) \overset{\text{def}}{=} g(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) = \mathcal{F}(\mathcal{Y}, \mathcal{X}), \\
(b) & \quad \mathcal{F}(\mathcal{X}, \mathcal{Y}) = \mathcal{F}(\mathcal{X}, \mathcal{Y}), \\
(c) & \quad g(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) + \mathcal{U}(\mathcal{X}) \mathcal{U}(\mathcal{Y}), \\
(d) & \quad g(\mathcal{X}, \mathcal{U}) = \mathcal{U}(\mathcal{X}),
\end{align*}$$

where $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ is a 2-form and $\mathcal{X}, \mathcal{Y}$ are arbitrary vector fields. Then $\mathbb{M}^n$ is called a Lorentzian paracontact Riemannian manifold [1,3].
Let $D$ be the Riemannian connection in $M^n$, then

\begin{equation}
(D_x u) Y = g(D_x U, Y).
\end{equation}

**Definition (4.1.1):** A Lorentzian paracontact manifold $M^n$ is called Lorentzian Para-Sasakian manifold if [3]

\begin{equation}
(D_x F)(Y) = g(X, Y)U + u(Y)X + 2u(X)u(Y)
\end{equation}

and

\begin{equation}
(D_x U) = \vec{X}.
\end{equation}

**Definition (4.1.2):** If in a Lorentzian paracontact Riemannian manifold $M^n$,

\begin{equation}
(D_x u) Y = -g(\bar{X}, Y) = -'F(\bar{X}, Y) = -'F(\bar{X}, Y),
\end{equation}

then $M^n$ is called a Lorentzian SP-Sasakian manifold.

**Definition (4.1.3):** Lorentzian paracontact Riemannian manifold is said to be LP-cosymplectic if

\begin{equation}
D_x F = 0 \quad \text{or} \quad (D_x u) Y = 0.
\end{equation}

In a LP-cosymplectic manifold, we have

\begin{equation}
(D_x 'F)(\bar{Y}, Z) + (D_x 'F)(Y, \bar{Z}) = 0,
\end{equation}

\begin{equation}
(D_x 'F)(\bar{Y}, \bar{Z}) + (D_x 'F)(Y, Z) = 0.
\end{equation}

**Definition (4.1.4):** A Lorentzian paracontact Riemannian manifold is called a LP-nearly cosymplectic manifold if [3]

\begin{equation}
(D_x F) X = 0, \quad \text{or} \quad (D_x U) = 0,
\end{equation}

which is equivalent to

\begin{equation}
(D_x F)(Y) + (D_y F)(X) = 0.
\end{equation}

Also, we have [1,3]

\begin{equation}
(D_x 'F)(Y, Z) + (D_y 'F)(X, Z) = 0.
\end{equation}
Let $M^n$ be an $n$-dimensional Riemannian manifold of class $C^\infty$ and let $F^\prime, U^\prime, u^\prime$ be $(1,1)$ tensor field, a vector field and a 1-form respectively. Let us define a connection $E$ in $M^n$ as

\begin{align}
&\text{(a)} \quad E_x Y = D_x Y + u^\prime(Y)X - g(X, Y)G, u^\prime + \mathcal{F}(X, Y) U^\prime, \\
&\text{(b)} \quad E_x g = 2u^\prime(X)g, \\
&\text{(c)} \quad g(F^\prime X, Y) = g(X, F^\prime Y) = \mathcal{F}(X, Y), \\
&\text{(d)} \quad g(G, u^\prime, X) = u^\prime(X),
\end{align}

(4.1.14)

where $\mathcal{F}(X, Y)$ is a 2-form and $G$ is a metric tensor, then $E$ is called a structure connection in $M^n$[4].

The torsion tensor $S(X, Y)$ of the connection $E$ is defined as

\begin{align}
S(X, Y) &= u^\prime(Y)X - u^\prime(X)Y + 2\mathcal{F}(X, Y)U^\prime.
\end{align}

(4.1.15)

If we put

\begin{align}
E_x Y &= D_x Y + H(X, Y),
\end{align}

(4.1.16)

where $H(X, Y)$ is a fundamental 2-form; then

\begin{align}
S(X, Y) &= H(X, Y) - H(Y, X).
\end{align}

(4.1.17)

Let us define

\begin{align}
\mathcal{H}(X, Y, Z) &= g(\mathcal{H}(X, Y), Z) = \mathcal{H}(X, Y, Z) - \mathcal{H}(Y, X, Z),
\end{align}

(4.1.18)

where

\begin{align}
\mathcal{H}(X, Y, Z) &= g(H(X, Y), Z).
\end{align}

(4.1.19)

Hence equation (4.1.14) a can be written as

\begin{align}
\mathcal{H}(X, Y, Z) + \mathcal{H}(X, Z, Y) = -2u^\prime(X)g(Y, Z).
\end{align}

(4.1.20)

Let us define a mapping

\begin{align}
P : T(M^n)^\prime \times T(M^n) \times T(M^n) \rightarrow \Omega(M^n)
\end{align}

66
such that

\[(4.1.21) \quad P(v, X, Y) = g(S(X, G. v), Y) + g(S(Y, G. v), X),\]

where \(T(M^\nu)\) is a set of all vector fields, \(T(M^\nu)\) is a set of all 1-forms and \(\mathcal{C}(M^\nu)\) is a set of all \(C^\nu\) functions on \(M^\nu\).

In view of (4.1.15) we have

\[(4.1.22) \quad P(v, X, Y) = 2u^\nu(G. v)g(X, Y) - u^\nu(X)v(Y) - u^\nu(Y)v(X) + 2v(F^0(X))g(U^\nu, Y) + 2v(F^0(Y))g(U^\nu, X).\]

Also in consequence of (4.1.18), (4.1.20) and (4.1.21) we have

\[(4.1.23) \quad P(v, X, Y) = 2u^\nu(G. v)g(X, Y) - 2u^\nu(X)v(Y) - 2u^\nu(Y)v(X) - v(H(X, Y) + H(Y, X)).\]

Thus from (4.1.22) and (4.1.23) we have for every 1-form

\[
v(H(X, Y) + H(Y, X)) + u^\nu(X)v(Y) + u^\nu(Y)v(X) + 2v(F^0(X))g(U^\nu, Y) + 2v(F^0(Y))g(U^\nu, X) = 0;
\]

consequently

\[(4.1.24) \quad H(X, Y) + H(Y, X) + u^\nu(X)Y + u^\nu(Y)X + 2F^0(X)g(U^\nu, Y) + 2F^0(Y)g(U^\nu, X) = 0.\]

From equations (4.1.15) and (4.1.17) we have

\[(4.1.25) \quad H(X, Y) - H(Y, X) = u^\nu(Y)X - u^\nu(X)Y + 2F^0(X, Y)U^\nu.\]

In view of (4.1.24) and (4.1.25) we have

\[(4.1.26) \quad H(X, Y) = -u^\nu(X)Y - F^0(X)g(U^\nu, Y) - F^0(Y)g(U^\nu, X) + F^0(X, Y)U^\nu.\]
Thus in consequence of (4.1.16) and (4.1.26), the structure connection $E$ in a Riemannian manifold is given by [4]

\[(4.1.27)\]

\[E_x Y = D_x Y - u^*(X) Y - \bar{X} u^*(Y) - \bar{Y} u^*(X) + g(\bar{X}, Y) U.\]

### 4.2 Structure connection:

Let $M'$ be a Lorentzian paracontact structure manifold, then the structure connection on $M'$ is given by

\[(4.2.1)\]

\[E_x Y = D_x Y - u(X) Y - \bar{X} u(Y) - \bar{Y} u(X) + g(\bar{X}, Y) U,\]

which is equivalent to

\[(4.2.2)\]

\[(E_x u) Y = (D_x u) Y + u(X) u(Y) - F(X, Y).\]

**Theorem (4.2.1):** On a Lorentzian paracontact Riemannian manifold we have the following:

- **(a)** $S(\bar{X}, Y) = 2F(X, Y) U,$
- **(b)** $S(X, U) = S(\bar{X}, U) = -\bar{X},$
- **(c)** $S(\bar{X}, U) = -\bar{X},$
- **(d)** $S(\bar{X}, Y) = u(Y) X + u(X) u(Y) U + 2F(X, Y) U,$
- **(e)** $S(\bar{X}, Y) - S(X, \bar{Y}) = u(Y) \bar{X} + u(X) \bar{Y},$
- **(f)** $u(S(X, Y)) = 2F(X, Y).$

**Proof:** The proof follows by virtue of the equations (4.1.1), (4.1.2), (4.1.3) and (4.1.15).

In a Lorentzian paracontact Riemannian manifold with structure connection $E$, we have
(a) \((E_x u) \bar{Y} = (D_x u) \bar{Y} - g(\bar{X}, \bar{Y}),\)

(b) \((E_x F)(Y, Z) = 2u(X)g(\bar{Y}, Z) + g((E_x F) Y, Z),\)

(4.2.4) \((c) (E_x F) Y = (D_x F) Y - g(X, Y)U - u(Y)X,\)

(d) \(E_x \bar{Y} = D_x \bar{Y} + g(X, Y)U.\)

**Theorem (4.2.2):** In a Lorentzian paracontact Riemannian manifold with structure connection \(E\), we have

(4.2.5) \((a) 'H(X, Y, Z) = u(Z) 'F(X, Y) - u(X) 'F(Y, Z) - u(Y) 'F(X, Y) - u(X) g(Y, Z),\)

(b) \('H(X, Y, Z) = 'S(\bar{X}, \bar{Y}, \bar{Z}) = 0.\)

**Proof:** The proof follows easily in view of equations (4.1.3)d and (4.1.26).

**Theorem (4.2.3):** If \(E\) is the structure connection in a Lorentzian paracontact Riemannian manifold \(M\), then we have

(4.2.6) \((a) (E_x F)(Y, Z) = (D_x F)(Y, Z) + 2u(X)F(Y, \bar{Z}) + u(Y)F(X, \bar{Z}) + u(Z)F(X, \bar{Y}) + 2u(X)F(Y, Z),\)

(b) \((E_x F)(\bar{Y}, \bar{Z}) = (D_x F)(\bar{Y}, \bar{Z}).\)

**Proof:** We know that [2]

\[ X(F(Y, Z)) = (E_x F)(Y, Z) + F(E_x Y, Z) + F(Y, E_x Z), \]

\[ X(F(Y, Z)) = (D_x F)(Y, Z) + F(D_x Y, Z) + F(Y, D_x Z). \]

From the above equations, we have

\((E_x F)(Y, Z) = (D_x F)(\bar{Y}, \bar{Z}) - g(H(X, Y), \bar{Z}) - g(\bar{Y}, H(X, Z)).\)

That is,

(4.2.7) \((E_x F)(Y, Z) = (D_x F)(Y, Z) - 'H(X, Y, \bar{Z}) - 'H(X, Z, \bar{Y}).\)
Also we have

\[ (4.2.8) \quad \mathcal{H}(X, Y, \overline{Z}) + \mathcal{H}(X, Z, \overline{Y}) = -2 u(X) \mathcal{F}(Y, \overline{Z}) - u(Y) \mathcal{F}(X, \overline{Z}) + u(Z) \mathcal{F}(X, \overline{Y}) - 2u(X) \mathcal{F}(Y, Z). \]

From equations (4.2.7) and (4.2.8), we have (4.2.6) a. Again barring X, Y and Z in (4.2.6) a we have (4.2.6) b.

**Theorem (4.2.4):** If \( M \) is a Lorentzian Para-Sasakian manifold with structure connection \( E \), then

\[ \begin{align*}
(4.2.9) \quad (a) \quad & E_X U = X + 2 \overline{X} - \overline{X}, \\
& (b) \quad (E_X F) Y = 2 u(X) u(Y), \\
& (c) \quad (E_X F) Y = (E_X F) \overline{Y} = 0.
\end{align*} \]

**Proof:** In view of definition (4.1.1) and equation (4.1.2), (4.2.4) c the theorem follows.

**Theorem (4.2.5):** In Lorentzian SP-Sasakian manifold with structure connection \( E \), we have

\[ (4.2.10) \quad (E_X u) \overline{Y} = -\{\mathcal{F}(X, Y) + \mathcal{F}(X, \overline{Y})\}. \]

**Proof:** In view of definition (4.1.2) and equation (4.2.4) a the result follows.

**Theorem (4.2.6):** The necessary condition for a Lorentzian Para-Sasakian manifold to be an LP-nearly cosymplectic manifold with structure connection \( E \), is that

\[ (4.2.11) \quad (E_X U) = X + \overline{X} - \overline{X}, \]
Proof: The proof follows from the definitions (4.1.1), (4.1.4) and equation (4.2.1).

Theorem (4.2.7): In a LP- casymplectic manifold with structure connection $E$, we have

\[ (E_x u)Y = -g(X, Y), \]

\[ (E_x F)Y = -g(X, Y)U - u(Y)X. \]

Proof: In view of definition (4.1.3) and equations (4.2.4) a, c, the proof is trivial.

Theorem (4.2.8): In a Lorentzian paracontact structure manifold with structure connection $E$, we have

\[ (E_x F)(\bar{Y}, Z) - (E_x F)(\bar{Y}, \bar{Z}) = (D_x F)(\bar{Y}, Z) - (D_x F)(\bar{Y}, \bar{Z}). \]

Proof: From equation (4.2.6) a, the proof follows.

4.3 Affine connection:

Let $B$ be an affine connection in a Lorentzian paracontact manifold $M$ and let a vector valued bilinear function $S^*$ be its torsion tensor, so that

\[ S^*(X, Y) = B_X Y = B_Y X - [X, Y], \]

where the square brackets $[ ]$ stand for lie brackets. The following conditions hold for the affine connection $B$ [5].

\[ B_{fx} Y - B_{yX} = F[F X, Y] + F(Y)u(X) + F(X)u(Y) + UY(u(X)) \]
and

\[(4.3.3)\quad B_x Y - B_{xy} F_X = F [X, FY] - F (Y) u (X) - F (X) u (Y)\]
\[+ UX (u(Y)).\]

The Nijenhuis tensor of \(F\) is given by

\[(4.3.4)\quad N(X, Y) = [FX, FY] - F [FX, Y] - F [X, FY] + F^2 [X, Y],\]

which in view of \((4.1.1)\) becomes

\[(4.3.5)\quad N(X, Y) = [FX, FY] - F [FX, Y] - F [X, FY] + [X, Y] + u ([X, Y]) U.\]

Theorem \((4.3.1)\): we have

\[N(X, Y) = u ([X, Y]) U + UY (u(X)) + UX (u(Y)).\]

Proof: In view of \((4.3.2)\), \((4.3.3)\) and \((4.3.5)\), the theorem follows.

Theorem \((4.3.2)\): A necessary condition for a Lorentzian paracontact structure manifold to be integrable is that

\[u ([X, Y]) U + UY (u(X)) + UX (u(Y)) = 0.\]

Proof: In consequence of the theorem \((4.3.1)\) and vanishing the Nijenhuis tensor of \(M^n\) gives the proof of the theorem.
REFERENCES

   On certain transformation in an LP- Sasaki em Manifold,
   Tensor N.S. 47.

   Structures on a differentiable manifold and their applications,
   Chandrama Prakashan, Allahabad.

   Lorentzian paracontact submanifolds,

   Structure connection in an almost contact metric manifold,

5. Upadhyay, M.D. and Dube, K.K. (1976)
   Almost contact hyperbolic (f, g, η, ξ)- structure,